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# Construction of Aspherical Manifolds from Special $G$-Manifolds

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## Introduction

Let $G$ be a compact connected Lie group acting smoothly and effectively on a manifold $X$. We say that $X$ is a (smooth) special $G$-manifold (see K. Jänich [6]) if for each $x \in X$ the slice representation $G_x \to GL(V_x)$ is the direct sum of a transitive and a trivial representation. In this case the orbit space $M=X/G$ is a differentiable manifold with boundary. K. Jänich showed that a special $G$-manifold $X$ is constructed by a Lie group $G$, an orbit space $M$ and an admissible orbit fine structure over $M$ (roughly speaking, isotropy groups of $G$ at $x \in X$).

Note that the following fact is known: If $G$ is abelian, then $S[U_A] \cong \prod [G] \cong [M; BG]$ (see [6, Corollary 1]). That is, the isomorphic class $[X]$ depends only on the isomorphic class of the $G$-principal bundle $P$, and the class $[X]$ corresponds to a homotopy class of maps of $M$ into the classifying space $BG$. But actually the homotopy groups of $X$ can not be computed directly even if the homotopy groups of $M$ are computable. In general also we do not know whether this $X$ is an *aspherical* (i.e., its universal covering is contractible) manifold or not.

In this paper we give a condition that the special $G$-manifold is aspherical. In this case it is known from the result of Conner and Raymond [1, Theorem 5.6] that $G$ is a toral group and all isotropy groups are finite. And under this condition it follows from Lemma 1 that the orbit structure $U_A$ over $M$ is a family of $U_\alpha$ which is isomorphic to $\boldsymbol{Z}_2$. And our main result is the following

THEOREM 1. *Let $T^k$ be a $k$-dimensional toral group ($k>0$), $M^m$ an $m$-dimensional compact connected differentiable manifold with boundary $\partial M=\bigcup_{\alpha\in A} B_\alpha$, where $B_\alpha$ is a connected component ($m>0$). Let* $(\boldsymbol{Z}_2)_A =$

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*$\{(Z_2)_\alpha\}_{\alpha \in A}$ be the orbit structure over $M^m$. Let $X^{k+m}$ be the special $T^k$-manifold over M constructed by Jänich's method. Then X is aspherical if and only if $M \bigcup_\partial M$ is aspherical.*

It follows that an aspherical special $T$-manifold $X$ over $M$ is an aspherical Seifert fibered manifold in the sense of Conner and Raymond [3]. We give some examples of aspherical special $T$-manifold of dimension 3 and 4 in Section 3.

## § 1. Prerequisites.

Let $G$ be a compact connected Lie group acting differentiably and effectively on a differentiable manifold $X$ and $\pi: X \to X/G = M$ be a natural projection. Let $G_x \to GL(V_x)$ be the induced representation of the isotropy group $G_x$ in the normal space $V_x$ of the orbit $Gx$ at a point $x \in X$. The representation of the compact Lie group in an $n$-dimensional real vector space is called *transitive* if its orbits different from $\{0\}$ are homeomorphic to $S^{n-1}$. A $G$-manifold $X$ is called *special* if for each $x \in X$ the representation $G_x \to GL(V_x)$ is the direct sum of a transitive and a trivial representation. If $X$ is a special $G$-manifold, then the orbit space $X/G = M$ has a "canonical" structure as a differentiable manifold with boundary. A pair $(X, \pi: X \to M)$ is called a *$G$-manifold over $M$.*

Now we consider the case that $G$ is a toral group $T$. Let $M$ be a connected compact differentiable manifold with boundary, and denote by $B_\alpha$ $(\alpha \in A)$ its boundary component and $M_0 = M - \partial M$. An *orbit structure* $U_A = \{U_\alpha\}_{\alpha \in A}$ *over* $M$ consists of closed subgroups $U_\alpha$, $\alpha \in A$, of $T$ such that for each $\alpha \in A$ there is a transitive representation of an isotropy group $U_\alpha$ at the zero point.

Given a special $T$-manifold $X$ over $M$, let $Y_\alpha = \pi^{-1}(B_\alpha)$ for each $\alpha \in A$, then $Y_\alpha$ is a compact differentiable $T$-invariant submanifold of $X$. Let $E_\alpha$ be the normal bundle of $Y_\alpha$ in $X$. Then $E_\alpha$ is a $T$-equivariant vector bundle under the induced operation of $T$ on $E_\alpha$, and there is an equivariant diffeomorphism from an open $T$-invariant neighborhood of a zero section of $E_\alpha$ onto an open $T$-invariant neighborhood of $Y_\alpha$ in $X$. Let the isotropy group at any point over $M_0$ be $\{e\}$, and $y_\alpha \in Y_\alpha$. Then the representation $G_{y_\alpha} \to GL(E_{\alpha, y_\alpha})$ is transitive and putting $U_\alpha = G_{y_\alpha}$ we have an orbit structure $U_A$ over $M$.

Next we will sketch Jänich's method for the construction of a special $T$-manifold $X$ from a $T$-principal bundle $(P, \tilde{\pi}: P \to M)$. Since there is a transitive representation of an isotropy group $U_\alpha$, $\alpha \in A$, we may take an orthogonal representation $U_\alpha \to O(k_\alpha)$ and then an isotropy group at a point $e_{k_\alpha}=(0, \cdots, 0, 1)\in S^{k_\alpha-1}$ is $\{e\}$ since $T$ is abelian. It follows that the orbit $S^{k_\alpha-1}$ is homeomorphic to $U_\alpha$ for each $\alpha\in A$, so that $U_\alpha$ is isomorphic to $S^1$ or $\boldsymbol{Z}_2$; hence $k_\alpha=2$ or 1 respectively. Then taking the representation space $\boldsymbol{R}^{k_\alpha}$ of $U_\alpha$-principal fiber bundle $T\to T/U_\alpha$, we can construct the $T$-equivariant vector bundle $F_\alpha=T\times_{U_\alpha}\boldsymbol{R}^{k_\alpha}$ over $T/U_\alpha$ with the $T$-invariant Riemannian metric and we have

$$E_\alpha=F_{\alpha\, T}\!\times P_\alpha$$

and

$$Y_\alpha=T/U_{\alpha\, T}\!\times P_\alpha$$

where $P_\alpha=P|B_\alpha$. With the canonical projection, $E_\alpha\to Y_\alpha$ is the $T$-equivariant vector bundle over $Y_\alpha$ with a $T$-invariant Riemannian metric. Let $E\to Y$ denote the disjoint union of $E_\alpha\to Y_\alpha$.

From now suppose $U_\alpha$ is isomorphic to $\boldsymbol{Z}_2$ for all $\alpha\in A$. Then the total space of the sphere bundle $SF_\alpha=T\times_{(\boldsymbol{Z}_2)_\alpha}S^0$ of $F_\alpha$ is isomorphic to $T$, and $SF_{\alpha\, T}\!\times P_\alpha$ is the sphere bundle $SE_\alpha$ of $E_\alpha$. Also there is the canonical equivariant diffeomorphism $i_\alpha\colon SE_\alpha\to P_\alpha$. Therefore $T$-manifold to construct is essentially $\{v\in E\,|\,\|v\|\leqq 1\}\bigcup_i P$. Choose a collar $\kappa$ which is a diffeomorphism of $\partial M\times I$ onto a closed neighborhood of $\partial M$ in $M$, where $\partial M\times\{0\}\to M$ is an inclusion. Let a map $\mathrm{pr}_1\colon\partial M\times I\to\partial M$ be a projection onto the first factor. Then $(\mathrm{pr}_1)^*(P|\partial M)\cong\kappa^*P$, and we choose such an isomorphism which is the identity map over $\partial M$. Therefore we have the following commutative diagram.

$$\begin{array}{ccccc}
\{v\in E\,|\,\|v\|<1\} & \supset & \{v\in E\,|\,0<\|v\|<1\} & \xrightarrow{(2)} & SE\times(0,1)\\
\big\downarrow{\scriptstyle (1)} & & \big\downarrow & & \big\downarrow\\
\partial M\times[0,1) & \supset & \partial M\times(0,1) & \longrightarrow & \partial M\times(0,1)
\end{array}$$

$$\begin{array}{ccccc}
\xrightarrow{i\times\mathrm{Id}} & (P|\partial M)\times(0,1) & \xrightarrow{\approx} & P|\kappa(\partial M\times(0,1))\subset P_0=\tilde{\pi}^{-1}(M_0) & \\
 & \big\downarrow & & \big\downarrow \qquad\qquad \big\downarrow{\scriptstyle \tilde{\pi}} & \\
\longrightarrow & \partial M\times(0,1) & \xrightarrow{\kappa} & \kappa(\partial M\times(0,1))\subset M_0 &
\end{array}$$

where (1) is given by a projection onto $\partial M$ and $\|\cdots\|$, and (2) is defined by $v\mapsto(v/\|v\|, \|v\|)$. Then $\pi\colon X\to M$ is constructed from the disjoint union $\{v\in E\,|\,\|v\|<1\}\to\partial M\times[0,1)$ and $P_0\to M_0$ by identifying each corresponding points under

$$\begin{array}{ccc}
\{v\in E\,|\,0<\|v\|<1\} & \longrightarrow & P_0\\
\big\downarrow & & \big\downarrow\\
\partial M\times(0,1) & \longrightarrow & M_0\,.
\end{array}$$

Then this construction yields the following classifications theorem of K. Jänich.

THEOREM. *For each orbit structure $(Z_2)_A$ over $M$ the isomorphism class of special $T$-manifolds over $M$ is classified by the $T^k$-principal bundle $P\to M$, that is*

$$S[(Z_2)_A]\approx\prod [T^k]\approx[M\colon BT^k]\approx H^2(M; Z^k)\ .$$

This theorem is proved in [6, pp. 309-312 and Corollary 1] (also cf. [4]). (Note that $S[(Z_2)_A]$ is the set of equivalence classes of special $T$-manifolds over $M$ and $\prod [T^k]$ is the set of isomorphic classes of $T$-principal bundle over $M$.)

## § 2. Construction theorem.

It is known that if $(G, X)$ is a compact connected Lie group acting effectively on a compact aspherical manifold then $G$ is a toral group and all isotropy groups are finite (Conner and Raymond [1, Theorem 5.6]). As above we have seen that if $X$ is a special $T$-manifold then $U_\alpha=S^1$ or $Z_2$. Therefore we have the following

LEMMA 1. *Let $(T, X)$ be a toral group $T$ acting differentiably and effectively on an aspherical special $T$-manifold $X$, then its orbit structure over $M$ is $U_A=(Z_2)_A$, where $(Z_2)_\alpha$ is isomorphic to $Z_2$ for each $\alpha\in A$.*

From now let $T$ be a toral group, $M^m$ an $m$-dimensional compact connected differentiable manifold with boundary $\partial M=\bigcup_{\alpha\in A} B_\alpha$ ($B_\alpha$ is a boundary component), and $(Z_2)_A$ an orbit structure over $M$. Then we shall investigate a special $T$-manifold over $M$ constructed by Jänich's method.

Let $\tilde{\pi}\colon P\to M^m$, $\tilde{\pi}'\colon P'\to M'^m$ be the same $T$-principal bundles ($P_\alpha= P|B_\alpha$), and $\tilde{E}_\alpha$ be defined by $(T\times R^1)_T\times P_\alpha$ which is diffeomorphic to $R^1\times P_\alpha$; $[(g, t), x]\mapsto(t, gx)$. Then a map $p_\alpha\colon \tilde{E}_\alpha\to E_\alpha$ defined by

$$p_\alpha([(g, t), x])=[[g, t], x]$$

is a double covering map and its covering transformation is $(t, x)\mapsto(-t, g_\alpha x)$ where $g_\alpha$ is the generator of $(Z_2)_\alpha$. And

$$S\tilde{E}_\alpha=(T\times S^0)_T\times P_\alpha\ .$$

Put

$$S_+\tilde{E}_\alpha=(T\times\{1\})_T\times P_\alpha\underset{i}{\cong}P_\alpha$$

and

$$S_-\widetilde{E}_\alpha=(T\times\{-1\})_T\times P_\alpha \underset{i}{\cong} P'_\alpha \ .$$

Choose a collar $\kappa$ which is a diffeomorphism from $\partial M\times I$ onto a closed neighborhood in $M$, where $\partial M\times\{0\}\to M$ is an inclusion. Let a map $\mathrm{pr}_1$: $\partial M\times I\to\partial M$ be a projection onto the first factor. Then $(\mathrm{pr}_1)^*(p|\partial M)\cong \kappa^*P$, $(\mathrm{pr}_1)^*(P'|\partial M)\cong\kappa^*P'$, and we choose such an isomorphism which is the identity map on $\partial M$. Therefore we have the following commutative diagram:

$$\begin{array}{ccccccc}
 & & \{v\in\widetilde{E}\,|\,0<t<1\} & \xrightarrow{(2)} & S_+\widetilde{E}\times(0,1) & & \\
\{v=[(g,t),x]\in\widetilde{E}\,|\,|t|<1\} & \supset & & & & & \\
 & \supset & \{v\in\widetilde{E}\,|\,0>t>-1\} & \xrightarrow{(2)'} & S_-\widetilde{E}\times(0,1) & & \\
\downarrow{\scriptstyle (1)} & & \downarrow & & \downarrow & & \\
\partial M\times[0,1) & \supset & \partial M\times(0,1) & \longrightarrow & \partial M\times(0,1) & & \\
 & \xrightarrow{i\times \mathrm{Id}} & (P|\partial M)\times(0,1) & \longrightarrow & P|\kappa(\partial M\times(0,1)) & \subset & P_0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & \xrightarrow{i\times \mathrm{Id}} & (P'|\partial M)\times(0,1) & \longrightarrow & P'|\kappa(\partial M\times(0,1)) & \subset & P'_0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & \longrightarrow & \partial M\times(0,1) & \xrightarrow{\kappa} & \kappa(\partial M\times(0,1)) & \subset & M_0
\end{array}$$

where (1) is given by a projection onto $\partial M$ and $\|\cdots\|$, and (2), (2)$'$ are defined by $v\mapsto(v/\|v\|, t)$ $(t>0)$, $v\mapsto(v/\|v\|, -t)$ $(t<0)$, respectively.

We define a manifold $P\bigcup_\partial P$ by the disjoint union $\{v\in\widetilde{E}\,|\,\|v\|<1\}\to \partial M\times[0, 1)$, $P_0\to M_0$ and $P'_0\to M'_0$ from identifying each corresponding points under

$$\begin{array}{ccc}
\{v\in\widetilde{E}\,|\,0<t<1\} & \longrightarrow & P_0 \\
\downarrow & & \downarrow \\
\partial M\times(0,1) & \longrightarrow & M_0
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\{v\in\widetilde{E}\,|\,0>t>-1\} & \longrightarrow & P'_0 \\
\downarrow & & \downarrow \\
\partial M\times(0,1) & \longrightarrow & M'_0
\end{array}$$

and define a projection $p$: $P\bigcup_\partial P\to X^{m+k}$ by

$$\begin{aligned}
p(x)&=x \qquad \text{for} \quad x\in P_0, \text{ or } P'_0 \\
p(v)&=p_\alpha(v) \quad \text{for} \quad v\in\{v\in\widetilde{E}_\alpha\,|\,\|v\|<1\}\ .
\end{aligned}$$

(Note that $S_+\widetilde{E}_\alpha\underset{i}{\cong}P_\alpha$; $[(g, 1), x]\mapsto gx$ and $S_-\widetilde{E}_\alpha\underset{i}{\cong}P'_\alpha$; $[(g, -1), x]\mapsto gg_\alpha x$). Then we have

LEMMA 2. *$p$ is a double covering map.*

Let $M'$ be a copy of an $m$-dimensional compact connected differentiable manifold $M$, and $M\bigcup_\partial M$ a differentiable manifold naturally obtained by attaching their boundaries. Then we define a map

$$\tilde{\pi}\bigcup_\partial\tilde{\pi}\colon P\bigcup_\partial P \longrightarrow M\bigcup_\partial M$$

by

$$(\tilde{\pi}\bigcup_\partial\tilde{\pi})(x)=\tilde{\pi}(x)\in M \qquad \text{for} \quad x\in P_0$$

$$(\tilde{\pi}\bigcup_\partial\tilde{\pi})(x')=\tilde{\pi}'(x')\in M' \quad \text{for} \quad x'\in P_0' ,$$

and a composite $\tilde{E}_\alpha \underset{p_\alpha}{\rightarrow} E_\alpha \rightarrow B_\alpha\times(-1, 1)\rightarrow M\bigcup_\partial M$ defined by

$$v=[(g, t), x] \longmapsto \begin{cases}\kappa(\tilde{\pi}(x), t) & \text{for} \quad t\geqq 0\\ \kappa(\tilde{\pi}'(x'), -t) & \text{for} \quad t<0 .\end{cases}$$

Then we have

LEMMA 3. *$\tilde{\pi}\bigcup_\partial\tilde{\pi}\colon P\bigcup_\partial P\rightarrow M\bigcup_\partial M$ is a $T^k$-principal bundle over $M\bigcup_\partial M$.*

Now we obtain

THEOREM 1. *Let $T^k$ be a $k$-dimensional toral group ($k>0$), $M^m$ an $m$-dimensional compact connected differentiable manifold with boundary ($m>0$) and $(Z_2)_A$ the orbit structure over $M^m$. Let $X^{k+m}$ be the special $T^k$-manifold over $M$ constructed as above. Then $X$ is aspherical if and only if $M\bigcup_\partial M$ is aspherical.*

PROOF. By Lemma 2 the manifold $X^{m+k}$ is aspherical if and only if $P\bigcup_\partial P$ is aspherical. By Lemma 3 there is an exact sequence of homotopy groups

$$\cdots\longrightarrow \pi_2(T^k, t_0)\longrightarrow \pi_2(P\bigcup_\partial P, x_0)\longrightarrow \pi_2(M\bigcup_\partial M, b_0)\longrightarrow \pi_1(T^k, t_0)\longrightarrow\cdots$$

($t_0\in T^k$, $x_0\in P\bigcup_\partial P$, and $(\tilde{\pi}\bigcup_\partial\tilde{\pi})(x_0)=b_0\in M\bigcup_\partial M$). If $M\bigcup_\partial M$ is aspherical, then it follows easily that $P\bigcup_\partial P$ is aspherical. If $P\bigcup_\partial P$ is aspherical then $\pi_i(M\bigcup_\partial M, b_0)=0$ ($i\geqq 3$) and $\pi_2(M\bigcup_\partial M, b_0)\rightarrow\pi_1(T^k, t_0)\cong \boldsymbol{Z}^k$ is injective, and so $\pi_2(M\bigcup_\partial M, b_0)\cong \boldsymbol{Z}^{k'}$ for some $k'$ ($0\leqq k'\leqq k$). Suppose $k'>0$ and consider the universal covering $\widetilde{M\bigcup_\partial M}$ of $M\bigcup_\partial M$ ($\widetilde{M\bigcup_\partial M}\ni \tilde{b}_0\mapsto b_0\in M\bigcup_\partial M$), then $\pi_i(\widetilde{M\bigcup_\partial M}, \tilde{b}_0)=0$ ($i\neq 2$) and $\pi_2(\widetilde{M\bigcup_\partial M}, \tilde{b}_0)\cong \boldsymbol{Z}^{k'}$, that is, $\widetilde{M\bigcup_\partial M}$ is a $K(\boldsymbol{Z}^{k'}, 2)$-space which has the same homotopy type as the $k'$-fold product of infinite dimensional complex projective spaces $\prod^{k'} CP^{\infty}$. In the cohomology level $H^i(\widetilde{M \bigcup_{\partial} M})=0$ $(i>m)$, since $\widetilde{M \bigcup_{\partial} M}$ is the finite dimensional manifold. But $\prod^{k'} CP^{\infty}$ is infinite dimensional, and also there is an integer $i$ $(>m)$ such that $H^i(\prod^{k'} CP^{\infty}) \neq 0$. This is a contradiction. Therefore $\pi_2(\widetilde{M \bigcup_{\partial} M}, \tilde{b}_0)=\pi_2(M \bigcup_{\partial} M, b_0)=0$ and $M \bigcup_{\partial} M$ is aspherical. q.e.d.

Since any closed 2-manifold except the 2-sphere and the real projective plane is aspherical [2, p. 40], we have

COROLLARY. *A special $T^k$-manifold $X^{2+k}$ over $M^2$ is aspherical if and only if $M^2$ is not diffeomorphic to $D^2$.*

## § 3. Examples.

In this section we shall investigate the aspherical special $T$-manifold over $M$ in the case of dimensions 3 and 4. It follows from the classification theorem of Jänich (see Section 1) that any 3-dimensional aspherical special $T^k$-manifold $(k=1, 2)$ is perfectly determined by Example 2 and Cases 1 and 3 (see the table at the end of this paper) up to the equivariant diffeomorphism in the sense of Neumann [7]. And some examples of 4-dimensional aspherical special $T^k$-manifold $(k=1, 2, 3)$ are given by Examples 1 $(m=3)$, 4, 5 $(m=2)$, 6 and Case 1, etc.

Now in general the aspherical special $T^k$-manifold $X$ over $M$ is constructed from the disjoint union

$$X=P_0 \cup \{v=([[g, t], (g', b)]) \in E=(T^k \times_{(Z_2)_A} R^1)_{T^k} \times P_A \mid \|t\| \leqq 1\}$$

with the identifying relation as indicated in Section 1, where $P \rightarrow M$ is the $T^k$-principal bundle and $M \bigcup_{\partial} M$ is aspherical. Let $K'=X-\{v \in E \mid \|v\|<1\}$, $K''_{\alpha}=\{v \in E_{\alpha} \mid \|v\| \leqq 1\}$, $K^0_{\alpha}=K' \cap K''_{\alpha}$ (which is homeomorphic to $P_{\alpha} \times \{1\}$) and $i'_1: K^0_1 \rightarrow K'$, $i'_2: K^0_2 \rightarrow K' \cup K''_1$, $\cdots$, $i'_n: K^0_n \rightarrow K' \cup (\bigcup_{\alpha=1}^{n-1} K''_{\alpha})$, $i''_{\alpha}: K^0_{\alpha} \rightarrow K''_{\alpha}$ be inclusion maps for each $\alpha \in A=\{1, \cdots, n\}$. Then by Van Kampen's theorem it follows that the fundamental group of an aspherical special $T^k$-manifold $X^{k+m}$ over $M^m$, $\pi_1(X, x_n)$ is isomorphic to the group which is obtained from the free product of $\pi_1(P, x_1)$, $\pi_1(Y_1, x_1)$, $\cdots$ and $\pi_1(Y_n, x_n)$ by adding the relations $i'_{\alpha *}(\omega_{\alpha})=i''_{\alpha *}(\omega_{\alpha})$ for all $\omega_{\alpha} \in \pi_1(P_{\alpha}, x_{\alpha})$, $(x_{\alpha} \in P_{\alpha})$, $\alpha=1, \cdots, n$.

**3.1.** The case of $T=SO(2)$.
Since $F_{\alpha}$ is the Möbius band, we have

EXAMPLE 1.

$$X^{m+1}=P_0 \bigcup_{S^1_* \times_{S^1} P_A} (Mb_{S^1} \times P_A)$$

is an aspherical special $SO(2)$-manifold over $M^m$ if $M\bigcup_\partial M$ is aspherical, where $S^1_*$ is the center circle of the Möbius band $Mb$ and $P\to M$ is the $SO(2)$-principal bundle.

Especially for $m=2$, we have

EXAMPLE 2.

$$X^3=S^1\times M_0^2 \bigcup_{S^1_*\times\partial M} Mb\times\partial M$$

is an aspherical special $SO(2)$-manifold over $M$, where $M$ is any 2-dimensional compact connected differentiable manifold with boundary except $D^2$ and $S^1_*$ is the center circle of the Möbius band $Mb$.

**3.2.** The case of $T=SO(2)\times SO(2)$.

In the case of the orbit structure $\{\{e\}\times Z_2\}_A$, we have

EXAMPLE 3.

$$X=P_0\cup\{v\in(S^1\times(S^1\times_{(Z_2)_A}R^1))_{T^2}\times P_A\,|\,\|t\|\leqq 1\}$$

is an aspherical special $T^2$-manifold over $M^m$ with the orbit structure $\{\{e\}\times Z_2\}_A$ if $M\bigcup_\partial M$ is aspherical, where $P\to M$ is the $T^2$-principal bundle.

Especially for $m=2$, we have

EXAMPLE 4.

$$X^4=S^1\times X^3\longrightarrow M^2$$

is an aspherical special $T^2$-manifold over $M$ with the orbit structure $\{\{e\}\times Z_2\}_A$, where $M$ is any 2-dimensional compact connected differentiable manifold with boundary except $D^2$ and $X^3$ is the manifold of Example 2.

Next we consider the case of the orbit structure $\{Z_2\times\{e\}, \{e\}\times Z_2\}$. Let $N$ be any $m$-dimensional compact connected differentiable manifold without boundary and $M^m=N^m-\mathrm{Int}\,(D_1^m\cup D_2^m)$ $(m\geqq 2,\ D_1^m\cap D_2^m=\varnothing)$ such that $M^m\bigcup_\partial M^m$ is aspherical. Then $M^m=N^m\#(D_1^m\#D_2^m)=N^m\#(I\times S^{m-1})$.

We shall construct an aspherical special $T^2$-manifold $M^m$. First it follows that the special $T^2$-manifold constructed over $D_1^m\#D_2^m$ with an orbit structure $U_A=\{Z_2\times\{e\}, \{e\}\times Z_2\}$ is

$$X^{m+2}=(Mb\times S^1 \bigcup_{S^1\times S^1} S^1\times Mb)\times S^{m-1}\longrightarrow I\times S^{m-1}$$

in the trivial (bundle) case, where $Mb\times S^1\bigcup_{S^1\times S^1}S^1\times Mb$ is the manifold $Mb\times S^1\cup S^1\times Mb$ intersecting canonically in $S^1\times S^1$. Then the restriction $(Mb\times S^1\bigcup_{S^1\times S^1}S^1\times Mb)\times S^{m-1}|_{\mathrm{Int}\,(I\times S^{m-1})}\to\mathrm{Int}\,(I\times S^{m-1})$ is the $T^2$-principal bundle. $P \to N$ be any $T^2$-principal bundle over $N$ and form

$$P' = \left(P - \operatorname{Int}\left(T^2 \times \frac{1}{2} D^m\right)\right) \bigcup_{\partial(T^2 \times D^m/2)} \left(T^2 \times (D_1^m \# D_2^m) - \operatorname{Int}\left(T^2 \times \frac{1}{2} D^m\right)\right)$$

where $D^m/2 = \{(x_1, \cdots, x_m) \in \boldsymbol{R}^m \mid x_1^2 + \cdots + x_m^2 \leqq 1/2\}$ is a disk regarded as imbedded in $D_1^m \# D_2^m$. Then

$$P' \longrightarrow M^m = N^m \# (D_1^m \# D_2^m)$$

is a $T^2$-principal bundle.

Note that the constructed $T^2$-principal bundle is trivial over the boundary $\partial M^m = \partial D_1^m \cup \partial D_2^m$.

Hence more generally we obtain the following

EXAMPLE 5. Let $N^m$ be any $m$-dimensional compact connected differentiable manifold without boundary, $M^m = N^m - \operatorname{Int}(\bigcup_{\alpha=1}^{2n} D_\alpha^m)$ where $m \geqq 2$ and $D_\alpha^m \cap D_{\alpha'}^m = \varnothing$ if $\alpha \neq \alpha'$ and $M \bigcup_\partial M$ be aspherical. Then the aspherical special $T^2$-manifold over $M^m$ with the orbit structure $\{(\boldsymbol{Z}_2 \times \{e\})_{2\alpha-1}, (\{e\} \times \boldsymbol{Z}_2)_{2\alpha}\}_{\alpha=1,\cdots,n}$ is

$$X = \left(P - \bigcup_{\alpha=1}^{n} \operatorname{Int}\left(T^2 \times \frac{1}{2} D_\alpha^m\right)\right)$$
$$\bigcup_{\bigcup_{\alpha=1}^{n} \partial(T^2 \times D_\alpha^m/2)} \left(\bigcup_{\alpha=1}^{n} \left(\left(Mb \times S^1 \bigcup_{S^1 \times S^1} S^1 \times Mb\right) \times S^{m-1} - \operatorname{Int}\left(T^2 \times \frac{1}{2} D_\alpha^m\right)\right)\right)$$

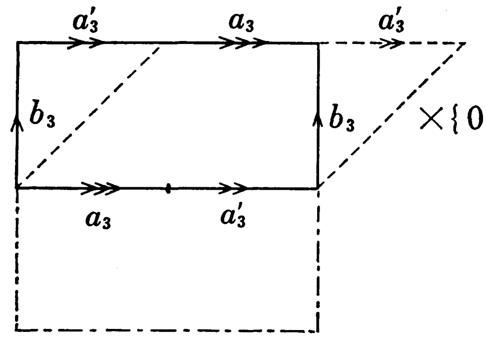
where $P \to N^m$ is any $T^2$-principal bundle.

As an example of another orbit structure, we have

EXAMPLE 6.

$$X^4 = ((T^2 \times M_0 \bigcup_{S^1 \times S_*^1 \times S_1^1} S^1 \times Mb \times S_1^1) \bigcup_{S_*^1 \times S^1 \times S_2^1} Mb \times S^1 \times S_2^1)$$
$$\bigcup_{S^1 \times S^1 \times S_3^1} (T^2 \times_{(\boldsymbol{Z}_2)_3} [-1, 1] \times S_3^1)$$

is an aspherical special $T^2$-manifold over $M = F - \bigcup_{\alpha=1}^{3} \operatorname{Int} D_\alpha^2$ $(\partial D_\alpha^2 = S_\alpha^1, \alpha \in A)$ with the orbit structure $(\boldsymbol{Z}_2)_A = \{\{e\} \times \boldsymbol{Z}_2 = \{(1, 1), (1, -1)\}_1, \boldsymbol{Z}_2 \times \{e\} = \{(1, 1), (-1, 1)\}_2, \{(1, 1), (-1, -1)\}_3\}$ over $M$, where $F$ is a closed surface of genus $g$. For example it follows from $(\boldsymbol{Z}_2)_3$ that $(g_1, g_2, t)$ is equivalent to $(-g_1, -g_2, -t)$ for each $(g_1, g_2, t) \in T^2 \times [-1, 1]$. Then the form of $Y_3$ is as follows:

$$Y_3=\pi^{-1}(S^1_3)=$$  $$\times\{0\}\times S^1_3\approx T^2\times S^1_3.$$


Then by Van Kampen's theorem it follows that the fundamental group of $X^4$ is

$$\begin{aligned}\pi_1(X, x_0)&=(((Z^2\oplus\pi_1(M, b_0)) \underset{Z^3}{*} Z^3) \underset{Z^3}{*} Z^3) \underset{Z^3}{*} Z^3\\ &=((((\langle a\rangle\oplus\langle b\rangle\oplus\pi_1(M)) \underset{Z^3}{*} (\langle a_1\rangle\oplus\langle b_1\rangle\oplus\langle c_1\rangle)) \underset{Z^3}{*} (\langle a_2\rangle\oplus\langle b_2\rangle\oplus\langle c_2\rangle))\\ &\quad \underset{Z^3}{*} (\langle a_3a_3'\rangle\oplus\langle a_3'b_3\rangle\oplus\langle c_3\rangle) \quad \text{(represented by its generators)}\end{aligned}$$

with the relations

$$\begin{aligned}a&=a_1=a_2^2=a_3a_3'\\ b&=b_1^2=b_2=b_3a_3'b_3a_3^{-1}\\ c_1\cdot c_2\cdot c_3&=x_1y_1x_1^{-1}y_1^{-1}\cdots x_gy_gx_g^{-1}y_g^{-1} \quad \text{if } F \text{ is orientable}\\ &=x_1x_1\cdots x_gx_g \quad \text{if } F \text{ is non-orientable}\,.\end{aligned}$$

Now we give some examples of aspherical special $T^k$-manifold for given orbit space $M$.

### Results

Let $Mb$ be the Möbius band and $Kl$ the Klein bottle.

Case 1. $M=I$,

$T=S^0$, $U_A=\{Z_2\}$, $X^1=S^1$

$T=S^1$, $U_A=\{Z_2\}$, $X^2=$ the Klein bottle

$T=T^2$, $U_A=\{Z_2\times\{e\}, \{e\}\times Z_2\}$, $X^3=Mb\times S^1 \underset{S^1\times S^1}{\cup} S^1\times Mb$

$U_A=\{Z_2\times\{e\}\}$ $X^3=Kl\times S^1$

$T=T^k$, $U_A=\{Z_2\times\{e\}\times\cdots\times\{e\}, \{e\}\times\cdots\times\{e\}\times Z_2\}$

$X=Mb\times T^{k-1} \underset{T^k}{\cup} T^{k-1}\times Mb$

$U_A=\{Z_2\times\{e\}\times\cdots\times\{e\}\}$ $X=Kl\times T^{k-1}$.

Case 2. $M=I\times A$ ($A$: an aspherical manifold without boundary)

$T=T^k$, $U_A=\{Z_2\times\{e\}\times\cdots\{e\}, \{e\}\times\cdots\times\{e\}\times Z_2\}$

$$X=Mb\times T^{k-1}\times A \bigcup_{T^k\times A} T^{k-1}\times Mb\times A$$

$U_A=\{Z_2\times\{e\}\times\cdots\times\{e\}\}$ $\quad X=Kl\times T^{k-1}\times A$ .

Case 3. $M=Mb$

$T=S^1$, $U=Z_2$ $\quad X^3=S^1\times Mb \bigcup_{S^1\times S^1} Mb\times S^1$

$T=T^k$, $U=Z_2\times\{e\}\times\cdots\times\{e\}$ $\quad X^{k+2}=(S^1\times Mb \bigcup_{S^1\times S^1} Mb\times S^1)\times T^{k-1}$ .

Case 4. $M=Mb\times A$ ($A$ is an aspherical manifold without boundary)

$T=T^k$, $U=Z_2\times\{e\}\times\cdots\times\{e\}$ $\quad X=(S^1\times Mb \bigcup_{S^1\times S^1} Mb\times S^1)\times T^{k-1}\times A$ .

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