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Exposed Points in Function Algebras

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In this paper we consider some properties of exposed points in the unit ball of function algebras. In § 1 we give some characterizations of exposed points in the unit ball U of certain function algebras. Also we consider conditions so that U can be equal to the closed convex hull of exposed points of U. In § 2, some examples are given.

Introduction

Let X be a compact Hausdorff space and A a *function algebra* on X, i.e., a uniformly closed subalgebra of $C(X)$ that contains the constants and separates points of X, where $C(X)$ denotes the Banach algebra of complex-valued continuous functions on X with the supremum norm. By U we denote the unit ball of A, i.e., $U=\{f\in A: \|f\|\leqq 1\}$. We recall the notion of exposed points of U. A function f in U is called an *exposed point* of U if there exists L in A^* such that $L(f)=1=\|L\|$ and $\operatorname{Re} L(g)<1$ for $g\in U$, $g\neq f$, where $\operatorname{Re} L(g)$ is the real part of $L(g)$. It is clear that every exposed point is an extreme point but the converse is not always true.

Characterizations of exposed points have been investigated in [1], [3], [7], [8], [9] and so on. Especially, Phelps [7] gave some interesting results on logmodular algebras. Moreover Fisher [3] and Serizawa [8] gave extensions of the Phelps' results. In this paper we give some generalizations of Phelps' and Fisher's results.

We here assume the following condition.

(*) *There exist (pairwise disjoint) closed sets X_i in X $(i=1, 2, \cdots)$ such that $A|_{X_i}$ is closed in $C(X_i)$ and $\bigcup_{i=1}^{\infty} X_i$ is dense in X.*

For each i, we denote by A_i the restriction of A to X_i. M_A and M_{A_i} will denote the maximal ideal space of A and A_i, respectively. Then

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A_i is a function algebra on a compact Hausdorff space X_i and there is a representing measure m_i for φ_i in M_{A_i} which is supported on X_i ([6; Chap. 7, p. 166]).

§ 1. The main results.

We say that A_i has the condition (α) if no non-zero function in A_i vanishes on a set E in X_i with $m_i(E)>0$.

THEOREM 1.1. *Let A be a function algebra on a compact Hausdorff space X with the condition* (*). *Let A_i and m_i be as above for each i. Suppose each A_i has the condition (α). If $f\in U$ and $m_i(F\cap X_i)>0$ for $i=1, 2, \cdots$, then f is an exposed point of U, where $F=\{x\in X: |f(x)|=1\}$.*

PROOF. Since $m_i(F\cap X_i)>0$ for each i, we define $L_i\in A_i^*$ by

$$L_i(g)=\frac{1}{m_i(F\cap X_i)}\int_{F\cap X_i} g\overline{f|_{X_i}}dm_i \quad (g\in A_i)\ .$$

Then $L_i(f|_{X_i})=1=\|L_i\|$. Now if $L_i(g)=1=\|L_i\|$ for $g\in A_i$, $\|g\|\leqq 1$, then

$$\int_{F\cap X_i} g\overline{f|_{X_i}}dm_i=m_i(F\cap X_i)\ .$$

Since $g\overline{f|_{X_i}}\in C(X_i)$ and $|g\overline{f|_{X_i}}|\leqq 1$ on X_i, $g\overline{f|_{X_i}}=1$ a.e. on $F\cap X_i$. So $g=f$ a.e. on $F\cap X_i$. By the condition (α) of A_i, $g=f$ on X_i. Hence $f|_{X_i}$ is an exposed point of the unit ball of A_i for each i. Furthermore if we put

$$L(g)=\sum_{i=1}^{\infty}\frac{1}{2^i}L_i(g|_{X_i}) \quad (g\in A)\ ,$$

then $L\in A^*$ and $L(f)=1=\|L\|$. For any $g\in A$ with $\|g\|\leqq 1$ and $g\neq f$, $g\neq f$ on X_j for some j, $1\leqq j<\infty$. In fact, if $g=f$ on X_i for all i, $g=f$ on X because of the density of $\bigcup_{i=1}^{\infty} X_i$ in X. So for the bounded linear functional L_j as above,

$$\operatorname{Re} L_j(g|_{X_j})<1\ .$$

Then $\operatorname{Re} L(g)=\operatorname{Re}\sum_{i=1}^{\infty}(1/2^i)L_i(g|_{X_i})<1$. Consequently, f is an exposed point of the unit ball U of A.

Next we consider conditions so that U can be the closed convex hull of its exposed points.

THEOREM 1.2. *Let A be a function algebra, generated by its inner* *functions, on a compact Hausdorff space X with the condition* (*). *Let A_i and m_i be as above for each i. Suppose each A_i has the condition* (α). *Then U is the closed convex hull of its exposed points.*

PROOF. Since A is generated by inner functions, a theorem in [2] (Theorem 2.2) implies that U is the closed convex hull of its inner functions. Now by Theorem 1.1, every inner function is an exposed point and thus the assertion holds.

A representing measure m for $\varphi \in M_A$ is said to be *dominant* if any representing measure for φ is absolutely continuous with respect to m ([4; Chap. 2, p. 44]).

In particular, we consider Theorem 1.2 under the following condition (**).

(**) *There exists a finite family $\{X_i\}_{i=1}^n$ of maximal antisymmetric sets of A with $X=\bigcup_{i=1}^n X_i$.*

Then we obtain the following result.

THEOREM 1.3. *Let A be a function algebra on a compact Hausdorff space X with the condition* (**). *Let $A_i=A|_{X_i}$. Let m_i be a dominant representing measure for $\varphi_i \in M_{A_i}$ ($1\leqq i\leqq n$). Suppose each A_i has the condition* (α). *Then U is the closed convex hull of its exposed points.*

PROOF. Let U_i be the unit ball of A_i and $\exp U_i$ be the set of exposed points of U_i for each i. A same method as Fisher [3; Theorem 3] shows that U_i is the closed convex hull of $\exp U_i$. For any $g\in U$, then $g|_{X_i}\in U_i$ for each i. Given $\varepsilon>0$. We can choose the functions $f_1^{(i)}, \cdots, f_{k(i)}^{(i)} \in \exp U_i$ and the constants $\lambda_1^{(i)}, \cdots, \lambda_{k(i)}^{(i)}$ such that $\lambda_j^{(i)}\geqq 0$, $\sum_{j=1}^{k(i)} \lambda_j^{(i)}=1$ and

$$\left\| g|_{X_i} - \sum_{j=1}^{k(i)} \lambda_j^{(i)} f_j^{(i)} \right\| < \varepsilon ,$$

for $i=1, 2, \cdots, n$. (The number $k(i)$ depends on index i.) Define the functions $\tilde{f}_{j_1\cdots j_n}$ and the constants $\nu_{j_1\cdots j_n}$ as follows:

$$\tilde{f}_{j_1\cdots j_n}=\begin{cases} f_{j_1}^{(1)} & \text{on } X_1 \\ \cdots & \cdots \\ f_{j_n}^{(n)} & \text{on } X_n \end{cases}$$

and

$$\nu_{j_1\cdots j_n}=\lambda_{j_1}^{(1)} \cdots \lambda_{j_n}^{(n)} ,$$

where $1 \leqq j_1 \leqq k(1)$, $\cdots$, and $1 \leqq j_n \leqq k(n)$. Then $\tilde{f}_{j_1\cdots j_n} \in C(X)$ and $\tilde{f}_{j_1\cdots j_n}|_{X_i} \in A_i$ for $1 \leqq i \leqq n$. So $\tilde{f}_{j_1\cdots j_n} \in A$ and $\|\tilde{f}_{j_1\cdots j_n}\| \leqq 1$. Furthermore $\tilde{f}_{j_1\cdots j_n}|_{X_i}$ are in $\exp U_i$ for each i. Thus $\tilde{f}_{j_1\cdots j_n}$ are exposed points of U. (This can be showed by the same argument as the proof of Theorem 1.1.) On the other hand, $\nu_{j_1\cdots j_n}$ are positive constants and $\sum_{j_1,\cdots,j_n} \nu_{j_1\cdots j_n} = 1$. So

$$\begin{aligned} \sum \nu_{j_1\cdots j_n} \tilde{f}_{j_1\cdots j_n} |_{X_i} &= \sum \nu_{j_1\cdots j_n} f_{j_i}^{(i)} \\ &= \sum_{j=1}^{k(i)} \lambda_j^{(i)} f_j^{(i)} . \end{aligned}$$

Hence

$$\begin{aligned} \|g - \sum \nu_{j_1\cdots j_n} \tilde{f}_{j_1\cdots j_n}\| &\leqq \max_{1 \leqq i \leqq n} \|g - \sum \nu_{j_1\cdots j_n} \tilde{f}_{j_1\cdots j_n}\|_{X_i} \\ &< \varepsilon . \end{aligned}$$

As ε is arbitrary, the theorem holds.

We obtain the Fisher's theorem as a special case of Theorems 1.1 and 1.3.

COROLLARY 1.4 ([3]). *Let A be a function algebra on a compact Hausdorff space X and m a dominant representing measure for $\varphi \in M_A$. Suppose that $m(\{g=0\})>0$, $g \in A$, implies $g=0$. Then a function $f \in U$ with $m(\{|f|=1\})>0$ is an exposed point of U and U is the closed convex hull of exposed points of U.*

PROOF. It is sufficient to show that A is an antisymmetric algebra, i.e., every real-valued function in A is constant. If $g \in A$ is real on X, then g is real on the closed support S_m of m. By the antisymmetric property of S_m ([6; Chap. 3, Theorem 6]), g is constant on S_m. By the assumption, g is constant on X. Thus A is antisymmetric and so this case is reduced to Theorems 1.1 and 1.3 where $i=1$.

If m is dominant, Serizawa's condition is equivalent to Fisher's. There is an algebra with a dominant representing measure which does not satisfy Serizawa's condition. (E.g., Example 2 in § 2.)

Under the condition (**) we consider the converse of Theorem 1.1.

PROPOSITION 1.5. *Let A be a function algebra on a compact Hausdorff space X with the condition (**). Let $A_i = A|_{X_i}$. Let m_i be a representing measure for $\varphi_i \in M_{A_i}$. Assume the property: if μ is a measure on X_i orthogonal to A_i, μ is absolutely continuous to m_i for each i. Then, if*

$f \in U$ is an exposed point of U, $m_i(F \cap X_i) > 0$ for $1 \leqq i \leqq n$, where $F = \{x \in X: |f(x)| = 1\}$.

PROOF. Suppose $m_j(F \cap X_j) = 0$ for some j, $1 \leqq j \leqq n$. Let μ be a measure on X_j orthogonal to A_j. Then $\mu(F \cap X_j) = 0$. So if f is an exposed point of U, for $f|_{X_j} \in A_j$ there is a function $g \in A_j$ such that $g = f$ on $F \cap X_j$, $g \neq f$ on X_j and $\|g\| = \|f|_{F \cap X_j}\|$ ([5]). Now let $L(f) = 1 = \|L\|$ for $L \in A^*$. Then there is a non-negative Baire measure ν on X such that $\nu(X) = 1$,

$$L(h) = \int_S h\bar{f} d\nu \qquad (h \in A) ,$$

where the closed support S of ν is contained in F. Put

$$h = \begin{cases} g & \text{on } X_j \\ f & \text{otherwise} . \end{cases}$$

Then $h \in C(X)$ and $h|_{X_i} \in A|_{X_i}$, $1 \leqq i \leqq n$. So $h \in A$, $\|h\| \leqq 1$ and $h \neq f$ on X. On the other hand,

$$\begin{aligned} L(h) &= \int_S h\bar{f} d\nu \\ &= \sum_{i=1}^{n} \int_{S \cap X_i} h\bar{f} d\nu \\ &= \sum_{i \neq j} \int_{S \cap X_i} |f|^2 d\nu + \int_{S \cap X_j} g\bar{f} d\nu \\ &= \int |f|^2 d\nu = 1 . \end{aligned}$$

Consequently, f is not an exposed point of U.

NOTE. In Theorem 1.1, suppose that X is separable and $A = C(X)$. Then, there is a countable dense set $\{x_{\alpha(1)}, x_{\alpha(2)}, \cdots\}$ in X. Let $X_i = \{x_{\alpha(i)}\}$, $A_i = C(X)|_{X_i} = C(X_i)$ and m_i be the unit point mass at $x_{\alpha(i)}$. Each X_i is a maximal set of antisymmetry of A. Then, if $f \in C(X)$ is a unimodular function, f is an exposed point of U and so f is an extreme point. On the other hand, Phelps [7] established the following: if there is a diffuse measure on X, the sets of extreme points and exposed points of U are equal. Indeed, in this case, $\mu = \sum_{i=1}^{\infty} (1/2^i) m_i$ is a diffuse measure. Moreover, if there is a diffuse measure on X, the unit ball of $C(X)$ is the closed convex hull of its exposed points.

§ 2. Examples.

EXAMPLE 1. Let A be the disk algebra or $R(K)$, where K is a compact subset of C and its interior is connected. By the theorems and proposition in § 1, exposed points of both algebras can be completely characterized.

EXAMPLE 2 ([7]). Let $X_1=\{z: |z|=1\}$, $X_2=\{z: |z-3|=1\}$ and $X=X_1\bigcup X_2$. Let A be the algebra of functions which are continuous on X having continuously analytic extensions to $\{z: |z|<1\}\bigcup\{z: |z-3|<1\}$. Let m_1 and m_2 be a representing measure on X for $z=0$ and $z=3$, respectively. Then each X_i is a maximal set of antisymmetry of A, because X_i is the closed support of m_i for $i=1, 2$. So a function $f\in A$ with $\|f\|\leqq 1$ is exposed point if and only if $m_i(F\bigcap X_i)>0$ for $i=1, 2$, where $F=\{z: |f(z)|=1\}$. And the unit ball of A is the closed convex of its exposed points.

EXAMPLE 3. Let $X=\{(z, t): |z|=1, 0\leqq t\leqq 1\}$ and A be a function algebra generated by z, t $(|z|=1, 0\leqq t\leqq 1)$. It is known that $X_\alpha=\{(z, t_\alpha): |z|=1\}$ is a maximal set of antisymmetry of A for each t_α, $0\leqq t_\alpha\leqq 1$. As the interval $[0, 1]$ is separable, there exists a countable dense set $\{t_{\alpha(1)}, t_{\alpha(2)}, \cdots\}$ in $[0, 1]$. Put $X_i=\{(z, t_{\alpha(i)}): |z|=1\}$. Then $\bigcup_{i=1}^\infty X_i$ is dense in X. Let $A_i=A|_{X_i}$ and m_i be a (unique) representing measure for $(0, t_{\alpha(i)})$ for each i. So X satisfies the condition (*) and each A_i has the condition (α). Thus Theorem 1.1 holds. And we can easily see that A is generated by inner functions z, e^{it} and $e^{-it}(|z|=1, 0\leqq t\leqq 1)$. Thus Theorem 1.2 also holds.

EXAMPLE 4. Let $X_1=\{z: |z|=1\}$, $X_2=\{z: 2\leqq z\leqq 3\}$ and $X=X_1\bigcup X_2$. Let A be the algebra of functions which are continuous on X and can be extended to be analytic in $\{z: |z|<1\}$. There is a countable dense set $\{t_{\alpha(1)}, t_{\alpha(2)}, \cdots\}$ in X_2. Now put $K_0=\{z: |z|=1\}$ and $K_i=\{t_{\alpha(i)}\}$ $(i\geqq 1)$. Then each K_i $(i\geqq 0)$ is a maximal set of antisymmetry of A and $\bigcup_{i=0}^\infty K_i$ is dense in X. So X has the condition (*). Let $A_i=A|_{K_i}$, m_0 the normalized Lebesgue measure and m_i the unit point mass at $t_{\alpha(i)}$ $(i\geqq 1)$. Each A_i has the condition (α). Now put the functions w_i $(1\leqq i\leqq 6)$ as follows: $w_1=z$ on X_1 and $w_1=1$ on X_2, $w_2=z$ on X_1 and $w_2=-1$ on X_2, $w_3=1$ on X_1 and $w_3=e^{it}$ on X_2, $w_4=-1$ on X_1 and $w_4=e^{it}$ on X_2, $w_5=1$ on X_1 and $w_5=e^{-it}$ on X_2, $w_6=-1$ on X_1 and $w_6=e^{-it}$ on X_2. Then A is generated by inner functions w_i, $i=1, \cdots, 6$. So Theorems 1.1 and 1.2 hold.

EXAMPLE 5. Let $X_1=\{(z,0): |z|=1\}$, $X_2=\{(0,t): 0\leqq t\leqq 1\}$ and $X=X_1\bigcup X_2$.

Let A be the algebra of functions which are continuous on X and which can be extended to be analytic in $\{(z, 0): |z|<1\}$. Then A is a function algebra on X. Since $[0, 1]$ is separable, there is a countable dense set $\{t_{\alpha(1)}, t_{\alpha(2)}, \cdots\}$ in $[0, 1]$, where $t_{\alpha(1)}=0$. Then put

$$\begin{aligned} K_1 &= \{(z, 0): |z|=1\} \cup \{(0, t_{\alpha(1)})\} \\ K_2 &= \{(0, t_{\alpha(2)})\} \\ &\cdots \quad \cdots \quad \cdots \\ K_i &= \{(0, t_{\alpha(i)})\} \\ &\cdots \quad \cdots \quad \cdots . \end{aligned}$$

We can see that each K_i is a maximal set of antisymmetry of A and $\bigcup_{i=1}^{\infty} K_i$ is dense in X. So X has the condition (*). Let $A_i=A|_{K_i}$. Let m_1 be a (unique) representing measure for $(a, 0)$ in M_A, $0\lneqq|a|<1$ and m_i $(i\geqq 2)$ the point mass at $(0, t_{\alpha(i)})$. Then each A_i has the condition (α). Thus Theorem 1.1 holds. But the unit ball of A is not the closed convex hull of its exposed points ([7]). In this case, if $f\in A$ is inner, f must be a constant of modulus 1 on X_1. A is not generated by inner functions.

EXAMPLE 6. Let $(X, \mathfrak{A}, m)$ be a probability measure space. Recall that a weak-*Dirichlet algebra A is an algebra of $L^\infty(m)$ such that (i) the constant functions lie in A; (ii) $A+\bar{A}$ is weak-*dense in $L^\infty(m)$; (iii) m is multiplicative on A. Let $H^\infty(m)$ be the weak-*closure of A in $L^\infty(m)$. As $H^\infty(m)$ is antisymmetric, we can apply the same method as Theorems 1.1 and 1.2 in § 1 for $i=1$. Then our statement is: Let A be a weak-*Dirichlet algebra such that (β) no non-zero function in $H^\infty(m)$ vanishes on a set of positive measure. For $f\in H^\infty(m)$ with $\|f\|\leqq 1$, $m(\{|f|=1\})>0$ implies that f is an exposed point of the unit ball U of $H^\infty(m)$. Moreover U is the closed convex hull of its exposed points (cf. [10]).

The assumption (β) of $H^\infty(m)$ is necessary. Let A be the algebra of continuous functions on the torus $T^2=\{(z, w): |z|=1, |w|=1\}$ which are uniform limits of polynomials in $z^n w^m$, where $(n, m)\in\{(n, m): m>0\}\cup\{(n, 0): n\geqq 0\}$. Denote by m the normalized Haar measure on T^2. Then A is a weak-*Dirichlet algebra of $L^\infty(m)$ that does not satisfy the assumption (β). Now take a function $g=zw$ in $H^\infty(m)$ and a subset E of T^2 with $0\lneqq m(E)\lneqq 1$. Let χ_E be a characteristic function of E. We put $f=\chi_E g$. Then f lies in the unit ball of $H^\infty(m)$ and $m(\{|f|=1\})>0$. But f is not an exposed point (indeed, not an extreme point).

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