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On the Sequential Approximation of Scalarly Measurable Functions by Simple Functions

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This paper is concerned with the approximation of weak*-measurable functions by means of simple functions. Let X be a Banach space, X^* the dual space of X, and let (S, Σ, μ) be a finite nonnegative complete measure space. In this paper we mainly treat with X^*-valued functions defined on the set S; hence we call (S, Σ, μ) the base measure space in the following. A function $f: S \to X^*$ is said to be weak*-measurable if for every $x \in X$ the numerical function $\langle x, f\rangle$ is μ-measurable. This kind of definition of measurability for X^*-valued functions does not assume the existence of approximate sequence of simple functions and is generically called a scalar measurability. On the other hand, a function $f: S \to X^*$ is said to be strongly measurable if there exists a sequence (f_n) of simple functions with $\lim \|f_n(s)-f(s)\|=0$ a.e.; hence the existence of an approximate sequence (f_n) of simple functions is involved in the definition itself. By use of the martingale argument, it is possible to find for every norm-bounded weak*-measurable function $f: S \to X^*$ a generalized sequence (f_α) of simple functions approximating f in the sense that $\lim_\alpha \|\langle x, f_\alpha\rangle - \langle x, f\rangle\|_{L^1(\mu)}=0$ for each $x \in X$. However, the weak*-measurability of a function f does not necessarily imply the existence of a sequence (f_n) of a countable number of simple functions that approximate f in the sense that

$$(*) \qquad \langle x, f_n\rangle \longrightarrow \langle x, f\rangle \quad \text{a.e.}$$

for each $x \in X$, where the null set on which the convergence does not hold may vary with x. More precisely, such sequential approximation of weak*-measurable functions by simple functions need not be possible unless the Banach space X or else the base measure space (S, Σ, μ) is suitably chosen. We will see later that even a weakly measurable function does not necessarily have approximate sequences if the base

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measure space is not perfect. The approximation problem as mentioned above arised both in the study of generalized derivatives of strongly absolutely continuous functions which take values in non-reflexive Banach spaces (such derivatives are, in many cases, properly weak*-measurable) and in the investigation of nonlinear differential equations in Banach spaces. The approximation problem is not only fundamental in the vector measure theory but also an interesting problem in the Banach space theory in connection with the study of so-called weak Radon-Nikodym property. In what follows, we say that an X^*-valued function f on S has the *sequential approximation property* with respect to the base measure space (S, Σ, μ) if it has an approximate sequence of simple functions satisfying (*).

The first purpose of this paper is to investigate some sufficient conditions for the range space of f and those for the base measure space of f in order that f have the sequential approximation property. We shall establish the following three results:

(i) If the range space X^* has the weak Radon-Nikodym property, then for every finite nonnegative complete measure space (S, Σ, μ) every weak*-measurable function has the sequential approximation property.

(ii) If the base measure space (S, Σ, μ) is separable (i.e., Σ is generated by a denumerable number of subsets of S), then every weak*-measurable function has the sequential approximation property.

(iii) If the base measure space (S, Σ, μ) is perfect, then every norm-bounded Pettis integrable function f has the sequential approximation property.

The proofs of the above theorems (i), (ii), (iii) are given by applying the martingale argument under the respective assumptions as mentioned in the statements.

Recently Geitz has given a characterization of Pettis integrable functions on a perfect measure space in terms of sequential approximation property. In fact, he showed that a norm-bounded weakly integrable function is Pettis integrable if and only if it has the sequential approximation property. Hence Theorem (iii) proves the necessity of his theorem (though our proof is different from the proof due to Geitz). Now our second purpose is to give a characterization of Gel'fand integrable functions in terms of sequential approximation property. Although our results do not characterize complete classes of functions with the sequential approximation property, we believe that these results are general enough for the discussion of important classes of vector measures.

§1. Preliminaries.

In what follows, given a pair $x \in X$ and $x^* \in X^*$ we write $\langle x, x^* \rangle$ for the value $x^*(x)$ of x^* at the point x.

Let (S, Σ, μ) be a finite nonnegative complete measure space. We say that the measure space (S, Σ, μ) is *separable* if the σ-field Σ is generated by a countable number of subsets of S. Recall that (S, Σ, μ) is separable iff $L^1(\mu)$ is a separable Banach space. Let Γ be a linear subset of X^* total on X. A function $f: S \to X$ is said to be *Γ-measurable* (resp. *Γ-integrable*), whenever $\langle x^*, f(\cdot) \rangle$ is measurable (resp. integrable) for every $x^* \in \Gamma$. The Γ-measurability is generically called a scalar measurability. If in particular, $X = Y^*$ for a Banach space Y and $\Gamma = Y$, we say that an X-valued Γ-measurable (Γ-integrable) function f is *weak*-measurable* (resp. *weak*-integrable*). If $\Gamma = X^*$, then f is said to be *weakly measurable* (resp. *weakly integrable*). If $\nu: \Sigma \to X$ is a vector measure, then $|\nu|(\cdot)$ denotes the total variation of ν. If $|\nu|$ is *σ-finite*, then ν is said to be of σ-finite variation. A weak*-measurable function $f: S \to X^*$ is said to be *weak* uniformly bounded* in the sense of Musiał ([11]) if there exists a finite number M such that for every $x \in X$, we have $|\langle x, f \rangle| \leqq M\|x\|$ a.e..

It is well-known (see [14]) that if $f(\cdot): S \to X$ is weakly integrable, then for each $E \in \Sigma$ there exists a $\nu(E) \in X^{**}$ such that $\langle x^*, \nu(E) \rangle = \int_E \langle x^*, f \rangle d\mu$ for all $x^* \in X^*$. Such an element $\nu(E)$ is called the *Dunford integral* of f over E and we write $\nu(E) = (D) - \int_E f d\mu$. If in particular $(D) - \int_E f d\mu \in X$ for each $E \in \Sigma$, then f is said to be *Pettis integrable* and we write $(P) - \int_E f d\mu$ for the integral $(D) - \int_E f d\mu$. It is known (See [4].) that if $f: S \to X^*$ is weak*-integrable, then for each set $E \in \Sigma$ there is a $\nu(E) \in X^*$ such that $\langle x, \nu(E) \rangle = \int_E \langle x, f \rangle d\mu$ for all $x \in X$. In this connection we sometimes say that $f: S \to X^*$ is *Gel'fand integrable* if it is weak*-integrable over S. The element $\nu(E)$ is called the *Gel'fand integral* of f over E; and we write

$$\nu(E) = (G) - \int_E f d\mu \,. \tag{1.1}$$

If in particular ν is of bounded variation, the integrand f is called a *Gel'fand derivative* of ν.

The following theorem due to Rybakov [13] will be frequently used in our argument below:

THEOREM R. *Let (S, Σ, μ) be a finite nonnegative complete measure* *space, X a Banach space, and let $\nu: \Sigma \to X^*$ be a μ-continuous measure of σ-finite variation. Then there exists a weak*-integrable function $f: S \to X^*$ such that $\nu(E) = (G) - \int_E f d\mu$ for $E \in \Sigma$.*

We now introduce two concepts treated in this paper.

DEFINITION 1. Let (S, Σ, μ) be a finite nonnegative complete measure space and let f be a weak*-measurable function from S to X^*. Then the function f is said to have the *sequential approximation property* with respect to (S, Σ, μ) if there exists a sequence (f_n) of X^*-valued simple functions such that for every $x \in X$, we have

$$\lim_{n \to \infty} \langle x, f_n \rangle = \langle x, f \rangle \quad \text{a.e.}, \tag{1.2}$$

where the exceptional set (i.e., the null set on which the convergence does not hold) for (1.2) is allowed to vary with x. Moreover, a dual Banach space X^* is said to have the *sequential approximation property* if for every finite nonnegative complete measure space (S, Σ, μ) and every bounded X^*-valued weak*-measurable function f on S, f has the sequential approximation property with respect to (S, Σ, μ).

DEFINITION 2. Let (S, Σ, μ) be a finite nonnegative complete measure space and let ν be a μ-continuous vector measure. Then ν is said to satisfy *condition* (SA) with respect to (S, Σ, μ) if there exists a sequence (f_n) of X^*-valued simple functions such that for every $x \in X$,

$$\lim_{n \to \infty} \langle x, f_n \rangle = g_x \quad \text{a.e.}, \tag{1.3}$$

where g_x denotes the Radon-Nikodym derivative $d/d\mu \langle x, \nu \rangle$ of the scalar-valued measure $\langle x, \nu \rangle$ on Σ and, in (1.3), the exceptional set is allowed to vary with x.

Suppose that a μ-continuous vector measure $\nu: \Sigma \to X^*$ satisfies condition (SA) with respect to (S, Σ, μ). Let (f_n) be a sequence of simple functions satisfying (1.3) and let ν_n be the indefinite Bochner integral of f_n. If ν has a weak*-integrable derivative f such that $\nu(E) = (G) - \int_E f d\mu$ for $E \in \Sigma$, then $\lim_n \langle x, f_n \rangle = \langle x, f \rangle$ a.e. for each $x \in X$; that is, the weak*-derivative f of ν has the sequential approximation property. Moreover, if $|\langle x, f_n \rangle| \leqq g_x$ a.e. for each $x \in X$ and each $n \geqq 1$, then $\lim_n |\langle x, \nu_n \rangle - \langle x, \nu \rangle|(S) = 0$ for each $x \in X$.

We here mention the relationship between the sequential approximation property for weak*-measurable functions and condition (SA) for vector measures.

PROPOSITION 1.1. *Let $f: S \to X^*$ be weak*-integrable and let ν be the Gel'fand indefinite integral defined through f by* (1.1). *Then f has the sequential approximation property if and only if ν satisfies condition* (SA).

Next, let $f: S \to X^*$ be weak*-measurable. Then one can find by use of the lattice properties of $L^1(\mu)$ a disjoint family of measurable sets $\{S_m \in \Sigma, m \geqq 1\}$ such that $S = \bigcup_{m=1}^{\infty} S_m$ and, for each m, $\chi_{S_m} \cdot f (= f^{(m)})$ is weak* uniformly bounded in the sense of Musiał [11]. Let $\Sigma(S_m) = \{E \in \Sigma: E \subseteqq S_m\}$ and let $\tilde{\nu}_m: \Sigma(S_m) \to X^* (m \geqq 1)$ be vector measures defined respectively through $f^{(m)}$ by (1.1), i.e., $\tilde{\nu}_m(E) = (G) - \int_E f^{(m)} d\mu$ for $E \in \Sigma(S_m)$. Now suppose that there exists a sequence $(f_{m,n})$ of simple functions such that the support of each $f_{m,n}$ is contained in S_m and $\lim_n \langle x, f_{m,n} \rangle = \langle x, f^{(m)} \rangle = d/d\mu \langle x, \tilde{\nu}_m \rangle$ a.e. for each $x \in X$. Then, by putting $f_n = \sum_{m=1}^{n} f_{m,n}$, we have $\lim_n \langle x, f_n \rangle = \langle x, f \rangle$ a.e. for each $x \in X$. So, in order to find an approximate sequence of simple functions for a given weak*-measurable function f, we may assume without loss of generality that f is weak*-uniformly bounded on S; and it suffices to show that the associated vector measure ν satisfies condition (SA) with respect to (S, Σ, μ).

Therefore, we will treat in the following only the approximation of μ-continuous vector measure ν in the sense of Definition 2, rather than that of weak*-measurable function f in the sense of Definition 1.

Finally, in connection with the study of Banach spaces with the sequential approximation property, we list three kinds of Radon-Nikodym properties. A Banach space X is said to have the Radon-Nikodym property (denoted simply as RNP) if for every finite nonnegative complete measure space (S, Σ, μ) and every μ-continuous X-valued measure $\nu: \Sigma \to X$ of bounded variation, there exists a strongly measurable function $f: S \to X$ such that $\nu(E) = (B) - \int_E f d\mu$, $E \in \Sigma$, where "$(B)-$" means that the integral is taken in the sense of Bochner. A Banach space X has the weak Radon-Nikodym property (hereafter denoted WRNP) if every X-valued measure ν on (S, Σ, μ) which is μ-continuous and of bounded variation has a Pettis integrable derivative $f: S \to X$, i.e., $\nu(E) = (P) - \int_E f d\mu$ for $E \in \Sigma$. The RNP is properly stronger than WRNP and a general dual Banach space does not necessarily have WRNP. Likewise, we may define the weak* Radon-Nikodym property (denoted as W*RNP), namely: A dual Banach space X^* has the weak* Radon-Nikodym property if every X^*-valued measure on (S, Σ, μ) which is μ-continuous and of bounded variation has a Gel'fand integrable derivative $f: S \to X^*$, i.e. $\nu(E)=(G)-\int_E f d\mu$ for $E \in \Sigma$. But, as is seen from Theorem (R) as mentioned above, every dual space X^* possesses W*RNP and it is meaningless to discuss the W*RNP for dual Banach spaces.

§2. Characterizations of the sequential approximation property.

Throughout this section, let (S, Σ, μ) be a finite, nonnegative complete measure space. In this section we shall give some general criteria for the sequential approximation property. By a separable σ-subfield Σ_0 of Σ we mean a σ-field generated by a denumerable number of elements of Σ (see [2]).

THEOREM 2.1. *Let $\nu: \Sigma \to X^*$ be a bounded μ-continuous measure. Then ν satisfies condition (SA) with respect to (S, Σ, μ) if and only if there exists a separable σ-subfield Σ_0 of Σ such that all of the μ-null sets are contained in Σ_0 and the Radon-Nikodym derivative $g_x = d/d\mu\langle x, \nu\rangle$ is Σ_0-measurable for each $x \in X$.*

PROOF. The necessity: Let (f_n) be a sequence of simple functions such that for each $x \in X$, $\langle x, f_n\rangle \to g_x$ a.e. as $n \to \infty$. Let $f_n = \sum_{A \in \pi_n} x^*_{n,A} \chi_A$, where $x^*_{n,A}$ are elements of X^* and π_n is a partition of S. Let Σ_n be the σ-subfield generated by π_n and let Σ_0 be the σ-subfield generated by $\bigcup_{n=1}^\infty \Sigma_n$. Then it follows that Σ_0 is a separable σ-subfield of Σ and for each $x \in X$, g_x is Σ_0-measurable.

The sufficiency: Let Σ_0 be a separable σ-subfield of Σ such that g_x is Σ_0-measurable for each $x \in X$. Then an increasing sequence (π_n) of Σ_0-partitions of S can be chosen so that Σ_0 may be generated by $\bigcup_{n=1}^\infty \pi_n$. Let Σ_n be the σ-subfield generated by π_n and put $f_n = f_{\pi_n} = \sum_{A \in \pi_n} (\nu(A)/\mu(A))\chi_A$. Then, the sequence $\{\langle x, f_n\rangle, \Sigma_n, n \geqq 1\}$ form a martingale for each $x \in X$. Moreover, since (f_n) is L^1-bounded and $\int_E |\langle x, f_n\rangle| d\mu \leqq |\langle x, \nu\rangle|(E)$ for each $E \in \Sigma_n$ ($|\langle x, \nu\rangle|(E)$ being the total variation of the measure $\langle x, \nu(\cdot)\rangle$ over E), one has $\lim_{\substack{\mu(E)\to 0 \\ E \in \Sigma_n}} \int_E |\langle x, f_n\rangle| d\mu = 0$. Now by the martingale mean convergence theorem (See e.g., [4, V. 2.4 and V. 2.8].) there exists a conditional expectation $\tilde{g}_x = E(g_x | \Sigma_0)$ of g_x relative to Σ_0 such that $\langle x, f_n\rangle \to \tilde{g}_x$ in $L^1(\mu)$ as $n \to \infty$. Since g_x is also Σ_0-measurable, we have $\tilde{g}_x = g_x$ a.e.. Hence $\langle x, f_n\rangle \to g_x$ in $L^1(\mu)$, and so $\langle x, f_n\rangle \to g_x$ a.e. by Theorem V. 2.8 of [4]. This means that ν satisfies condition (SA) with respect to (S, Σ, μ). q.e.d.

REMARK. We note that the above proof shows that, whenever ν

satisfies condition (SA) with respect to (S, Σ, μ), we can take an approximate sequence (f_n) of simple functions in such a way that for each $x \in X$, $\langle x, f_n \rangle \to g_x$ a.e. and the range of each f_n is contained in the closed linear subspace spanned by $\nu(\Sigma)$. The following result is obvious from the above theorem:

THEOREM 2.2. *If (S, Σ, μ) is a separable measure space, then every X^*-valued μ-continuous measure of bounded variation satisfies condition* (SA).

We next list some well-known facts (See [6, VI. 8.1].) which will be used in the sequel:

Let (S, Σ, μ) be a measure space and let T be any bounded linear operator of the Banach space X into $L^1(\mu)$. Then there is a uniquely determined set function $\nu: \Sigma \to X^*$ such that

(2.1) for every x in X the set function $\langle x, \nu \rangle$ is μ-continuous and countably additive on Σ, and

$$Tx = d/d\mu \langle x, \nu \rangle \quad \text{for } x \text{ in } X. \tag{2.2}$$

Moreover the norm of T satisfies

$$\sup_{E \in \Sigma} \| \nu(E) \| \leqq \| T \| \leqq 4 \cdot \sup_{E \in \Sigma} \| \nu(E) \| . \tag{2.3}$$

Conversely, if $\nu: \Sigma \to X^*$ is a vector measure satisfying (2.1) and if T is defined by (2.2), then T becomes a bounded linear operator from X into $L^1(\mu)$ which satisfies (2.3). Furthermore, T is weakly compact if and only if $\nu(\cdot)$ is countably additive on Σ with respect to the strong topology of X^*.

Now the sequential approximation property is characterized in terms of the above-mentioned linear operator T in the following way:

THEOREM 2.3. *Let $\nu: \Sigma \to X^*$ be a bounded μ-continuous vector measure. Let T be a continuous linear operator from X to $L^1(\mu)$ defined through ν by* (2.2). *Then ν satisfies condition* (SA) *with respect to (S, Σ, μ) if and only if T has a norm-separable range.*

PROOF. The necessity: Suppose that ν satisfies condition (SA) with respect to (S, Σ, μ). Then, in virtue of a remark mentioned before Theorem 2.2, one finds a countably generated σ-subfield Σ_0 of Σ such that each $g \in T(X)$ is Σ_0-measurable. Hence $T(X) \subseteq L^1(S, \Sigma_0, \mu | \Sigma_0)$. On the other hand, $L^1(S, \Sigma_0, \mu | \Sigma_0)$ is norm-separable; hence so is $T(X)$.

The sufficiency: Suppose that $T(X)$ is separable. Then there exists a countable subset D of $T(X)$ such that D is norm-dense in $T(X)$, and so there is a countably generated σ-subfield Σ_0 of Σ such that each $g \in D$ is Σ_0-measurable. Hence for each $g \in T(X)$ there is a sequence $(g_n) \subseteqq D$ with $\|g_n - g\|_1 \to 0$ as $n \to \infty$. Now one can choose a subsequence (g_{n_k}) of (g_n) such that $g_{n_k} \to g$ a.e.. Since each $g_{n_k}(k \geqq 1)$ is Σ_0-measurable, g is also Σ_0-measurable. Thus, $T(X) \subseteqq L^1(S, \Sigma_0, \mu|\Sigma_0)$. Consequently, it follows from Theorem 2.1 that ν satisfies condition (SA) with respect to (S, Σ, μ). q.e.d.

Let us consider the class $\mathfrak{M}$ of all the X^*-valued vector measures which are bounded and μ-continuous with respect to a fixed finite non-negative measure space (S, Σ, μ). The next result shows that the subclass consisting of X^*-valued measures satisfying condition (SA) with respect to (S, Σ, μ) is closed in $\mathfrak{M}$ under the weak* convergence.

THEOREM 2.4. *Let (ν_n) be a sequence of X^*-valued μ-continuous and bounded vector measure on Σ satisfying condition* (SA) *with respect to (S, Σ, μ). Suppose that for each $E \in \Sigma$ the sequence $(\nu_n(E))$ is weak* convergent. If we define a set function $\nu: \Sigma \to X^*$ by $\nu(E) = w^*-\lim_n \nu_n(E)$ for each $E \in \Sigma$, then ν becomes an X^*-valued, μ-continuous and bounded vector measure satisfying condition* (SA).

PROOF. Let T_n and T be linear operators from into $L^1(\mu)$ associated respectively through (2.2) with ν_n and ν. For each $E \in \Sigma$, $\nu_n(E) \to \nu(E)$ in the weak* topology, so that $\sup_n \|\nu_n(E)\| < +\infty$ for each $E \in \Sigma$. Hence from Nikodym's boundedness theorem ([4, p. 14]) it follows that $\sup_n \|\nu_n\|(S) (= M) < +\infty$, where $\|\nu_n\|(S)$ denotes the semi-variation of ν_n (see [4], p. 2). Also, $\|T_n x\|_1 = |\langle x, \nu_n \rangle|(S) \leqq \|\nu_n\|(S)\|x\| \leqq M\|x\|$, and hence $(T_n x)$ is L^1-bounded for each $x \in X$. This means that $(T_n x)$ converges to Tx in the weak topology of $L^1(\mu)$ (see [6], IV. 8.7). Let $Y = \overline{sp}(\bigcup_{n=1}^{\infty} T_n(X))$. Y is separable since each $T_n(X)$ is separable. So, $Tx \in Y$. Consequently, $T(X) \subseteqq Y$ and $T(X)$ turns out to be norm-separable. q.e.d.

§3. Scalarly measurable functions on perfect measure spaces.

In this section we discuss some sufficient conditions on the base measure space (S, Σ, μ) in order that vector measures on Σ satisfy condition (SA). If the range space X^* of functions under consideration is supposed to be arbitrary, the sequential approximation property depends upon the choice of the base measure spaces.

Firstly, if (S, Σ, μ) is separable, then every X^*-valued, μ-continuous vector measure of bounded variation satisfies condition (SA) with respect to (S, Σ, μ). In view of this we may give a characterization of Gel'fand integrable functions in terms of sequential approximation property.

THEOREM 3.1. *Let (S, Σ, μ) be a separable measure space and $f: S \to X^*$ be a norm-bounded function. Then f is Gel'fand integrable if and only if there exists a sequence (f_n) of simple functions such that $\langle x, f_n \rangle \to \langle x, f \rangle$ a.e. for each $x \in X$.*

Secondly, we consider the case in which (S, Σ, μ) is perfect. A finite nonnegative complete measure space (S, Σ, μ) is said to be perfect if for every μ-measurable function $\phi: S \to \boldsymbol{R}$ there is a Borel set B in $\boldsymbol{R}$ such that $B \subseteq \phi(S)$ and $\mu(\phi^{-1}(B)) = \mu(S)$. Perfect measure spaces play an important role in the measure theory; and it should be noted that the class of perfect measure spaces is considerably broad.

We begin by stating a slight modification of a result due to Stegall [8], Proposition 3J.

THEOREM S. *Let (S, Σ, μ) be a perfect measure space and let $\nu: \Sigma \to X^{**}$ be an X^{**}-valued Lipschitz μ-continuous measure, i.e., $\sup_{E \in \Sigma} \|\nu(E)\|/\mu(E) (=M) < +\infty$. Let $K = \{d/d\mu\langle x^*, \nu \rangle : x^* \in X^*, \ \|x^*\| \leqq 1\}$. Suppose that the space $\boldsymbol{R}^S$ is endowed with the pointwise topology, and that the following condition (**) holds:*

(**) Every sequence in K has a μ-measurable cluster point in $\boldsymbol{R}^S$. Then $\nu(\Sigma)$ is relatively compact in X^{**}. In particular, if the Gel'fand derivative of ν is an X-valued function, then condition (**) is automatically satisfied and $\nu(\Sigma)$ is always relatively compact in X^{**}.

PROOF. First we show that K is relatively compact in $L^1(\mu)$. To this end it suffices to prove a sequence (g_n) in K has a $L^1(\mu)$-convergence subsequence. First K is relatively compact in the topology of convergence in measure by [7], Corollary 2G. Hence there exists a subsequence (g_{n_k}) of (g_n) and an element $g \in L^1(\mu)$ such that $g_{n_k} \to g$ in measure. By the dominated convergence theorem (See [6, III. 7].), we have $g_{n_k} \to g$ in $L^1(\mu)$ and so K is relatively compact in $L^1(\mu)$. Thus the operator $T: X^* \to L^1(\mu)$ defined by $Tx^* = d/d\mu\langle x^*, \nu \rangle$ is compact, and so is $T^*: L^\infty(\mu) \to X^{**}$. Thus, $\{\nu(E): E \in \Sigma\} = \{T^*(\chi_E): E \in \Sigma\}$ is relatively compact in X^{**}. Next, suppose that the Gel'fand derivative of ν is an X-valued function, i.e., there exists a function $f: S \to X$ such that $\nu(E) = (G) - \int_E f d\mu$ for every $E \in \Sigma$. Then condition (**) is automatically satisfied. In fact, let (g_n) be a sequence in K. Then we can write $g_n = \langle x_n^*, f \rangle$ for some x_n^* with $\|x_n^*\| \leqq 1$ for each $n \geqq 1$. If $\mathscr{F}$ is non-principal ultrafilter on N, then $\lim_{n\to\mathscr{F}} x_n^*$ converges in the weak* topology. Let x_0^* be its weak* limit in X^*. Then $\lim_{n\to\mathscr{F}} g_n(s) = \langle x_0^*, f(s)\rangle$ $(=g(s))$ for every $s \in S$, and so (g_n) has a μ-measurable cluster point g in the space R^S. This means that condition (**) is satisfied. q.e.d.

From this fact and Theorem 2.3 we obtain the next theorem.

THEOREM 3.2. *Let (S, Σ, μ) be a perfect measure space and let $f: S \to X^*$ be norm-bounded and Pettis integrable. Set $\nu(E) = (P) - \int_E f d\mu$ in X^*. Then ν satisfies condition* (SA) *with respect to (S, Σ, μ).*

PROOF. First the assumption implies that $\nu(\Sigma)$ is relatively compact. Let $T: X \to L^1(\mu)$ be an operator defined through ν by (2.2). We demonstrate that the dual operator $T^*: L^\infty(\mu) \to X^*$ is compact. To this end, it is sufficient to show that the set $P = \{T^*(g): \sum_{i=1}^n \alpha_i \chi_{E_i}, 0 \leqq \alpha_i \leqq 1, E_i \in \Sigma; E_i \cap E_j = \emptyset (i \neq j), n \geqq 1\}$ is contained in $\operatorname{co} \nu(\Sigma)$, the closed convex hull of $\nu(\Sigma)$. Suppose $g = \sum_{i=1}^n \alpha_i \chi_{E_i}$, where $E_1, \cdots, E_n$ are pairwise disjoint members of Σ and $0 \leqq \alpha_1 \leqq \alpha_2 \leqq \cdots \leqq \alpha_n \leqq 1$. Then $T^*(g) = \sum_{i=1}^n \alpha_i \nu(E_i) = \alpha_1 \nu(\bigcup_{i=1}^n E_i) + (\alpha_2 - \alpha_1)\nu(\bigcup_{i=2}^n E_i) + (\alpha_3 - \alpha_2)\nu(\bigcup_{i=3}^n E_i) + \cdots + (\alpha_n - \alpha_{n-1})\nu(E_n) \in \operatorname{co} \nu(\Sigma)$ since $\alpha_1 + \sum_{i=2}^n (\alpha_i - \alpha_{i-1}) = \alpha_n \leqq 1$. So, by Schauder's theorem, T is compact as well. Thus, we infer from Theorem 2.3 that ν satisfies condition (SA) with respect to (S, Σ, μ). q.e.d.

REMARK. Eventually, a stronger result holds:

THEOREM 3.2′. *If (S, Σ, μ) is a perfect measure space and $f: S \to X^*$ is norm-bounded and Pettis integrable, then there exists a sequence (f_n) of X^*-valued simple functions such that for each x^{**} in X^{**} $\lim_{n\to\infty} \langle x^{**}, f_n\rangle = \langle x^{**}, f\rangle$ a.e..*

In fact, the function f, viewed as an X^{***}-valued function (X^{***} being the third dual of X), is also Pettis integral. Thus $\nu: \Sigma \to X^{***}$ satisfies condition (SA) with respect to (S, Σ, μ) by Theorem 2.1. On the other hand ν is an X^*-valued vector measure: hence by Remark mentioned before Theorem 2.2 there exists a sequence (f_n) of X^*-valued simple functions such that for every x^{**} in X^{**}

$$\lim_{n\to\infty} \langle x^{**}, f_n\rangle = \langle x^{**}, f\rangle \quad \text{a.e. .}$$

Furthermore, a close inspection of the above proof reveals that there is no necessity for the function f to be the dual space valued function.

It turns out that we obtain (as a Corollary to the previous result) a more general conclusion:

COROLLARY 3.3. *Let* (S, Σ, μ) *be a perfect measure space and* f *a bounded function from* S *into* X. *If* f *is Pettis integrable, then there exists a sequence* (f_n) *of* X*-valued simple functions such that for each* x^* *in* X^*,

$$\lim_{n\to\infty} \langle x^*, f_n\rangle = \langle x^*, f\rangle \quad \text{a.e.} \tag{3.1}$$

After the first draft of this paper was prepared, the author was informed by R. Geitz [9] that the converse of Corollary 3.3 holds, namely: If there exists a sequence (f_n) of X-valued simple functions satisfying (3.1), then f is Pettis integrable. He obtains the same conclusion as in Corollary 3.3. But our proof gives another proof of one direction of his theorem.

Finally, we give a counterexample which shows that Theorem 3.2 does not necessarily hold if the base measure space (S, Σ, μ) is not perfect. The example we are going to show below was implicitly given by Fremlin and Talagrand [8]. They constructed a function f such that the range of an indefinite Pettis integral is not relatively compact; and it is seen from Theorem S that the base measure space is not perfect. In fact, their function provides a concise example of vector measures without condition (SA).

EXAMPLE. Let W be an uncountable set and $\mathscr{P}W$ the power set of W. On $\mathscr{P}W$ a natural compact Hausdorff topology can be given by employing sub-basic open sets $\{a : t \in a\}$ and $\{a : t \notin a\}$ where t runs through W. Let Z_2 be an Abelian group under an additional operation of integers modulo a prime number 2. If $\mathscr{P}W$ is identified with $(Z_2)^W$, then $\mathscr{P}W$ is understood to be a topological group. Therefore, there exists a unique normalized Haar measure λ such that $\lambda(\{a \subseteq W : b \subseteq a\}) = 2^{-\mathrm{card}(b)}$ for all finite $b \in \mathscr{P}W$. Now as a special case of Theorem 10 of [14], one finds an extention $\bar{\lambda}$ of λ to a σ-field of subsets of $\mathscr{P}W$ that contains all filters $\mathscr{F}$ on W such that $\bar{\lambda}(\mathscr{F}) = 1$ whenever $\lambda^*(\mathscr{F}) = 1$, where $\lambda^*(\cdot)$ denotes the outer measure induced from λ. Let $\mu = \bar{\lambda} \times \bar{\lambda}$ be the product measure on $\mathscr{P}W \times \mathscr{P}W$ and define $f: \mathscr{P}W \times \mathscr{P}W \to l^\infty(W)$ by

$$f(a, b) = \chi_a - \chi_b ,$$

where χ_a, χ_b denote characteristic functions of a, b, respectively. Then, as shown in Example 2D of [8], f is Pettis integral. Let T be a bounded linear operator from $l^1(W)$ to $L^1(\mu)$ defined by $T\phi=\langle\phi, f\rangle$ for $\phi\in l^1(W)$. We then demonstrate that $T(l^1(W))$ is non-separable. In fact, let $t_1, t_2\in W$, $t_1\neq t_2$, and let

$$A=\{(a, b): t_1\in a,\ t_2\notin a;\ t_1\in b,\ t_2\in b\}\ .$$

Then, by use of the basic properties of Haar measure, one gets

$$\begin{aligned}\| T\delta_{t_1}-T\delta_{t_2}\|_1\geqq\mu(A)&=\bar{\lambda}(\{a : t_1\in a,\ t_2\notin a\})\times\bar{\lambda}(\{b : t_1\in b,\ t_2\in b\})\\ &=\lambda(\{a : t_1\in a,\ t_2\notin a\})\times\lambda(\{b : t_1\in b,\ t_2\in b\})\\ &=\frac{1}{4}\times\frac{1}{4}=\frac{1}{16}>0\ ,\end{aligned}$$

where δ_t denotes the element of $l^1(W)$ which takes the value 1 at t and the value 0 elsewhere. This means that $T(l^1(W))$ is non-separable since W is uncountable. Consequently, we infer from Theorem 2.2 that f has no approximate sequences of simple functions.

§4. Banach spaces with the sequential approximation property.

As is seen from Theorem 2.3, if X is separable then X^* has the sequential approximation property. More generally, if X is a weakly compactly generated space (that is, a Banach space with a weakly compact fundamental set and is simply called a WCG space), then X^* has the sequential approximation property. Note that the class of WCG spaces is much broader than that of reflexive spaces.

THEOREM 4.1. *If a Banach space X is* WCG, *then the dual space X^* has the sequential approximation property.*

PROOF. Let W be a weakly compact fundamental subset of X. Then, in view of Theorem 2.3, it is sufficient to show that $T(W)$ is relatively compact in $L^1(\mu)$. To this end, take any sequence (g_n) with $g_n\in T(W)$. Then for each $n\geqq 1$ there exists an x_n in W such that $Tx_n=g_n$. Since W is weakly compact, there is a weakly convergent subsequence of (x_n). For simplicity in notation we assume that (x_n) itself is weakly convergence and x_0 is the weak limit. Then Tx_n $(=g_n)$ converges as $n\to\infty$. Now let (S, Σ, μ) be any measure space and ν: $\Sigma\to X^*$ any μ-continuous measure of bounded variation. Let f: $S\to X^*$ be a Gel'fand derivative of ν. Since $\langle x_n, f(s)\rangle\to\langle x_0, f(s)\rangle$ for all s in S, we may apply Theorem IV 8.12 of [6] to get the norm convergence of (Tx_n) in $L^1(\mu)$. Hence $T(W)$ is norm-compact. q.e.d.

If X^* has RNP, then it is clear that X^* has the sequential approximation property. More generally, we have the following result:

THEOREM 4.2. *If* X^* *has* WRNP, *then* X^* *has the sequential approximation property.*

PROOF. Suppose that X^* has WRNP. Then X can not contain any isomorphic copy of l^1 by Janicka's Theorem (see [10] or [11]). Since each X^*-valued measure of bounded variation has a relatively compact range (See Musiał [11], Corollary 10.), we obtain the desired result from Theorem 2.2. q.e.d.

COROLLARY 4.3. *Let* (S, Σ, μ) *be a finite nonnegative complete measure space. If* X^* *has* WRNP, *then every weak*-measurable function has the sequential approximation property with respect to* (S, Σ, μ).

PROOF. Employ a countable partition $\{S_m: m\geqq 1\}$ of S as mentioned after Proposition 1.1. q.e.d.

Finally we give another characterization of Gel'fand integrable functions in terms of sequential approximation property.

THEOREM 4.4. *Let* $f: S \to X^*$ *be norm-bounded. If* X^* *has* WRNP, *then* f *is Gel'fand integrable if and only if there exists a sequence* (f_n) *of simple functions such that* $\langle x, f_n\rangle \to \langle x, f\rangle$ *a.e. for each* $x\in X$.

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