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A Kummer Congruence for the Hurwitz-Herglotz Function

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Introduction

There has been much interest in the arithmetic properties of singular values of modular functions. Among the various topics studied are properties these singular values share with the Bernoulli numbers. Recently, Katayama [6] established analogues of the von Staudt-Clausen theorem for singular values of the Hurwitz-Herglotz function (defined below). In this paper, it will be shown that the singular values of the Hurwitz-Herglotz function also satisfy a Kummer congruence. Kummer congruences for singular values of Eisenstein series and for Eisenstein series of higher level have been demonstrated by H. Lang [8], [9].

§ 1. Kummer congruences.

Let k and r be integers with $k>r\geqq 1$ and let p be prime with $p-1\nmid k$. The classical Kummer congruence [7] states that the Bernoulli numbers satisfy

$$(1.1)\qquad \sum_{s=0}^{r}(-1)^{s}\binom{r}{s}\frac{B_{k+s(p-1)}}{k+s(p-1)}\equiv 0(\mathrm{mod}\ p^{r})\ .$$

In the study of p-adic L-series, Iwasawa introduced generalized Bernoulli numbers B_{χ_f}, associated to characters χ_f of conductor f. Carlitz [3] proved a Kummer congruence for these generalized Bernoulli numbers. He showed that with k, p, and r as before and with $p\nmid f$, then

$$(1.2)\qquad \sum_{s=0}^{r}(-1)^{s}\binom{r}{s}\frac{B_{\chi_f}^{k+s(p-1)}}{k+s(p-1)}\equiv 0(\mathrm{mod}\ p^{r})\ .$$

A related Kummer congruence was established by Vandiver [11] who

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demonstrated that if k, p, and r are as before and a and f positive integers, then

$$\sum_{s=0}^{r}(-1)^s\binom{r}{s}f^{k+s(p-1)}\frac{B_{k+s(p-1)}\left(\frac{a}{f}\right)}{k+s(p-1)}\equiv 0 \pmod{p^r} \tag{1.3}$$

where $B_j(x)$ denotes the jth Bernoulli function.

Now let $G_k=\sum'^{\infty}_{m,n=-\infty}1/(m\omega_1+n\omega_2)^k$ denote the Eisenstein series of order k where $\omega_1, \omega_2 \in C$ and $\omega=\omega_2/\omega_1$, $\operatorname{Im}\omega>0$. Let $A_p(\omega)$ denote the penultimate coefficient in the multiplicator equation

$$x^{p+1}-A_1(\omega)x^p+\cdots-A_p(\omega)x+(-1)^{(p-1)/2}p=0$$

satisfied by $x=p(\eta^2(p\omega)/\eta^2(\omega))$, where η is the Dedekind eta-function, and $p\geqq 5$ is prime. Let $C_k(w)=(1/\eta^{2k}(\omega))G_k(\omega)$. Then H. Lang [9] showed, using the Kummer congruence (1.1), that if α belongs to an imaginary quadratic field, $\operatorname{Im}\alpha>0$, and if k and r are integers with $k>r$ and $p\geqq 5$ prime with $p-1\nmid k$, then

$$\sum_{s=0}^{r}(-1)^s\binom{r}{s}\frac{C_{k+s(p-1)}(\alpha)}{k+s(p-1)}A_p^{r-s}(\alpha)\equiv 0 \pmod{p^r} .$$

This is the Kummer congruence for Eisenstein series.

Now let $k\geqq 3$, $f\geqq 1$, a_1, and a_2 be integers and let $\omega\in C$ with $\operatorname{Im}\omega>0$. The Eisenstein series of level f and order k is defined by

$$G_k(\omega;\, a_1, a_2, f)=\sum_{\substack{m\equiv a_1(\bmod f)\\ m\equiv a_2(\bmod f)}}'\frac{1}{(m\omega+n)^k} .$$

With χ_f a conductor $f>1$, define

$$G_k(\omega;\, a_1, a_2, \chi_f)=-\tau(\bar{\chi}_f)\frac{f^{k-1}k!}{(2\pi i)^k\eta^{2k}(\omega)}\sum_{h(\bmod f)}\chi_f(h)G_k(\omega;\, a_1h, a_2h, f)$$

where $\tau(\bar{\chi}_f)$ is the Gauss sum for the conjugate character χ_f. Using Carlitz's Kummer congruence (1.2), Lang [8] showed that for integers k and r, $k>r$, $k\geqq 3$ and $p\geqq r$ prime, $p\nmid f$, and for α belonging to an imaginary quadratic field with $\operatorname{Im}\alpha>0$, then

$$\sum_{s=0}^{r}(-1)^{s(1+(p-1)/2)}\binom{r}{s}\frac{C_{k+s(p-1)}(\alpha)}{k+s(p-1)}A^{r-s}(\alpha)\equiv 0 \pmod{p^r} .$$

This is the Kummer congruence for Eisenstein series of higher level.

The Hurwitz-Herglotz function $H_k(\omega;\, u,\, v)$ is defined by

$$H_k(\omega;\, u,\, v)=\frac{k!}{(2\pi i)^k \eta^{2k}(\omega)}\sum_{m,n=-\infty}^{\infty}{}' \frac{e^{2\pi i(mu+nv)}}{(m\omega+n)^k}\,.$$

In this paper it will be shown, using Vandiver's Kummer congruence (1.3), that Kummer congruence holds for singular values of the Hurwitz-Herglotz function.

THEOREM 1.4. *Let $p\geqq 5$ be prime and let $a, b, f, k,$ and r be integers with $u=a/f$, $v=b/f$, $(a, b, f)=1$, $k\geqq 3$, $k>r\geqq 1$, and $p-1\nmid k$, then if α belongs to an imaginary quadratic field with* $\operatorname{Im}\alpha>0$,

$$\sum_{s=0}^{r}(-1)^{s(1+(p-1)/2)}\binom{r}{s} f^{k+s(p-1)}\frac{H_{k+s(p-1)}(\alpha;\, u,\, v)}{k+s(p-1)}A^{r-s}(\alpha)\equiv 0(\operatorname{mod} p^r)\,.$$

As Katayama [6] remarks, Siegel [10] has shown that the values at positive integers of $L(s, \chi\cdot\lambda)=\sum_{(\mathfrak{a},\mathfrak{f})=1}(\chi\cdot\lambda(\mathfrak{a})/(N\mathfrak{a})^s)$, where K is an imaginary quadratic field, $\mathfrak{f}$ is an integral ideal of K, χ is a primitive ray-class character mod $\mathfrak{f}$ and λ is a Grössencharacter, may be expressed in terms of the Hurwitz-Herglotz function. Consequently, Theorem 1.4 also give a Kummer congruence for the values of this L-series.

§2. Facts about certain zeta functions.

In the proof of Theorem 1.4 it will be important to calculate q-expansions. The following facts will be needed. For proofs of these facts see Apostol [1].

The Hurwitz zeta function $\zeta(s, a)$ is defined for $\operatorname{Re}(s)>1$ by

$$\zeta(s, a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s}\,.$$

This function may be analytically continued to a function which is analytic for all s except for a simple pole at $s=1$. The Hurwitz zeta function satisfies the following function equation.

THEOREM 2.1. *If $h, k\in \boldsymbol{Z}$ with $1\leqq h\leqq k$, then for all s*

$$\zeta\Big(1-s, \frac{h}{k}\Big)=\frac{2\Gamma(s)}{(2\pi k)^s}\sum_{r=1}^{k}\cos\Big(\frac{\pi s}{2}-\frac{2\pi rh}{k}\Big)\zeta\Big(s, \frac{r}{k}\Big)\,.$$

The values at negative integers of the Hurwitz zeta function are given by

THEOREM 2.2. *For every integer* $n \geqq 0$

$$\zeta(-n, a) = -\frac{B_{n+1}(a)}{n+1} .$$

The periodic zeta function $F(x, s)$ is defined by

$$F(x, s) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n x}}{n^s}$$

for x real and $\operatorname{Re}(s) > 1$. The periodic zeta function and the Hurwitz zeta function are related by the following two results.

THEOREM 2.3. (Hurwitz's formula). *If* $0 < a \leqq 1$ *and* $\operatorname{Re}(s) > 1$, *then*

$$\zeta(1-s, a) = \frac{\Gamma(s)}{(2\pi)^s} \left\{ e^{-\pi i s/2} F(a, s) + e^{\pi i s/2} F(-a, s) \right\} .$$

If $a \neq 1$, *this is also valid for* $\operatorname{Re}(s) > 0$.

LEMMA 2.4. *Let* h *and* k *be integers with* $k \geqq 1$, *then for* $\operatorname{Re} s > 1$,

$$F\left(\frac{h}{k}, s\right) = k^{-s} \sum_{r=1}^{k} e^{2\pi i r h/k} \zeta\left(s, \frac{r}{k}\right) .$$

The following lemma gives a result that will be useful in the sequel.

LEMMA 2.5. *If* $a, f,$ *and* k *are positive integers, then*

$$F\left(-\frac{a}{f}, k\right) + (-1)^k F\left(\frac{a}{f}, k\right) = \frac{-B_k\left(\frac{a}{f}\right)}{k!} (2\pi i)^k .$$

PROOF. From Lemma 2.4

$$\begin{aligned} F\left(-\frac{a}{f}, k\right) &= f^{-k} \sum_{r=1}^{f} e^{-2\pi i a r/f} \zeta\left(k, \frac{r}{f}\right) \\ &= f^{-k} \sum_{r=1}^{f} \cos\left(\frac{2\pi a r}{f}\right) \zeta\left(k, \frac{r}{f}\right) \\ &\quad + i f^{-k} \sum_{r=1}^{f} \cos\left(\frac{\pi}{2} + \frac{2\pi a r}{f}\right) \zeta\left(k, \frac{r}{f}\right) . \end{aligned} \tag{2.6}$$

Likewise

(2.7)
$$F\left(\frac{a}{f}, k\right)=f^{-k}\sum_{r=1}^{f} e^{2\pi iar/f}\zeta\left(k, \frac{r}{f}\right)$$
$$=f^{-k}\sum_{r=1}^{f}\cos\left(\frac{2\pi ar}{f}\right)\zeta\left(k, \frac{r}{f}\right)$$
$$-if^{-k}\sum_{r=1}^{f}\cos\left(\frac{\pi}{2}+\frac{2\pi ar}{f}\right)\zeta\left(k, \frac{r}{f}\right).$$

When k is even, it follows from (2.6) and (2.7) that

(2.8)
$$F\left(-\frac{a}{f}, k\right)+(-1)^k F\left(\frac{a}{f}, k\right)=2f^{-k}\sum_{r=1}^{f}\cos\left(\frac{2\pi ar}{f}\right)\zeta\left(k, \frac{r}{f}\right).$$

While when k is odd, (2.6) and (2.7) yield

(2.9)
$$F\left(-\frac{a}{f}, k\right)+(-1)^k F\left(\frac{a}{f} k\right)=2if^{-k}\sum_{r=1}^{f}\cos\left(\frac{\pi}{2}+\frac{2\pi ar}{f}\right)\zeta\left(k, \frac{r}{f}\right).$$

The function equation for the Hurwitz zeta function, Theorem 2.1, demonstrates that (2.8) and (2.9) yield

(2.10)
$$F\left(-\frac{a}{f}, k\right)+(-1)^k F\left(\frac{a}{f}, k\right)=\frac{(2\pi i)^k}{(k-1)!}\zeta\left(1-k, \frac{a}{f}\right).$$

Applying Theorem 2.2, one concludes that

$$F\left(-\frac{a}{f}, k\right)+(-1)^k F\left(\frac{a}{f}, k\right)=-\frac{(2\pi i)^k}{k!}B_k\left(\frac{a}{f}\right).$$

§ 3. The q-expansion.

Note that with $u=a/f$, $v=b/f$, and $(a, b, f)=1$ one has

(3.1)
$$f^k H_k(\omega; u, v)=\frac{f^k k!}{(2\pi i)^k \eta^{2k}(\omega)}\sum_{m,n=-\infty}^{\infty}{}' \frac{e^{2\pi i(mu+nv)}}{(m\omega+n)^k}$$
$$=\frac{f^k k!}{(2\pi i)^k \eta^{2k}(\omega)}\sum_{0\leq\mu,\nu<f}\ \sum_{\substack{m\equiv\mu(\mathrm{mod}\, f)\\ n\equiv\nu(\mathrm{mod}\, f)}}{}' \frac{e^{-2\pi i(mu+nv)}}{(m\omega+n)^k}$$
$$=\frac{f^k k!}{(2\pi i)^k \eta^{2k}(\omega)}\sum_{0\leq\mu,\nu<f} e^{-(2\pi i/f)(a\mu+b\nu)}\sum_{\substack{m\equiv\mu(\mathrm{mod}\, f)\\ n\equiv\nu(\mathrm{mod}\, f)}}{}' \frac{1}{(m\omega+n)^k}$$
$$=\frac{f^k k!}{(2\pi i)^k \eta^{2k}(\omega)}\sum_{0\leq\mu,\nu<f} e^{-(2\pi i/f)(a\mu+b\nu)}G_k(\omega; \mu, \nu, f).$$

Hecke [4] has established that the Eisenstein series of higher level have the following q-expansion,

$$(3.2)\qquad G_k(\omega;\,\mu,\nu,f)=\delta\left(\frac{\nu}{f}\right)\sum_{m_1\equiv\mu(\mathrm{mod}\,f)}{}' \frac{1}{m_1} \\ +\frac{(-2\pi i)^k k}{f^k k!}\sum_{\substack{mm_2>0\\ m_2\equiv\nu(\mathrm{mod}\,f)}} m^{k-1}(\operatorname{sgn} m)e^{(2\pi i/f)m\mu}q^{mm_2/f}$$

where $\delta(x)=\begin{cases}0 \text{ if } x\notin Z\\ 1 \text{ if } x=Z\end{cases}$ and $q=e^{2\pi i\omega}$.

From (3.1) and (3.2) it now follows that

$$f^kH_k(\omega;\,u,v)=\eta^{-2k}(\omega)\sum_{0\le\mu,\nu<f}e^{-(2\pi i/f)(a\mu+b\nu)} \\ \times\left\{\frac{f^k k!}{(2\pi i)^k}\delta\left(\frac{\nu}{f}\right)\sum_{m_1\equiv\mu(\mathrm{mod}\,f)}{}'\frac{1}{m_1^k}+(-1)^k k\sum_{\substack{mm_2>0\\ m_2\equiv\nu(\mathrm{mod}\,f)}}(\operatorname{sgn} m)e^{(2\pi i/f)m\mu}q^{mm_2/f}\right\}.$$

To calculate the constant term note

$$\sum_{0\le\mu,\nu<f}e^{-(2\pi i/f)(a\mu+b\nu)}\delta\left(\frac{\nu}{f}\right)\sum_{m_1\equiv\mu(\mathrm{mod}\,f)}{}'\frac{1}{m_1^k}=\sum_{\mu=0}^{f-1}e^{-2\pi i a\mu/f}\sum_{m_1\equiv\mu(\mathrm{mod}\,f)}{}'\frac{1}{m_1^k}$$
$$=\sum_{\mu=0}^{f-1}\sum_{m=-\infty}^{\infty}{}'\frac{e^{-(2\pi i a\mu/f)}}{(fm+\mu)^k}=\sum_{n=-\infty}^{\infty}{}'\frac{e^{-(2\pi i/f)an}}{n^k}$$
$$=\sum_{n=1}^{\infty}\frac{e^{-(2\pi i/f)an}}{n^k}+(-1)^k\sum_{n=1}^{\infty}\frac{e^{(2\pi i/f)an}}{n^k}$$
$$=F\left(-\frac{a}{f},k\right)+(-1)^kF\left(\frac{a}{f},k\right)=-B_k\left(\frac{a}{f}\right)(2\pi i)^k/k!\,,$$

using Lemma 2.5. Hence

$$(3.3)\qquad f^kH_k(\omega;\,u,v)=\eta^{-2k}(\omega)\Big[-B_k\left(\frac{a}{f}\right)f^k \\ +\sum_{0\le\mu,\nu<f}e^{-(2\pi i/f)(a\mu+b\nu)}(-1)^k k\sum_{\substack{mm_2>0\\ m_2\equiv\nu(\mathrm{mod}\,f)}}m^{k-1}(\operatorname{sgn} m)e^{(2\pi i/f)m\mu}q^{mm_2/f}\Big].$$

§ 4. The Kummer congruence.

The following lemma is proved in an almost identical manner to Satz 1 of [8]; its proof is omitted here.

LEMMA 4.1. *Let α belong to an imaginary quadratic field with* $\operatorname{Im}\alpha>0$. *Then for positive integers a, b, f, and k with $k\geqq 3$, $u=a/f$, and $v=b/f$, and $(a,b,f)=1$, then $f^kH_k(\alpha,u,v)$ is an algebraic integer.*

With Vandiver's Kummer congruence (1.4) and (3.3) in mind, form the sum

$$(4.2)\qquad \sum_{s=0}^{r}(-1)^s\binom{r}{s}\frac{f^{k+s(p-1)}H_{k+s(p-1)}(\omega;u,v)}{k+s(p-1)}\eta^{2k+2s(p-1)}(\omega)$$
$$=-\sum_{s=0}^{r}(-1)^s\binom{r}{s}\frac{B_{k+s(p-1)}\left(\frac{a}{f}\right)}{k+s(p-1)}$$
$$+\sum_{0\leq\mu,\nu<f}e^{-(2\pi i/f)(a\mu+b\nu)}\sum_{\substack{mm_2>0\\ m_2\equiv\nu(\operatorname{mod} f)}}(\operatorname{sgn} m)e^{(2\pi i/f)m\mu}q^{mm_2/f}$$
$$\times\sum_{s=0}^{r}(-1)^{k+s(p-1)}\binom{r}{s}m^{k+s(p-1)-1}\,.$$

Note that

$$(4.3)\qquad \sum_{s=0}^{r}(-1)^{k+s(p-1)}\binom{r}{s}m^{k+s(p-1)-1}=m^{k-1}((-m)^{p-1}-1)^r\equiv 0(\operatorname{mod} p^r)\,.$$

Let $R(q,p)$ denote the ring of q-expansions of the form

$$\sum a_m q^{m/12f}$$

with algebraic and p-integral coefficients, with only a finite number of nonzero terms with $m<0$. Hence one can use (1.3), (4.2) and (4.3) to write

$$(4.4)\qquad \sum_{s=0}^{r}(-1)^s\binom{r}{s}f^{k+s(p-1)}\frac{H_{k+s(p-1)}(\omega;u,v)}{k+s(p-1)}\eta^{2k+2s(p-1)}(\omega)\equiv 0(\operatorname{mod} p^r)R(q,p)\,.$$

Since the coefficients of the q-expansions of η^2 and η^{-2} are integers, one has

$$(4.5)\qquad \sum_{s=0}^{r}(-1)^s\binom{r}{s}f^{k+s(p-1)}\frac{H_{k+s(p-1)}(\omega;u,v)}{k+s(p-1)}\eta^{-2(p-1)(r-s)}(\omega)\equiv 0(\operatorname{mod} p^r)R(q,p)\,.$$

For $p\geqq 5$, Herglotz (5) has shown that

$$\eta^{-2(p-1)}(\omega)\equiv(-1)^{(p-1)/2}A_p(\omega)\ (\operatorname{mod} p)R(q,p)\,.$$

Note that the q-expansion, in powers of $q^{1/6}$, of $A_p(\omega)$ has integer coefficients and that $A_p(\alpha)$ for α belonging to an imaginary quadratic field with $\operatorname{Im}\alpha>0$ is an algebraic integer.

One can replace $\eta^{-2(p-1)}(\omega)$ by $(-1)^{(p-1)/2}A_p(\omega)$ in (4.5) (by the result of Carlitz [2, page 427]) to obtain

$$(4.6)\qquad \sum_{s=0}^{r}(-1)^{s(1+(p-1)/2)}\binom{r}{s}f^{k+s(p-1)}\frac{H_{k+s(p-1)}(\omega;u,v)}{k+s(p-1)}A_p^{r-s}(\omega)$$
$$\equiv 0\ (\operatorname{mod} p^r)R(q,p)\,.$$

The following lemma is proved in an identical manner to the corresponding result of [8]; the proof will be omitted here.

LEMMA 4.7. *Let* $p \geqq r$ *be prime,* a, b, f, k, *and* r *be integers with* $u=a/f$, $v=b/f$, $(a, b, f)=1$, $k \geqq 3$, *and* $k>r$, *then if* α *belongs to an imaginary quadratic field with* $\operatorname{Im}\alpha>0$, *then*

$$\sum_{s=0}^{r}(-1)^{s(1+(p-1)/2)}\binom{r}{s}f^{k+s(p-1)}\frac{H_{k+s(p-1)}(\alpha; u, v)}{k+s(p-1)}A_p^{r-s}(\alpha)$$

is algebraic and is p-integral.

Hence it can finally be concluded from (4.6) and Lemma 4.7 that the Kummer congruence for the Hurwitz-Herglotz function is valid.

THEOREM 1.4 (The Kummer congruence for the Hurwitz-Herglotz function). *Let* $p \geqq 5$ *be prime and let* a, b, f, k *and* r *be integers with* $u=a/f$, $v=b/f$, $(a, b, f)=1$, $k \geqq 3$, $k>r \geqq 1$, *and* $p-1 \nmid k$ *,then if* α *belongs to an imaginary quadratic field with* $\operatorname{Im}\alpha>0$,

$$\sum_{s=0}^{r}(-1)^{s(1+(p-1)/2)}\binom{r}{s}f^{k+s(p-1)}\frac{H_{k+s(p-1)}(\alpha; u, v)}{k+s(p-1)}A_p^{r-s}(\alpha) \equiv 0 \pmod{p^r} .$$

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