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Classification of Periodic Maps on Compact Surfaces: I

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Introduction.

A homeomorphism $f: M\rightarrow M$ of a space M onto itself is called a periodic map on M with period n if $f^n=$identity and $f^k\neq$identity $(1\leqq k<n)$. We say that a periodic map f on M is *equivalent* to a periodic map f' on M' if there exists a homeomorphism $h: M\rightarrow M'$ such that $fh=hf'$. In this paper, we will obtain classification of orientation-preserving periodic maps on compact orientable surfaces. Classification of orientation-reversing periodic maps on compact orientable surfaces and periodic maps on compact non-orientable surfaces will be given in the forthcoming paper [5].

We will consider a pair (f, M) where M is a compact connected surface and f is a periodic map on M with period n. Let $\mathscr{S}_k=\mathscr{S}_k(f)=\{x\in M; f^k(x)=x, f^i(x)\neq x\ (1\leqq i<k)\}$ and $\mathscr{S}=\mathscr{S}(f)=\bigcup_{k=1}^{n-1}\mathscr{S}_k(f)=\{x\in M; 1\leqq\exists k<n, f^k(x)=x\}$, say *a singular set* of f. Let P_n be a set of (f, M) satisfying the condition that $\mathscr{S}(f)$ consists of finite points in $\dot{M}$ (may be empty). For an element (f, M), we obtain its orbit space M/f from M by the identification of x with $f(x)$ for $x\in M$.

PROPOSITION 1 (Whyburn [4]). *The orbit space M/f is a compact surface.*

Let $p: M\rightarrow M/f$ be a canonical map. Then p is an n-fold cyclic branched covering map of M/f with a branched set $p(\mathscr{S}(f))$. For a compact connected surface X and a set S of finite points in $\dot{X}$, we denote by $P_n(X, S)$ a set of elements (f, M) of P_n satisfying the following conditions;

(1) the orbit space M/f is homeomorphic to X,

(2) the canonical map $p: M\rightarrow X$ is an n-fold cyclic branched covering map with a branched set S.

Suppose that (f, M) is equivalent to (f', M'). Clearly there exists a

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homeomorphism $g\colon M/f \to M'/f'$ such that $g(p(\mathscr{S}(f)))=p'(\mathscr{S}(f'))$ and $gp=ph$ where $p'\colon M' \to M'/f'$ is a canonical map and $h\colon M\to M'$ is a homeomorphism satisfying $fh=hf'$. Let $X=M/f$ and $S=p(\mathscr{S}(f))$. Then $(f, M)\in P_n(X, S)$ and $(f', M')\in P_n(X, S)$. We denote by $\mathscr{P}_n(X, S)$ a set of equivalence classes of $P_n(X, S)$. Then for classification of P_n, we will determine a complete set of equivalence classes of $P_n(X, S)$ (see Theorem 1), and prove that:

THEOREM 2. *Assume that X is a compact orientable surface of genus g and that the boundary ∂X consists of l components, and furthermore that a set S consists of m points in $\dot{X}$. Let $n=p_1^{e_1}p_2^{e_2}\cdots p_s^{e_s}$ be the prime decomposition of n. Then the number of elements of $\mathscr{P}_n(X, S)$ is given by;*

(I) $C_0^*(n; l, m)=C_0(n; l, m)/2+Q_0(n; l, m)/2$ *if* $g\geqq 1$,

(II) $C_0^*(n; l, m)-\sum_{i=1}^{s} C_0^*(n/p_i; l, m)+\sum_{1\leqq i<j\leqq s} C_0^*(n/p_ip_j; l, m)+\cdots+(-1)^j\sum_{1\leqq i_1<i_2<\cdots<i_j\leqq s} C_0^*(n/(p_{i_1}p_{i_2}\cdots p_{i_j}); l, m)+\cdots+(-1)^s C_0^*(n/(p_1 p_2\cdots p_s); l, m)$ $(=\sum_{q|n}\mu(q)C_0^*(n/q; l, m))$, *if* $g=0$, *where $\mu(q)$ is the Möbius function.* *($C_0(n; l, m)$ and $Q_0(n; l, m)$ are given in* § 3 *in detail.)*

Let P_n^+ be a subset of P_n such that M is an orientable surface and f is an orientation-preserving periodic map. Then, we will obtain, as a consequence, classification of P_n^+ (see Theorem 5 and Theorem 6). Especially, assume that n is a prime number, and let $P_n^+(g, l, l_1, m)$ be a set of elements (f, M) of P_n^+ satisfying the following conditions;

(1) M is a compact orientable surface of genus g and the boundary ∂M consists of l components,

(2) f is an orientation preserving periodic map on M such that its singular set $\mathscr{S}(f)$ consists of m points in $\dot{M}$,

(3) the number of setwise fixed boundary components of M by f is l_1.

Denote by $\mathscr{P}_n^+(g, l, l_1, m)$ a set of equivalence classes of $P_n^+(g, l, l_1, m)$. Then we will prove that:

THEOREM 3. *Suppose that n is an odd prime number. Then $P_n^+(g, l, l_1, m)\neq\emptyset$ if and only if g, l, l_1 and m satisfy the following conditions* (I), (II) *and* (III):

(I) $l-l_1\equiv 0 \pmod{n}$,

(II) $l_1+m\neq 1$,

(III) $g+n\times\min\{l_1+m, 1\}-((n-1)/2)(l_1+m)-1$ *is a non-negative integer and a multiple of n.*

Furthermore, the number of elements of $\mathscr{P}_n^+(g, l, l_1, m)$ is equal to $C(n; l, m)/2+Q(n; l, m)/2$. ($C(n; l, m)$ and $Q(n; l, m)$ are given in § 4 in detail.)

THEOREM 4. *Suppose that* $n=2$. *Then* $P_2^+(g, l, l_1, m) \neq \emptyset$ *if and only if* g, l, l_1 *and* m *satisfy the following conditions* (I), (II′), *and* (III);

(I) $l-l_1 \equiv 0 \pmod 2$,

(II′) l_1+m *is even*,

(III) $g+2\times\min\{l_1+m, 1\}-(l_1+m)/2 \geqq 1$; *odd*.

Furthermore, the number of elements of $\mathscr{P}_2^+(g, l, l_1, m)$ *is equal to* 1, *that is, an involution* $(f, M) \in P_2^+(g, l, l_1, m)$ *is unique up to equivalence.*

In case of $m=0$, Theorem 4 is given by Asoh [1].

In § 1, we will give a model of (X, S) and reduce an equivalence relation of $P_n(X, S)$. In § 2, using the homeotopy group of (X, S), we will determine the equivalence classes of $P_n(X, S)$ (see Theorem 1) and in § 3, we will prove Theorem 2. In § 4, we will have classification of orientation-preserving periodic maps.

The author will like to express his sincere gratitude to Prof. Sin'ichi Suzuki, and especially, to his colleague Mr. Teruhiko Hilano for his helpful conversation in § 3.

§ 1. A model for X and the reductions of equivalence relation for $P_n(X, S)$.

Let X be a compact connected orientable surface of genus g and let the boundary ∂X consist of l components $\tilde{d}_1, \tilde{d}_2, \cdots, \tilde{d}_l$. For the sake of convenience, we first take a model for X in the 3-dimensional Euclidean space R^3 as shown in Fig. 1, and simple oriented loops $a_1, a_2, \cdots, a_g$,

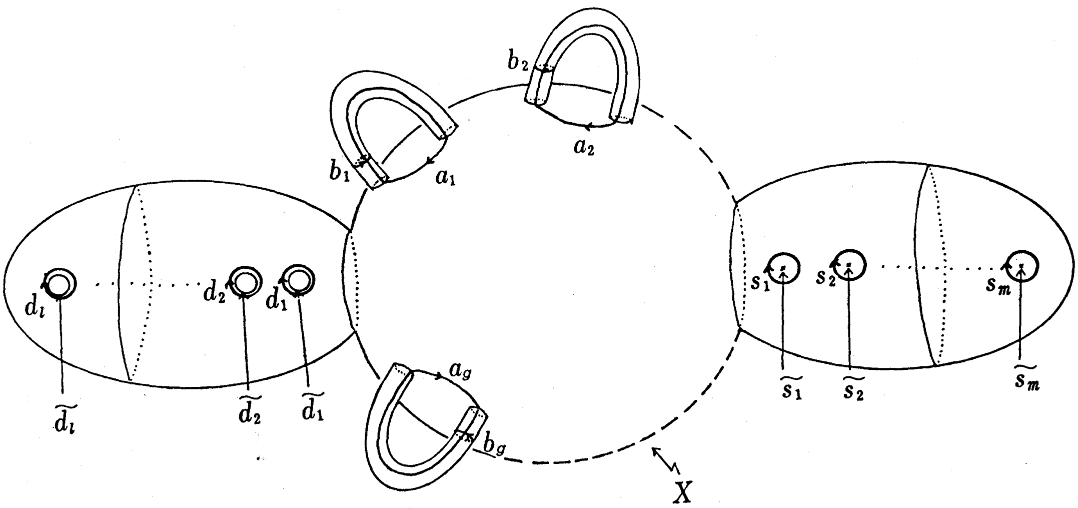

FIGURE 1

$b_1, b_2, \cdots, b_g, d_1, d_2, \cdots, d_l$ on X as shown in Fig. 1. Let S be finite points $\tilde{s}_1, \tilde{s}_2, \cdots, \tilde{s}_m$ in $\dot{X}$, and take simple oriented loops $s_1, s_2, \cdots, s_m$ on X as shown in Fig. 1.

To avoid a multiplicity of brackets, we refer to loops rather than to homology classes. Then the first integral homology group of $X-S$ is given by;

$$H_1(X-S)=\left|\begin{matrix} a_1, b_1, a_2, b_2, \cdots, a_g, b_g, \\ d_1, d_2, \cdots, d_l, \\ s_1, s_2, \cdots, s_m \end{matrix} ; \begin{matrix} d_1+d_2+\cdots+d_l \\ +s_1+s_2+\cdots+s_m=0 \end{matrix}\right|. \tag{1.1}$$

To determine equivalence classes of $P_n(X, S)$, the following is useful;

DEFINITION 1. Let $[H_1(X-S); Z_n]^*$ be a set of homomorphisms ω of the first integral homology group $H_1(X-S)$ onto the cyclic group Z_n of order n such that $\omega(s_i)\neq 0$ for every $s_i \in H_1(X-S)$. We say that two elements ω_1 and ω_2 of $[H_1(X-S); Z_n]^*$ are *$\mathscr{A}$-equivalent*, denoted by $\omega_1 \underset{\mathscr{A}}{\sim} \omega_2$, if there exists a homeomorphism h of (X, S) onto (X, S) such that $\omega_1 h_* = \omega_2$, where h_* is the automorphism of $H_1(X-S)$ induced by $h|_{X-S}$.

To avoid a multiplicity of $_*$, we also use $^*h$ as h_*, if there is no confusion.

Using a branched covering theory, we obtain the following, in a similar way to P. A. Smith [2]:

PROPOSITION 2. *There is a one-to-one correspondence between the set of equivalence classes of $P_n(X, S)$ and the set of $\mathscr{A}$-equivalence classes of $[H_1(X-S); Z_n]^*$.*

Let $Z_n(g; l, m)$ be a set of systems of inegers $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta}, \boldsymbol{\theta})=(\alpha_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_g, \beta_g, \delta_1, \delta_2, \cdots, \delta_l, \theta_1, \theta_2, \cdots, \theta_m)$ satisfying the following conditions;

(0) $0\leqq\alpha_i<n$, $0\leqq\beta_i<n$, $0\leqq\delta_j<n$, and $1\leqq\theta_k<n$ $(i=1, 2, \cdots, g;\ j=1, 2, \cdots, l;\ k=1, 2, \cdots, m)$,

(1) $\delta_1+\delta_2+\cdots+\delta_l+\theta_1+\theta_2+\cdots+\theta_m\equiv 0 \pmod n$,

(2) g.c.d. $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_g, \beta_g, \delta_1, \delta_2, \cdots, \delta_l, \theta_1, \theta_2, \cdots, \theta_m\}\equiv 1$ (mod n), where g.c.d. means the greatest common divisor.

REMARK. In case where an element of systems of integers $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta}, \boldsymbol{\theta})$ is not an integer satisfying the condition (0), we regard it as a representative of Z_n.

Now ω is an element of $[H_1(X-S); Z_n]^*$. If $\omega(a_i)=\alpha_i$, $\omega(b_i)=\beta_i$, $\omega(d_j)=\delta_j$, and $\omega(s_k)=\theta_k$ $(i=1, 2, \cdots, g;\ j=1, 2, \cdots, l;\ k=1, 2, \cdots, m)$, then

ω is represented by an element $(\alpha_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_g, \beta_g, \delta_1, \delta_2, \cdots, \delta_l, \theta_1, \theta_2, \cdots, \theta_m)$ of $Z_n(g; l, m)$, say $\Sigma(\omega)$. Conversely, for an element $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta}, \boldsymbol{\theta})$ of $Z_n(g; l, m)$, there exist uniquely an element ω of $[H_1(X-S); Z_n]^*$ such that $\sum(\omega)=(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta}, \boldsymbol{\theta})$. So Σ is a one-to-one correspondence between $[H_1(X-S); Z_n]^*$ and $Z_n(g; l, m)$. We will define the equivalence relation on $Z_n(g; l, m)$ by the equivalence relation $\underset{\mathscr{A}}{\sim}$ on $[H_1(X-S); Z_n]^*$, as follows:

DEFINITION 2. We say that two elements $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta}, \boldsymbol{\theta})$ and $(\boldsymbol{\alpha}', \boldsymbol{\beta}', \boldsymbol{\delta}', \boldsymbol{\theta}')$ of $Z_n(g; l, m)$ are *equivalent*, denoted by $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta}, \boldsymbol{\theta})\sim(\boldsymbol{\alpha}', \boldsymbol{\beta}', \boldsymbol{\delta}', \boldsymbol{\theta}')$, if $\Sigma^{-1}((\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta}, \boldsymbol{\theta}))$ is $\mathscr{A}$-equivalent to $\Sigma^{-1}((\boldsymbol{\alpha}', \boldsymbol{\beta}', \boldsymbol{\delta}', \boldsymbol{\theta}'))$.

We have clearly that Σ is a one-to-one correspondence between the set of $\mathscr{A}$-equivalence classes of $[H_1(X-S); Z_n]^*$ and the set of equivalence classes of $Z_n(g; l, m)$.

§ 2. Determination of the equivalence classes of $P_n(X, S)$.

To determine the equivalence classes of $P_n(X, S)$, we use the following result of S. Suzuki [3]:

PROPOSITION 3. *There exist homeomorphisms* ρ, ρ_{1i}, τ_1, μ_1, θ_{12} *of* (X, S) *onto itself such that automorphisms of* $H_1(X-S)$ *induced by them are given by*:

$$\rho(a_i)=a_{i+1}\,,\quad \rho(b_i)=b_{i+1}\,,\quad (i=1, 2, \cdots, g)\,;$$
$$\rho_{1i}(a_1)=a_i\,,\quad \rho_{1i}(b_1)=b_i\,,\quad \rho_{1i}(a_i)=a_1\,,\quad \rho_{1i}(b_i)=b_1\,;$$
$$\tau_1(a_1)=a_1-b_1\,,\quad \tau_1(b_1)=b_1\,;$$
$$\mu_1(a_1)=b_1\,,\quad \mu_1(b_1)=-a_1\,;$$
$$\theta_{12}(a_1)=a_1-a_2\,,\quad \theta_{12}(b_1)=b_1\,,\quad \theta_{12}(a_2)=a_2\,,\quad \theta_{12}(b_2)=b_1+b_2\,;$$

where the remaining generators of (1.1) *are unchanged.*

LEMMA 1. *For an element* $\Sigma(\omega)=(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta}, \boldsymbol{\theta})\in Z_n(g; l, m)$, *we have the followings*:

$$\Sigma(\omega\rho)=(\alpha_2, \beta_2, \alpha_3, \beta_3, \cdots, \alpha_g, \beta_g, \alpha_1, \beta_1, \boldsymbol{\delta}, \boldsymbol{\theta})\,,$$
$$\Sigma(\omega\rho_{1i})=(\alpha_i, \beta_i, \alpha_2, \beta_2, \cdots, \alpha_{i-1}, \beta_{i-1}, \alpha_1, \beta_1, \alpha_{i+1}, \beta_{i+1}, \cdots, \alpha_g, \beta_g, \boldsymbol{\delta}, \boldsymbol{\theta})\,,$$
$$\Sigma(\omega\tau_1)=(\alpha_1-\beta_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_g, \beta_g, \boldsymbol{\delta}, \boldsymbol{\theta})\,,$$
$$\Sigma(\omega\mu_1)=(\beta_1, -\alpha_1, \alpha_2, \beta_2, \cdots, \alpha_g, \beta_g, \boldsymbol{\delta}, \boldsymbol{\theta})\,,$$
$$\Sigma(\omega\theta_{12})=(\alpha_1-\alpha_2, \beta_1, \alpha_2, \beta_2+\beta_1, \alpha_3, \beta_3, \cdots, \alpha_g, \beta_g, \boldsymbol{\delta}, \boldsymbol{\theta})\,.$$

By these results, we have the following lemma:

LEMMA 2. *$(\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g, \delta, \theta) \sim (0, \gamma, 0, \cdots, 0, \delta, \theta)$, where* $\gamma =$ g.c.d. $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_g, \beta_g\}$.

PROOF. (I) We will prove that $(\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g, \delta, \theta) \sim (0, \gamma_1, 0, \gamma_2, \cdots, 0, \gamma_g, \delta, \theta)$, where $\gamma_i =$ g.c.d. $\{\alpha_i, \beta_i\}$. First it will be shown that $(\alpha_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_g, \beta_g, \delta, \theta) \sim (0, \gamma_1, \alpha_2, \beta_2, \cdots, \alpha_g, \beta_g, \delta, \theta)$. If $\alpha_1 = 0$, there is nothing to do. If $\beta_1 = 0$, we have merely to apply μ_1^{-1}. So we may assume that $\alpha_1\beta_1 \neq 0$. Then we have the following;

$$\begin{aligned}
&\exists\, q_1, r_1 \in N \text{ such that } \alpha_1 = q_1\beta_1 + r_1, \quad 0 < r_1 < \beta_1,\\
&\exists\, q_2, r_2 \in N \text{ such that } \beta_1 = q_2 r_1 + r_2, \quad 0 < r_2 < r_1,\\
&\exists\, q_3, r_3 \in N \text{ such that } r_1 = q_3 r_2 + r_3, \quad 0 < r_3 < r_2,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&\exists\, q_{t-2}, r_{t-2} \in N \text{ such that } r_{t-4} = q_{t-2} r_{t-3} + r_{t-2}, \quad 0 < r_{t-2} < r_{t-3},\\
&\exists\, q_{t-1}, r_{t-1} \in N \text{ such that } r_{t-3} = q_{t-1} r_{t-2} + r_{t-1}, \quad 0 < r_{t-1} < r_{t-2},\\
&\exists\, q_t, r_t \in N \text{ such that } r_{t-2} = q_t r_{t-1} + r_t, \quad r_t = 0.
\end{aligned}$$

Let $h_1 = \tau_1^{q_1}\mu_1^{-1}\tau_1^{-q_2}\mu_1\tau_1^{q_3} \cdots \mu_1^{(-1)^{i+1}}\tau_1^{(-1)^{i+1}q_i} \cdots \mu_1^{(-1)^{t+1}}\tau_1^{(-1)^{t+1}q_t}$. Then, $\omega \underset{\mathscr{S}}{\simeq} \omega h_1$. $\Sigma(\omega h_1) = (0, \gamma_1, \alpha_2, \beta_2, \cdots, \alpha_g, \beta_g, \delta, \theta)$ is equivalent to $\Sigma(\omega)$. By the same way, we have an automorphism h_2 of $H_1(X-S)$ induced by the composition of homeomorphisms τ_1 and μ_1 such that $\Sigma(\omega h_1\rho_{12}h_2) = (0, \gamma_2, 0, \gamma_1, \alpha_3, \beta_3, \cdots, \alpha_g, \beta_g, \delta, \theta)$ is equivalent to $\Sigma(\omega)$. Repeating the same procedures, we have an automorphism h of $H_1(X-S)$ induced by the composition of homeomorphisms τ_1, μ_1 and ρ_{1i} such that $\Sigma(\omega h) = (0, \gamma_1, 0, \gamma_2, \cdots, 0, \gamma_g, \delta, \theta)$ is equivalent to $\Sigma(\omega)$.

(II) If $\gamma_1 = \gamma_2 = \cdots = \gamma_g = 0$, there is nothing to do. So we may assume that $\gamma_1\gamma_2 \cdots \gamma_g \neq 0$. Applying ρ_{1i}, if necessary, we may assume that γ_1 is the smallest positive integer in $\{\gamma_1, \gamma_2, \cdots, \gamma_g\}$. Then there are non-negative integers q_i and r_i such that $\gamma_i = q_i\gamma_1 + r_i$ $(2 \leqq i \leqq g)$ and $0 \leqq r_i < \gamma_1$. Let $h_1' = \theta_{12}^{-q_2}(\rho_{12}\rho_{13}\rho_{12}\theta_{12}^{-q_3}\rho_{12}\rho_{13}\rho_{12}) \cdots (\rho_{12}\rho_{1g}\rho_{12}\theta_{12}^{-q_g}\rho_{12}\rho_{1g}\rho_{12})$. Then, we have $\omega \underset{\mathscr{S}}{\simeq} \omega h_1'$. $\Sigma(\omega h_1') = (0, \gamma_1, 0, r_2, 0, r_3, \cdots, 0, r_g, \delta, \theta)$ is equivalent to $\Sigma(\omega)$. If r_s is the smallest positive integer in $\{\gamma_1, r_2, r_3, \cdots, r_g\}$, then we apply ρ_{1s}; and by the same way, we have an automorphism h_2' of $H_1(X-S)$ induced by the composition of homeomorphisms ρ_{1i} and θ_{12} such that $\Sigma(\omega h_1'\rho_{1s}h_2') = (0, r_s, 0, r_2', 0, r_3', \cdots, 0, r_g', \delta, \theta)$ is equivalent to $\Sigma(\omega)$. Repeating the same procedures, we have an automorphism h' of $H_1(X-S)$ induced by the composition of homeomorphisms ρ_{1i} and θ_{12} such that $\Sigma(\omega h') = (0, \gamma, 0, 0, \cdots, 0, 0, \delta, \theta)$ is equivalent to $\Sigma(\omega)$, where $\gamma =$ g.c.d. $\{\gamma_1, \gamma_2, \cdots, \gamma_g\}$, completing the proof.

We use some more typical homeomorphisms of surfaces.

DEFINITION 3. Let $A'=\{(r, \theta); r \leqq 6\}$, $A=\{(r, \theta); r \leqq 5\}$, $A_1=\{(r, \theta); r \leqq 1\}$, $B_+=\{(r, \theta); (r\cos\theta-3)^2+r^2\sin^2\theta \leqq 1\}$, $B_-=\{(r, \theta); (r\cos\theta+3)^2+r^2\sin^2\theta \leqq 1\}$ be subsets in R^2 as shown in Fig. 2. We define a homeomorphism $\varphi: A' \to A'$ by putting

$$\varphi((r, \theta))=(r, \theta+\pi) \quad \text{if} \quad r \leqq 5 ,$$
$$\varphi((r, \theta))=(r, \theta+(6-r)\pi) \quad \text{if} \quad 5 \leqq r \leqq 6 ,$$

and define a homeomorphism $\psi: A-A_1 \to A-A_1$ by putting

$$\psi((r, \theta))=(r, \theta+2(r-1)\pi) \quad \text{if} \quad 1 \leqq r \leqq 2 ,$$

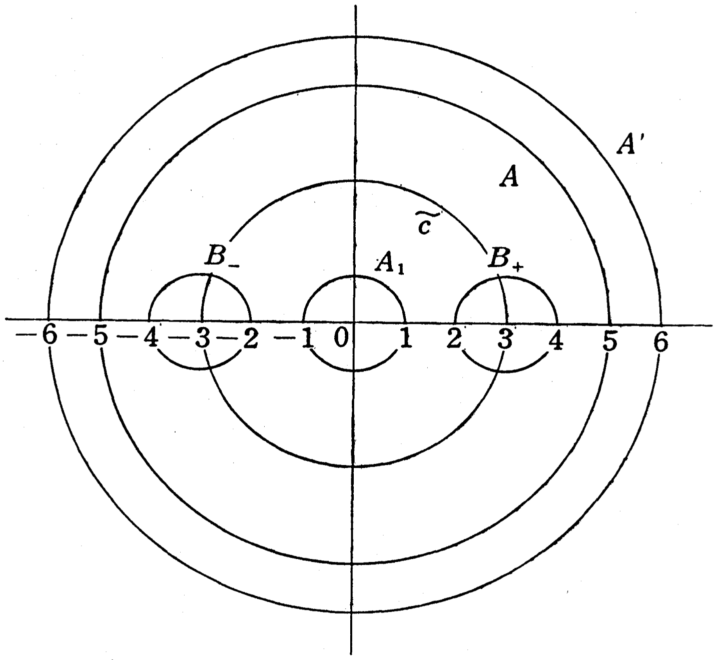

FIGURE 2

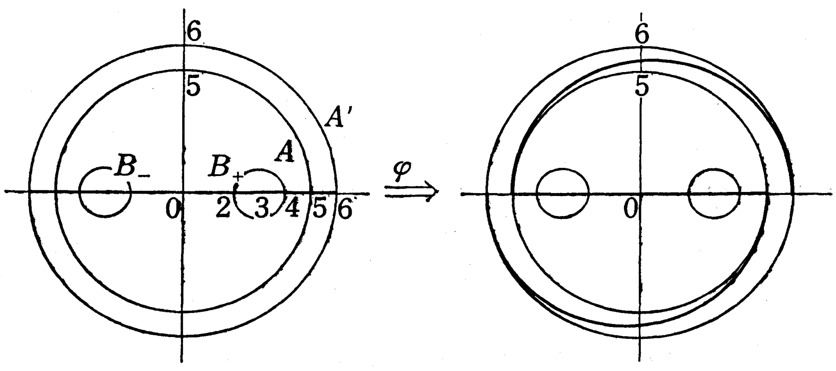

FIGURE 3

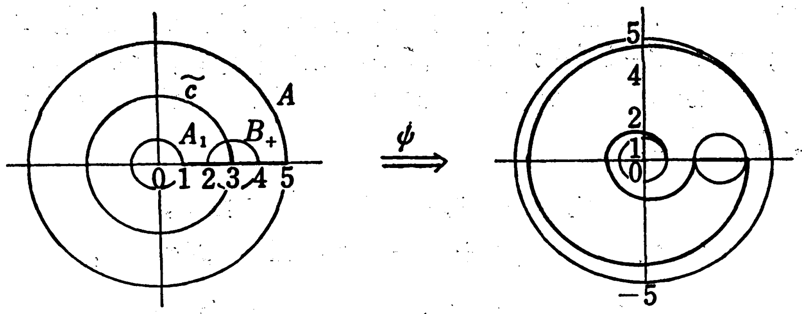

FIGURE 4

$$\psi((r, \theta))=(r, \theta+2(5-r)\pi) \quad \text{if} \quad 4\leqq r\leqq 5\,,$$
$$\psi((r, \theta))=(r, \theta) \quad \text{if} \quad 2\leqq r\leqq 4\,,$$

which are the same maps as ρ'_{12} and $\dot{\sigma}_{c_{2}\zeta_{1}}$ in S. Suzuki [3], as shown in Fig. 3 and Fig. 4.

(1) ∂_i $(2\leqq i\leqq l)$: Let h be an embedding of $A'-(\dot{B}_+\cup\dot{B}_-)$ in $X-S$ such that $h(A'-(\dot{B}_+\cup\dot{B}_-))\cap\partial X=h(\partial B_+)\cup h(\partial B_-)=\tilde{d}_1\cup\tilde{d}_i$ and $h(A'-(\dot{B}_+\cup\dot{B}_-))\cap\{a_1, b_1, \cdots, a_g, b_g, d_1, d_2, \cdots, d_l\}=\{d_1, d_i\}$. Then we have a homeomorphism ∂_i of (X, S) onto itself defined by $\partial_i=h\varphi h^{-1}$ on $h(A'-(\dot{B}_+\cup\dot{B}_-))$ and by $\partial_i=$ the identity on $X-h(A'-(\dot{B}_+\cup\dot{B}_-))$; see Fig. 5.

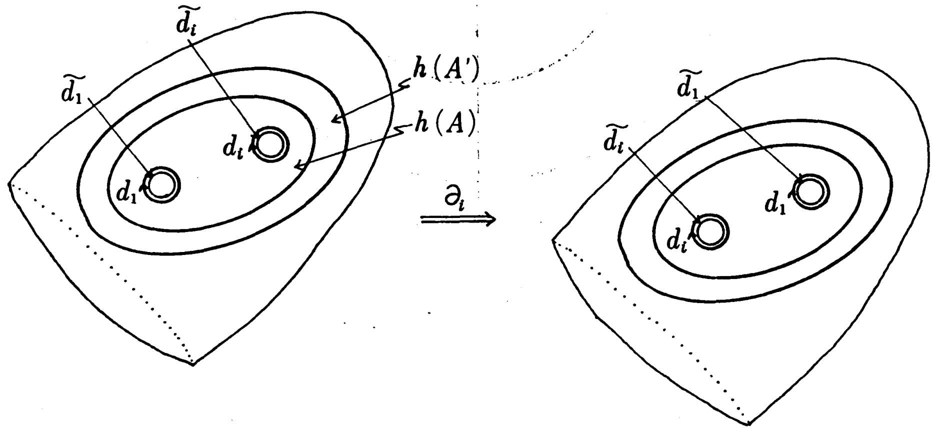

FIGURE 5

(2) σ_j $(2\leqq j\leqq m)$: Let h be an embedding of A' in X such that $h(A')\cap S=h((3, 0))\cup h((-3, 0))=\tilde{s}_1\cup\tilde{s}_j$ and $h(A')\cap\{a_1, b_1, a_2, b_2, \cdots, a_g, b_g, s_1, s_2, \cdots, s_m\}=\{s_1, s_j\}$. Then we have a homeomorphism σ_j of (X, S) onto itself defined by $\sigma_j=h\psi h^{-1}$ on $h(A')$ and by $\sigma_j=$the identity on $X-h(A')$; see Fig. 6.

(3) ∂_a: We take a 2-cell Δ and identify $\partial\Delta$ with a component $\tilde{d}_1$ of ∂X. We obtain the surface $X\cup\Delta$ of genus g with $l-1$ boundary

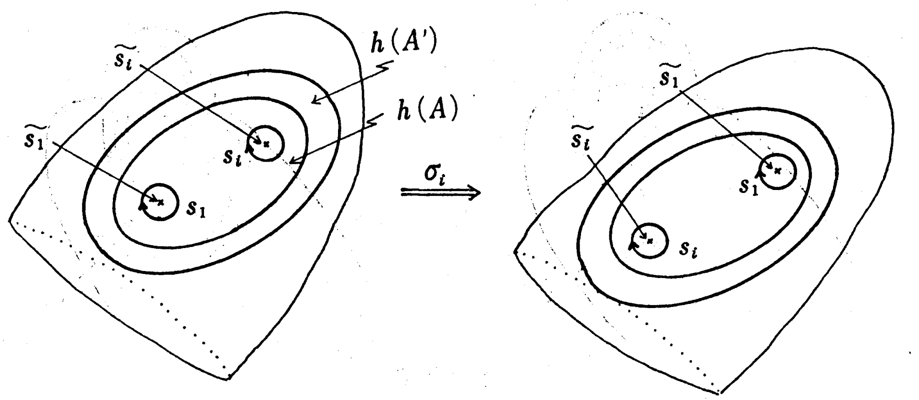

FIGURE 6

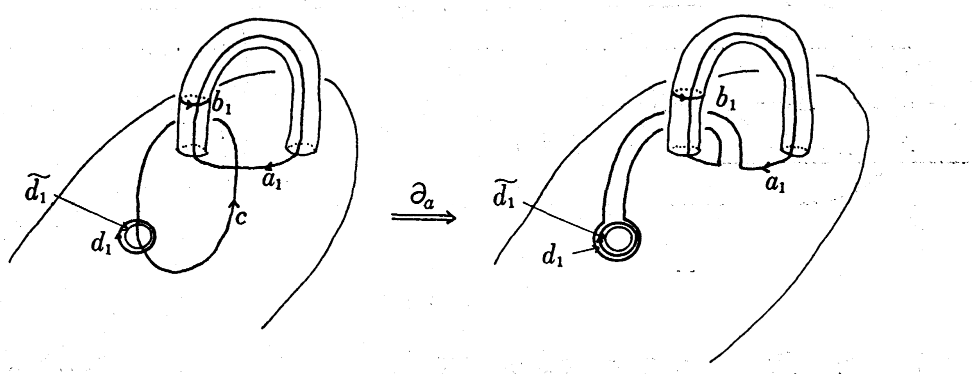

FIGURE 7

components. Let c be a simple loop on $X \cup \Delta$ passing through the center of Δ such that $c \cap \{a_1, b_1, \cdots, a_g, b_g, d_1, d_2, \cdots, d_l, s_1, s_2, \cdots, s_m\} = c \cap \{a_1, d_1\}$ and c intersects transversally at one point with a_1; see Fig. 7. Let h be an embedding of $A - A_1$ in $X \cup \Delta - S$ satisfying the following conditions; (1) $h(\tilde{c}) = c$, (2) $h(A - A_1)$ is a regular neighborhood of c, (3) $h(A - A_1) \cap \{a_1, b_1, \cdots, a_g, b_g, d_1, d_2, \cdots, d_l\} = h(A - A_1) \cap \{a_1, d_1\}$ and (4) $h(B_+) = \Delta$, where $\tilde{c} = \{(r, \theta); r = 3\}$. Then, we have a homeomorphism ∂_a of (X, S) onto itself defined by $\partial_a = h\psi h^{-1}$ on $h(A - A_1)$ and by $\partial_a =$ the identity on $X - h(A - A_1)$; see Fig. 7.

(4) $\underline{\sigma_a}$: We take a 2-cell Δ in $\mathring{X}$ such that $\mathring{\Delta} \supset s_1$, $\Delta \cap \{a_1, b_1, \cdots, a_g, b_g, s_1, s_2, \cdots, s_m\} = \{s_1\}$ and $\Delta \cap S = \{\tilde{s}_1\}$. Let c be a simple loop on X passing through $\tilde{s}_1$ such that $c \cap \{a_1, b_1, \cdots, a_g, b_g, d_1, d_2, \cdots, d_l, s_1, s_2, \cdots, s_m\} = c \cap \{a_1, s_1\}$ and that c intersects transversally at one point with a_1 (see Fig. 8). Let h be an embedding of $A - A_1$ into $\mathring{X}$ satisfying the conditions; (1) $h(\tilde{c}) = c$, (2) $h(A - A_1)$ is a regular neighborhood of c, (3) $h(A - A_1) \cap \{a_1, b_1, \cdots, a_g, b_g, s_1, s_2, \cdots, s_m\} = h(A - A_1) \cap \{a_1, s_1\}$ and (4) $h(B_+) = \Delta$. Then, we have a homeomorphism σ_a of (X, S) onto itself defined by $\sigma_a = h\psi h^{-1}$ on $h(A - A_1)$ and by $\sigma_a =$ the identity on $X - h(A - A_1)$; see Fig. 8.

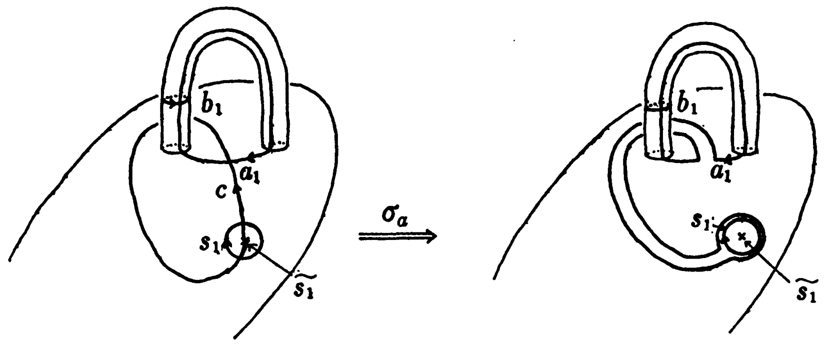

FIGURE 8

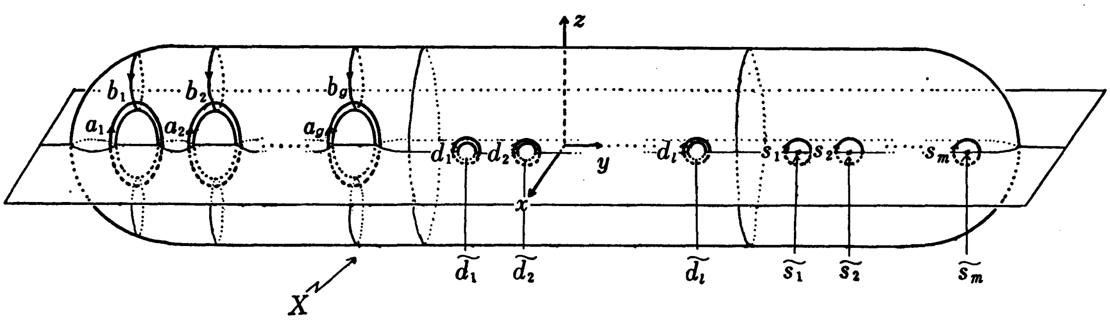

FIGURE 9

(5) η: For the sake of convenience, we take a model for X in the 3-dimensional Euclidean space R^3 as shown in Fig. 9. Let η be a homeomorphism of (X, S) defined by $\eta(x, y, z)=(x, y, -z)$.

Then, by S. Suzuki [3] and in an elementary way, we have the followings:

PROPOSITION 4. *The homeotopy group of* (X, S) *is generated by* ρ, ρ_{1i} $(2\leqq i\leqq g)$, τ_1, μ_1, θ_{12}, ∂_j $(2\leqq j\leqq l)$, σ_k $(2\leqq k\leqq m)$, ∂_a, σ_a *and* η.

LEMMA 3. (1) *The automorphisms of* $H_1(X-S)$ *induced by them are given by;*

$$\partial_i(d_1)=d_i\,, \qquad \partial_i(d_i)=d_1\,;$$
$$\sigma_i(s_1)=s_i\,, \qquad \sigma_i(s_i)=s_1\,;$$
$$\partial_a(a_1)=a_1-d_1\,;$$
$$\sigma_a(a_1)=a_1-s_1\,;$$
$$\eta(a_i)=-a_i \quad (1\leqq i\leqq g)\,, \qquad \eta(d_j)=-d_j \quad (1\leqq j\leqq l)\,,$$
$$\eta(s_k)=-s_k \quad (1\leqq k\leqq m)\,.$$

(2) *For an element* $\Sigma(\omega)=(\alpha, \beta, \delta, \theta)$ *of* $Z_n(g; l, m)$, *we have*

$$\Sigma(\omega\partial_i)=(\boldsymbol{\alpha}, \boldsymbol{\beta}, \delta_i, \delta_2, \delta_3, \cdots, \delta_{i-1}, \delta_1, \delta_{i+1}, \cdots, \delta_l, \boldsymbol{\theta})$$
$$\Sigma(\omega\sigma_i)=(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta}, \theta_i, \theta_2, \theta_3, \cdots, \theta_{i-1}, \theta_1, \theta_{i+1}, \cdots, \theta_m)$$
$$\Sigma(\omega\partial_a)=(\alpha_1-\delta_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_g, \beta_g, \boldsymbol{\delta}, \boldsymbol{\theta})$$
$$\Sigma(\omega\sigma_a)=(\alpha_1-\theta_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_g, \beta_g, \boldsymbol{\delta}, \boldsymbol{\theta})$$
$$\Sigma(\omega\eta)=(-\alpha_1, \beta_1, -\alpha_2, \beta_2, \cdots, -\alpha_g, \beta_g, -\delta_1, -\delta_2, \cdots, -\delta_l, -\theta_1, -\theta_2, \cdots, -\theta_m)\ .$$

By using the Lemmas 1 and 3, we have the following lemma:

LEMMA 4. $(0, \gamma, 0, 0, \cdots, 0, 0, \delta_1, \delta_2, \cdots, \delta_l, \theta_1, \theta_2, \cdots, \theta_m)\sim(0, 1, 0, 0, \cdots, 0, 0, \delta_1, \delta_2, \cdots, \delta_l, \theta_1, \theta_2, \cdots, \theta_m)$.

PROOF. Since g.c.d. $\{\gamma, \delta_1, \delta_2, \cdots, \delta_l, \theta_1, \theta_2, \cdots, \theta_m\}\equiv 1 \pmod n$, there exist integers $z_0, z_1, \cdots, z_l, z'_1, z'_2, \cdots, z'_m$ such that $z_0\gamma+z_1\delta_1+z_2\delta_2+\cdots+z_l\delta_l+z'_1\theta_1+z'_2\theta_2+\cdots+z'_m\theta_m\equiv 1 \pmod n$. Let $h=\tau_1^{-z_0}\partial_a^{-z_1}(\partial_2\partial_a^{-z_2}\partial_2)\cdots(\partial_l\partial_a^{-z_l}\partial_l)\sigma_a^{-z'_1}(\sigma_2\sigma_a^{-z'_2}\sigma_2)\cdots(\sigma_m\sigma_a^{-z'_m}\sigma_m)$, then $\omega\underset{\mathfrak{F}}{\sim}\omega h$. Hence $\Sigma(\omega h)=(1, \gamma, 0, 0, \cdots, 0, 0, \boldsymbol{\delta}, \boldsymbol{\theta})$ is equivalent to $\Sigma(\omega)$. $\Sigma(\omega h\mu_1\tau_1^{-\gamma}\mu_1^2)=(0, 1, 0, 0, \cdots, 0, 0, \boldsymbol{\delta}, \boldsymbol{\theta})$ is equivalent to $\Sigma(\omega)$.

Since the symmetric group $\mathfrak{S}_u$ ($u=l$ or m) is generated by the set of transpositions $\{(1, i)\}_{i=2}^{u}$, we have the following:

LEMMA 5. *For any permutation λ of $\{1, 2, \cdots, l\}$ and λ' of $\{1, 2, \cdots, m\}$, we have* $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \delta_1, \delta_2, \cdots, \delta_l, \boldsymbol{\theta})\sim(\boldsymbol{\alpha}, \boldsymbol{\beta}, \delta_{\lambda(1)}, \delta_{\lambda(2)}, \cdots, \delta_{\lambda(l)}, \boldsymbol{\theta})$, *and* $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta}, \theta_1, \theta_2, \cdots, \theta_m)\sim(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta}, \theta_{\lambda'(1)}, \theta_{\lambda'(2)}, \cdots, \theta_{\lambda'(m)})$.

To determine a complete set of equivalence classes of $Z_n(g; l, m)$, we will define an equivalence relation $\underset{\eta}{\sim}$ as follows:

DEFINITION 4. (I) An element $(\boldsymbol{\delta}, \boldsymbol{\theta})=(\delta_1, \delta_2, \cdots, \delta_l, \theta_1, \theta_2, \cdots, \theta_m)$ of $Z_n(0; l, m)$ is *η-equivalent* to an element $(\boldsymbol{\delta}', \boldsymbol{\theta}')=(\delta'_1, \delta'_2, \cdots, \delta'_l, \theta'_1, \theta'_2, \cdots, \theta'_m)$ of $Z_n(0; l, m)$, denoted by $(\boldsymbol{\delta}, \boldsymbol{\theta})\underset{\eta}{\sim}(\boldsymbol{\delta}', \boldsymbol{\theta}')$, if (i) $(\boldsymbol{\delta}, \boldsymbol{\theta})=(\boldsymbol{\delta}', \boldsymbol{\theta}')$ or (ii) (a) if $\delta_1=\delta_2=\cdots=\delta_j=0<\delta_{j+1}$ and $\delta'_1=\delta'_2=\cdots=\delta'_{j'}=0<\delta'_{j'+1}$, then $j=j'$, (b) $n-\delta_l=\delta'_{j+1}$, $n-\delta_{l-1}=\delta'_{j+2}, \cdots, n-\delta_{l-i+1}=\delta'_{j+i}, \cdots, n-\delta_{j+1}=\delta'_l$, and (c) $n-\theta_m=\theta'_1$, $n-\theta_{m-1}=\theta'_2, \cdots, n-\theta_{m-i+1}=\theta'_i, \cdots, n-\theta_1=\theta'_m$.

(II) An element $(0, 1, 0, 0, \cdots, 0, 0, \boldsymbol{\delta}, \boldsymbol{\theta})$ of $Z_n(g; l, m)$ is *η-equivalent* to an element $(0, 1, 0, 0, \cdots, 0, 0, \boldsymbol{\delta}', \boldsymbol{\theta}')$ of $Z_n(g; l, m)$, denoted by $(0, 1, 0, 0, \cdots, 0, 0, \boldsymbol{\delta}, \boldsymbol{\theta})\underset{\eta}{\sim}(0, 1, 0, 0, \cdots, 0, 0, \boldsymbol{\delta}', \boldsymbol{\theta}')$, if $(\boldsymbol{\delta}, \boldsymbol{\theta})\underset{\eta}{\sim}(\boldsymbol{\delta}', \boldsymbol{\theta}')$.

THEOREM 1. *A complete set of equivalence classes on $Z_n(g; l, m)$ is represented by*

$$(1)\qquad \mathscr{X}_n(g; l, m)=\left\{\begin{array}{l}(0, 1, 0, 0, \cdots, 0, 0, \delta_1, \delta_2, \cdots, \delta_l, \theta_1, \theta_2, \cdots, \theta_m)\,; \\ 0\leqq\delta_1\leqq\delta_2\leqq\cdots\leqq\delta_l<n\,, \\ 1\leqq\theta_1\leqq\theta_2\leqq\cdots\leqq\theta_m<n\,, \\ \delta_1+\delta_2+\cdots+\delta_l+\theta_1+\theta_2+\cdots+\theta_m\equiv 0 \pmod n\end{array}\right\}\Big/\eta$$

$$\text{if}\quad g\geqq 1\,.$$

$$(2)\qquad \mathscr{X}_n(0; l, m)=\left\{\begin{array}{l}(\delta_1, \delta_2, \cdots, \delta_l, \theta_1, \theta_2, \cdots, \theta_m)\,; \\ \text{g.c.d.}\,\{\delta_1, \delta_2, \cdots, \delta_l, \theta_1, \theta_2, \cdots, \theta_m\}\equiv 1 \pmod n \\ 0\leqq\delta_1\leqq\delta_2\leqq\cdots\leqq\delta_l<n\,, \\ 1\leqq\theta_1\leqq\theta_2\leqq\cdots\leqq\theta_m<n\,, \\ \delta_1+\delta_2+\cdots+\delta_l+\theta_1+\theta_2+\cdots+\theta_m\equiv 0 \pmod n\end{array}\right\}\Big/\eta\,.$$

PROOF. By Lemmas 4 and 5, any element $\Sigma(\omega)=(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta}, \boldsymbol{\theta})$ of $Z_n(g; l, m)$ is equivalent to an element of a set $\mathscr{X}_n(g; l, m)$. Hence it is sufficient to prove that two distinct elements of the set $\mathscr{X}_n(g; l, m)$ are not equivalent.

Let $\Sigma(\omega)=(0, 1, 0, 0, \cdots, 0, 0, \delta_1, \delta_2, \cdots, \delta_l, \theta_1, \theta_2, \cdots, \theta_m)$ and $\Sigma(\omega')=(0, 1, 0, 0, \cdots, 0, 0, \delta'_1, \delta'_2, \cdots, \delta'_l, \theta'_1, \theta'_2, \cdots, \theta'_m)$ be equivalent elements of $Z_n(g; l, m)$. Then, by Proposition 4, there exists a homeomorphism h of (X, S) onto itself which is a composition of elements in $\{\rho, \rho_{1i}, \tau_1, \mu_1, \theta_{12}, \partial_j, \sigma_k, \partial_a, \sigma_a, \eta\}$ such that $\omega'=\omega h_*$, where h_* is the automorphism of $H_1(X-S)$ induced by $h|_{X-S}$. By Lemmas 1 and 3, we note (i) $\{\delta_1, \delta_2, \cdots, \delta_l\}=\{\delta'_1, \delta'_2, \cdots, \delta'_l\}$ and $\{\theta_1, \theta_2, \cdots, \theta_m\}=\{\theta'_1, \theta'_2, \cdots, \theta'_m\}$ or (ii) $\{\delta_1, \delta_2, \cdots, \delta_l\}\equiv\{-\delta'_1, -\delta'_2, \cdots, -\delta'_l\}$ and $\{\theta_1, \theta_2, \cdots, \theta_m\}\equiv\{-\theta'_1, -\theta'_2, \cdots, -\theta'_m\}$. Hence we have $(\boldsymbol{\delta}, \boldsymbol{\theta})\underset{\eta}{\sim}(\boldsymbol{\delta}', \boldsymbol{\theta}')$, since $0\leqq\delta_1\leqq\delta_2\leqq\cdots\leqq\delta_l<n$, $0\leqq\delta'_1\leqq\delta'_2\leqq\cdots\leqq\delta'_l<n$, $1\leqq\theta_1\leqq\theta_2\leqq\cdots\leqq\theta_m<n$, and $1\leqq\theta'_1\leqq\theta'_2\leqq\cdots\leqq\theta'_m<n$.

Let $\Sigma(\omega)=(\boldsymbol{\delta}, \boldsymbol{\theta})$ and $\Sigma(\omega')=(\boldsymbol{\delta}', \boldsymbol{\theta}')$ be equivalent elements of $Z_n(0; l, m)$. By the same way, we have $(\boldsymbol{\delta}, \boldsymbol{\theta})\underset{\eta}{\sim}(\boldsymbol{\delta}', \boldsymbol{\theta}')$.

§ 3. Proof of Theorem 2.

To determine the number of elements of $\mathscr{P}_n(X, S)$, we will first take the set $D_0(n; l, m)=\{(\delta_1, \delta_2, \cdots, \delta_l, \theta_1, \theta_2, \cdots, \theta_m);\ \delta_i, \theta_j\in N, 0\leqq\delta_1\leqq\delta_2\leqq\cdots\leqq\delta_l<n, 1\leqq\theta_1\leqq\theta_2\leqq\cdots\leqq\theta_m<n, \delta_1+\delta_2+\cdots+\delta_l+\theta_1+\theta_2+\cdots+\theta_m\equiv 0 \pmod n\}$ and compute the number $C_0(n; l, m)$ of elements of $D_0(n; l, m)$.

Let $f_j(x, y)=\sum_{i=0}^{\infty} y^i x^{ij}$ be a formal power series, and $F_f(x, y)=\prod_{j=0}^{n-1} f_j(x, y)$, $F_s(x, z)=\prod_{j=1}^{n-1} f_j(x, z)$ and $F(x, y, z)=F_f(x, y)F_s(x, z)$. Then, we have $F(x, y, z)$ as a generating function. Hence $C_0(n; l, m)$ is equal to the sum $\sum_{i=0}^{l+m-1} K(i)$ of the coefficients $K(i)$ of the terms $x^{in}y^l z^m$ in $F(x, y, z)$. Therefore, $C_0(n; l, m)$ is equal to the coefficient of the term

$y^l z^m$ in $(\sum_{i=1}^{n} F(\zeta_i, y, z))/n$, where $\zeta_1 = \cos 2\pi/n + i \sin 2\pi/n$ and $\zeta_i = \zeta_1^i$. If d is a divisor of n, and if ζ is a primitive d-th root of unity ($\zeta = \zeta_i$ for some i), then $F(\zeta, y, z) = (1-y^d)^{-d'}(1-z)(1-z^d)^{-d'}$, where d' is the natural number n/d. Hence the coefficient of the term $y^l z^m$ in $F(\zeta, y, z)$ is equal to

$$\begin{cases} \dbinom{\frac{l}{d}+d'-1}{d'-1}\dbinom{\frac{m}{d}+d'-1}{d'-1} & \text{if } l \equiv 0 \pmod n \text{ and } m \equiv 0 \pmod n, \\ \dbinom{\frac{l}{d}+d'-1}{d'-1}\dbinom{\frac{m-1}{d}+d'-1}{d'-1} & \text{if } l \equiv 0 \pmod n \text{ and } m \equiv 1 \pmod n, \\ 0 & \text{otherwise}, \end{cases}$$

where $\binom{a}{b} = a!/((a-b)!\,b!)$.

Therefore, $C_0(n; l, m)$ is given as follows:

Let $d_1, d_2, \cdots, d_s$ be all common divisors of l, m and n except 1 and $d_i' = n/d_i$ ($i = 1, 2, \cdots, s$). Also $c_1, c_2, \cdots, c_t$ be every common divisor of l, $m-1$ and n except 1 and $c_j' = n/c_j$ ($j = 1, 2, \cdots, t$). Then, we have

$$C_0(n; l, m) = \frac{1}{n}\left\{\binom{l+n-1}{n-1}\binom{m+n-2}{n-2} + \sum_{i=1}^{s} \varphi(d_i)\binom{\frac{l}{d_i}+d_i'-1}{d_i'-1}\binom{\frac{m}{d_i}+d_i'-1}{d_i'-1} - \sum_{j=1}^{t} \varphi(c_j)\binom{\frac{l}{c_j}+c_j'-1}{c_j'-1}\binom{\frac{m-1}{c_j}+c_j'-1}{c_j'-1}\right\},$$

where $\varphi(d)$ is the Euler function.

Let $Q_0(n; l, m)$ be the number of elements (δ, θ) of $D_0(n; l, m)$ satisfying that $\{(\delta', \theta') \in D_0(n; l, m); (\delta', \theta') \underset{\eta}{\sim} (\delta, \theta)\} = \{(\delta, \theta)\}$. Then, $Q_0(n; l, m)$ is equal to

$$\begin{cases} \dbinom{\left[\frac{n}{2}\right]+\left[\frac{m}{2}\right]-1}{\left[\frac{m}{2}\right]}\dbinom{\left[\frac{n}{2}\right]+\left[\frac{l}{2}\right]}{\left[\frac{l}{2}\right]} & \text{if (1) } m \text{ is even or (2) } n \text{ is even, } m \text{ is odd and } l \geqq 1; \\ 0 & \text{otherwise}; \end{cases}$$

where $[c/2]$ is the largest integer not greater than $c/2$.

THEOREM 2. (1) *The number* $C_0^*(n; l, m)$ *of elements of* $\mathscr{P}_n(X, S)$

is given by

$$C_0^*(n;l,m)=\frac{1}{2}C_0(n;l,m)+\frac{1}{2}Q_0(n;l,m) \quad \textit{if} \quad g\geqq 1 .$$

(2) *Suppose* $g=0$. *Let* $n=p_1^{e_1}p_2^{e_2}\cdots p_s^{e_s}$, *where* p_i *is a prime number and* e_i *is a positive integer* $(i=1,2,\cdots,s)$. *Then, the number of elements of* $\mathscr{P}_n(X,S)$ *is given by*

$$\begin{aligned} C_0^*(n;l,m)-\sum_{i=1}^{s} C_0^*\Big(\frac{n}{p_i};l,m\Big)+\sum_{1\leq i<j\leq s} C_0^*\Big(\frac{n}{p_ip_j};l,m\Big)+\cdots \\ +(-1)^j\sum_{1\leq i_1<i_2<\cdots<i_j\leq s} C_0^*\Big(\frac{n}{p_{i_1}p_{i_2}\cdots p_{i_j}};l,m\Big)+\cdots \\ +(-1)^s C_0^*\Big(\frac{n}{p_1p_2\cdots p_s};l,m\Big)\quad \Big(=\sum_{q|n}\mu(q)C_0^*\Big(\frac{n}{q};l,m\Big)\Big).\end{aligned}$$

PROOF. (1) By Theorem 1, $C_0^*(n;l,m)$ is equal to the number of elements of $D_0(n;l,m)/\eta$. Hence $C_0^*(n;l,m)=\{C_0(n;l,m)-Q_0(n;l,m)\}/2+Q_0(n;l,m)=C_0(n;l,m)/2+Q_0(n;l,m)/2$.

(2) We take an integer $q=p_{i_1}p_{i_2}\cdots p_{i_j}$, where $1\leqq i_1<i_2<\cdots<i_j\leqq s$. Then we consider the subset $D_0(n,q;l,m)$ of $D_0(n;l,m)$ satisfying that q is a divisor of g.c.d. $\{\delta_1,\delta_2,\cdots,\delta_l,\theta_1,\theta_2,\cdots,\theta_m,n\}$. The correspondence $(\delta_1,\delta_2,\cdots,\delta_l,\theta_1,\theta_2,\cdots,\theta_m)\to(\delta_1/q,\delta_2/q,\cdots,\delta_l/q,\theta_1/q,\theta_2/q,\cdots,\theta_m/q)$ defines a bijection from $D_0(n,q;l,m)$ to $D_0(n/q;l,m)$. Hence $C_0^*(n/q;l,m)$ is equal to the number of elements of $D_0(n,q;l,m)$. We have Theorem 2 (2).

§ 4. Classification of orientation-preserving periodic maps on compact orientable surfaces.

For a compact orientable surface M, we will determine the number of equivalence classes of periodic maps with period n on M.

For brevity, we first assume that n is a prime number. Let $P_n^+(g,l,l_1,m)$ be a set defined in Introduction. For $(f,M)\in P_n^+(g,l,l_1,m)$ its orbit space M/f is a compact orientable surface of genus $\{2g-2-(n-1)(l_1+m)+2n\}/2n$ with $(l-l_1)/n+l_1$ boundary components, and a canonical map $p\colon M\to M/f$ is a branched covering of M/f with branched set $p(\mathscr{S})$ consisting of m points. Hence we have $(f,M)\in P_n(X,S)$, where the genus of X is $\{2g-2-(n-1)(l_1+m)+2n\}/2n$, the boundary ∂X consists of $(l-l_1)/n+l_1$ components, and S consists of m points in $\dot{X}$.

It is necessary for $P_n^+(g,l,l_1,m)\neq\emptyset$ that

(a) $l-l_1\equiv 0 \pmod n$,

(b) $g-1-((n-1)/2)(l_1+m)\equiv 0 \pmod n$,

and (c) $g-1-((n-1)/2)(l_1+m)\geqq 0$, if $n\neq 2$;

(a), (b′) $2g-2-(l_1+m)\equiv 0 \pmod 4$,

and (c′) $2g-2-(l_1+m)+4\geqq 0$, if $n=2$.

To determine the number of elements of $\mathscr{P}_n^+(g, l, l_1, m)$, we will take the subset $D(n; l_1, m)$ of $D_0(n; l_1, m)$ satisfying that $1\leqq\delta_1$. In the same way as in § 3, the number $C(n; l_1, m)$ of elements of $D(n; l_1, m)$ is given by;

$$\begin{cases} \dfrac{1}{n}\left\{\dbinom{l_1+n-2}{n-2}\dbinom{m+n-2}{n-2}+n-1\right\} & \text{if } l_1\equiv 0,\ m\equiv 0 \pmod n \text{ or } l_1\equiv 1,\ m\equiv 1 \pmod n\,, \\ \dfrac{1}{n}\left\{\dbinom{l_1+n-2}{n-2}\dbinom{m+n-2}{n-2}-n+1\right\} & \text{if } l_1\equiv 0,\ m\equiv 1 \pmod n \text{ or } l_1\equiv 1,\ m\equiv 0 \pmod n\,, \\ \dfrac{1}{n}\dbinom{l_1+n-2}{n-2}\dbinom{m+n-2}{n-2} & \text{otherwise}\,. \end{cases}$$

Let $Q(n; l_1, m)$ be the number of elements of $D(n; l_1, m)$ satisfying that $(\delta_1, \delta_2, \cdots, \delta_{l_1}, \theta_1, \theta_2, \cdots, \theta_m)=(n-\delta_{l_1}, n-\delta_{l_1-1}, \cdots, n-\delta_2, n-\delta_1, n-\theta_m, n-\theta_{m-1}, \cdots, n-\theta_2, n-\theta_1)$. Clearly, $Q(n; l_1, m)$ is equal to;

$$\begin{cases} \dbinom{\left[\frac{n-1}{2}\right]+\frac{l_1}{2}-1}{\frac{l_1}{2}}\dbinom{\left[\frac{n-1}{2}\right]+\frac{m}{2}-1}{\frac{m}{2}} & \text{if } l_1 \text{ and } m \text{ is even}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

THEOREM 3. *Suppose that n is an odd prime number. Then we have $P_n^+(g, l, l_1, m)\neq\varnothing$ if and only if we have the following conditions* (1), (2) *and* (3);

(1) $l-l_1\equiv 0 \pmod n$,

(2) $l_1+m\neq 1$,

(3) $g+n\times\min\{l_1+m, 1\}-((n-1)/2)(l_1+m)-1$ *is a non-negative integer and is a multiple of n.*

Furthermore, the number of elements of $\mathscr{P}_n^+(g, l, l_1, m)$ is equal to $C(n; l_1, m)/2+Q(n; l_1, m)/2$.

PROOF. (Necessity) (I) The case $l_1+m\geqq 2$. By (1) and (3), we have $(f, M)\in P_n(X, S)$, where X is a compact surface of genus $g(X)=\{2g-2-(n-1)+2n\}/2n$ with $(l-l_1)/n+l_1$ boundary components and S is m points in $\mathring{X}$. If $g(X)>0$, then $\mathscr{P}_n^+(g, l, l_1, m)$ is a one-to-one correspondence to the subset of $\mathscr{X}_n(g(X), (l-l_1)/n+l_1, m)$ satisfying that $\delta_1=\delta_2=\cdots=\delta_{(l-l_1)/n}=0<\delta_{(l-l_1)/n+1}\leqq\cdots\leqq\delta_{(l-l_1)/n+l_1}<n$. If $g(X)=0$, then $\mathscr{P}_n^+(g, l, l_1, m)$ is a one-to-one correspondence to the subset of $\mathscr{X}_n(0, (l-l_1)/n+l_1, m)$ satisfying that (i) g.c.d. $\{\delta_1, \delta_2, \cdots, \delta_{(l-l_1)/n+l_1}, \theta_1, \theta_2, \cdots, \theta_m\}\equiv 1$, and (ii) $\delta_1=\delta_2=\cdots=\delta_{(l-l_1)/n}=0<\delta_{(l-l_1)/n+1}\leqq\cdots\leqq\delta_{(l-l_1)/n+l_1}<n$. But the condition (i) is always satisfied, since $l_1+m\geqq 2$ and n is prime. Hence $\mathscr{P}_n^+(g, l, l_1, m)$ is a one-to-one correspondence to $D(n; l_1, m)/\eta$. Therefore the number of elements of $\mathscr{P}_n^+(g, l, l_1, m)$ is equal to $C(n; l_1, m)/2+Q(n; l_1, m)/2$. Clearly, we have $C(n; l_1, m)/2+Q(n; l_1, m)/2>0$, since $l_1+m\geqq 2$.

(II) The case $l_1+m=0$. By (3), we see $g(X)=\{2g-2-(n-1)+2n\}/2n=(2g+n-1)/2n>0$. In the same way as in case (I), the number of elements of $\mathscr{P}_n^+(g, l, l_1, m)$ is equal to $C(n; 0, 0)/2+Q(n; 0, 0)/2=(n-1+1)/2n+1/2=1$.

(Sufficiency) The condition (1) is clearly the same as (a). Suppose that $l_1+m=1$. By (b) and (c), we have $(f, M)\in P_n(X, S)$, where $g(X)=\{2g-2-(n-1)+2n\}/2n=(2g+n-1)/2n>0$. In the same way, the number of elements of $\mathscr{P}_n^+(g, l, l_1, m)$ is equal to $C(n; l_1, m)/2+Q(n; l_1, m)/2=\{(n-1)-(n-1)\}/n=0$. Hence, we have $l_1+m\neq 1$. So we will prove the condition (3). If $l_1+m\geqq 2$, then we get $g+n\times\min\{l_1+m, 1\}-((n-1)/2)(l_1+m)-1=g+n-((n-1)/2)(l_1+m)-1$, which follows (3) from (b) and (c). If $l_1+m=0$, then $g+n\times\min\{l_1+m, 1\}-((n-1)/2)(l_1+m)-1=g-1$ is a multiple of n, by (b). Hence $g-1\geqq 0$ since $g\geqq 0$. Hence the condition (3) is obtained.

THEOREM 4. *Suppose that* $n=2$. *Then, we have* $P_2^+(g, l, l_1, m)\neq\emptyset$ *if and only if we have the following conditions;*

(1) $l-l_1\equiv 0 \pmod 2$,

(2) l_1+m *is even,*

(3) $g+2\times\min\{l_1+m, 1\}-(l_1+m)/2\geqq 1$; *odd.*

Furthermore, the number of elements of $\mathscr{P}_2^+(g, l, l_1, m)$ *is equal to* 1.

PROOF. By (b′), l_1+m is even. Hence, in a similar way as in the proof of Theorem 3, we have Theorem 4.

In general, let n be a positive integer. We denote by $P_n^+(\tilde{g}, \tilde{l}, \tilde{m}, \tilde{l}, \tilde{m})$ a set of elements of $(f, M)\in P_n$ satisfying the following conditions;

(1) M is a compact orientable surface of genus $\tilde{g}$ with $\tilde{l}$ boundary components $D_1, D_2, \cdots, D_{\tilde{l}}$,

(2) f is an orientation-preserving periodic map on M such that its singular set $\mathscr{S}(f)$ consists of $\tilde{m}$ points $\tilde{S}_1, \tilde{S}_2, \cdots, \tilde{S}_{\tilde{m}}$ in $\mathring{M}$,

(3) $\tilde{\mathfrak{l}}=(\tilde{l}_a)_{a|n}$ is a vector of non-negative integers $\tilde{l}_a$, where $\tilde{l}_a$ is the number of elements of the set $D(a)=\{D_i;\ f^a(D_i)=D_i$ and $f^b(D_i)\neq D_i$ for $1\leqq b<a\}$ for each divisor a of n,

(4) $\tilde{\mathfrak{m}}=(\tilde{m}_a)_{\substack{a|n\\ a\neq n}}$ is a vector of non-negative integers $\tilde{m}_a$, where $\tilde{m}_a$ is the number of elements of the set $S(a)=\{\tilde{S}_j;\ f^a(\tilde{S}_j)=\tilde{S}_j$ and $f^b(\tilde{S}_j)\neq\tilde{S}_j$ for $1\leqq b<a\}$ for each divisor a of n except n. We denote by $\mathscr{P}_n^+(\tilde{g}, \tilde{l}, \tilde{m}, \tilde{\mathfrak{l}}, \tilde{\mathfrak{m}})$ a set of equivalence classes of $P_n^+(\tilde{g}, \tilde{l}, \tilde{m}, \tilde{\mathfrak{l}}, \tilde{\mathfrak{m}})$.

Using the orbit space M/f and the branched cover $p: M\rightarrow M/f$, we have the following:

Proposition 5. *If $P_n^+(\tilde{g}, \tilde{l}, \tilde{m}, \tilde{\mathfrak{l}}, \tilde{\mathfrak{m}})\neq\emptyset$, then we have*

(1) $\tilde{l}=\sum_{a|n}\tilde{l}_a$ *and* $\tilde{m}=\sum_{\substack{a|n\\ a\neq n}}\tilde{m}_a$,

(2) $\tilde{l}_a\equiv 0 \pmod a$ *for each divisor a of n, and* $\tilde{m}_a\equiv 0 \pmod a$ *for each divisor a of n except n,*

(3) $\sum_{\substack{a|n\\ a\neq n}}(1-n/a)(\tilde{l}_a+\tilde{m}_a)$ *is even,*

(4) $\tilde{g}-1+(1/2)\sum_{\substack{a|n\\ a\neq n}}(1-n/a)(\tilde{l}_a+\tilde{m}_a)+n$ *is a non-negative integer and is a multiple of n.*

Under the conditions (1), (2), (3) and (4) in Proposition 5, we will determine the number of elements of $\mathscr{P}_n^+(\tilde{g}, \tilde{l}, \tilde{m}, \tilde{\mathfrak{l}}, \tilde{\mathfrak{m}})$. We take vectors $\mathfrak{l}=(l_a)_{\substack{a|n\\ a\neq n}}$ of non-negative integers and $\mathfrak{m}=(m_a)_{\substack{a|n\\ a\neq n}}$ of non-negative integers, where $l_a=\tilde{l}_a/a$ and $m_a=\tilde{m}_a/a$ for each divisor a of n except n. For n, $\mathfrak{l}$ and $\mathfrak{m}$, we take the set

$$D(n;\mathfrak{l},\mathfrak{m})=\left\{\begin{array}{l}(\delta_1,\delta_2,\cdots,\delta_{l^*},\theta_1,\theta_2,\cdots,\theta_m)\ ;\\ (0)\quad \delta_i,\theta_j\in N\ ,\\ (1)\quad 1\leqq\delta_1\leqq\delta_2\leqq\cdots\leqq\delta_{l^*}<n\ ,\\ (2)\quad 1\leqq\theta_1\leqq\theta_2\leqq\cdots\leqq\theta_m<n\ ,\\ (3)\quad \delta_1+\delta_2+\cdots+\delta_{l^*}+\theta_1+\theta_2+\cdots+\theta_m\equiv 0 \pmod n\ ,\\ (4)\quad l_a \text{ is the number of elements of the set}\\ \qquad\ \{\delta_i;\ \text{g.c.d.}\,\{\delta_i,n\}=a\}\ ,\\ (5)\quad m_a \text{ is the number of elements of the set}\\ \qquad\ \{\theta_j;\ \text{g.c.d.}\,\{\theta_j,n\}=a\}\ ,\end{array}\right\}$$

where $l^*=\sum_{\substack{a|n\\ a\neq n}}l_a$ and $m=\sum_{\substack{a|n\\ a\neq n}}m_a$. Then the number $C(n;\mathfrak{l},\mathfrak{m})$ of elements of $D(n;\mathfrak{l},\mathfrak{m})$ is given as follows:

Let $n=p_1^{e_1}p_2^{e_2}\cdots p_s^{e_s}$ be the prime decomposition of n, where p_i is a prime number and e_i is a positive integer, and put $a=p_1^{f_1}p_2^{f_2}\cdots p_s^{f_s}$, where $0\leqq f_i\leqq e_i$. Without loss of generality, we assume that $0\leqq f_1<e_1$, $0\leqq f_2<e_2$, $\cdots$, $0\leqq f_v<e_v$, $f_{v+1}=e_{v+1}$, $f_{v+2}=e_{v+2}$, $\cdots$, $f_s=e_s$ for some v $(0\leqq$

$v\leqq s$). Let $g_a(x, y_a, z_a)=g_a(x, y, z)=\prod_{j=1}^{n/a-1}(1+yx^{ja}+y^2x^{2ja}+\cdots)(1+zx^{ja}+z^2x^{2ja}+\cdots)$ be a formal power series, and $f_a(x, y_a, z_a)=f_a(x, y, z)=g_a(x, y, z)$ $(\prod_{i=1}^{v} g_{p_i a}^{-1}(x, y, z))\,(\prod_{1\leq i<j\leq v} g_{p_i p_j a}(x, y, z))\cdots(\prod_{1\leq j_1<j_2<\cdots<j_t\leq v} g_{p_{j_1}p_{j_2}\cdots p_{j_t}a}^{(-1)^t}(x, y, z))$ $\cdots(g_{p_1p_2\cdots p_v a}^{(-1)^v}(x, y, z))$. Let $\boldsymbol{y}=(y_a)_{\substack{a|n\\ a\neq n}}$, $\boldsymbol{z}=(z_a)_{\substack{a|n\\ a\neq n}}$ be vectors of variables y_a, z_a, respectively, for each divisor a of n except n, and put $F(x, \boldsymbol{y}, \boldsymbol{z})=\prod_{\substack{a|n\\ a\neq n}} f_a(x, y_a, z_a)$. Then we have $F(x, \boldsymbol{y}, \boldsymbol{z})$ as a generating function. Let d be a divisor of n, and ζ be a primitive dth root of unity. Then, $C(n; \mathfrak{l}, \mathfrak{m})=(1/n)\sum_{d|n}\varphi(d)C_d(n; \mathfrak{l}, \mathfrak{m})$, where $C_d(n; \mathfrak{l}, \mathfrak{m})$ is the coefficient of the term $\prod_{\substack{a|n\\ a\neq n}} y_a^{l_a}z_a^{m_a}$ of $F(\zeta, \boldsymbol{y}, \boldsymbol{z})$.

Let $Q(n; \mathfrak{l}, \mathfrak{m})$ be the number of $(\boldsymbol{\delta}, \boldsymbol{\theta})$ of the subset of $D(n; \mathfrak{l}, \mathfrak{m})$ satisfying that $(\boldsymbol{\delta}, \boldsymbol{\theta})=(\delta_1, \delta_2, \cdots, \delta_{l^*}, \theta_1, \theta_2, \cdots, \theta_m)=(n-\delta_{l^*}, n-\delta_{l^*-1}, \cdots, n-\delta_2, n-\delta_1, n-\theta_m, n-\theta_{m-1}, \cdots, n-\theta_2, n-\theta_1)$. Clearly, $Q(n; \mathfrak{l}, \mathfrak{m})$ is equal to

$$\begin{cases}\displaystyle\prod_{\substack{a|n\\ 0<a<n/2}}\binom{\frac{\varphi(n/a)}{2}+\frac{l_a}{2}-1}{\frac{l_a}{2}}\binom{\frac{\varphi(n/a)}{2}+\frac{m_a}{2}-1}{\frac{m_a}{2}}, & \text{if } l_a \text{ is even and } m_a \text{ is even for each divisor } a \text{ of } n \text{ such that } 0<a<n/2\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

We take an integer $q=p_{i_1}p_{i_2}\cdots p_{i_t}$, where $1\leqq i_1<i_2<\cdots<i_t\leqq s$. Then we consider the subset $D(n, q; \mathfrak{l}, \mathfrak{m})$ of $D(n; \mathfrak{l}, \mathfrak{m})$ satisfying that q is a divisor of g.c.d. $\{\delta_1, \delta_2, \cdots, \delta_{l^*}, \theta_1, \theta_2, \cdots, \theta_m, n\}$. If a divisor a of n is not a multiple of q, then we have $l_a=0$ and $m_a=0$ in $D(n, q; \mathfrak{l}, \mathfrak{m})$. Let a is a multiple of q. We take $l_{a'}^{(q)}=l_a$ and $m_{a'}^{(q)}=m_a$; and let us consider a vector $\mathfrak{l}^{(q)}=(l_{a'}^{(q)})_{\substack{a'|n'\\ a'\neq n'}}$ and $\mathfrak{m}^{(q)}=(m_{a'}^{(q)})_{\substack{a'|n'\\ a'\neq n'}}$ where $a'=a/q$ and $n'=a/q$. Then, the one-to-one correspondence from $D(n, q; \mathfrak{l}, \mathfrak{m})$ to $D(n'; \mathfrak{l}^{(q)}, \mathfrak{m}^{(q)})$ is given by; $(\delta_1, \delta_2, \cdots, \delta_{l^*}, \theta_1, \theta_2, \cdots, \theta_m)\to(\delta_1/q, \delta_2/q, \cdots, \delta_{l^*}/q, \theta_1/q, \theta_2/q, \cdots, \theta_m/q)$. Hence the number of elements of $D(n, q; \mathfrak{l}, \mathfrak{m})$ is equal to $C(n/q; \mathfrak{l}^{(q)}, \mathfrak{m}^{(q)})$. Therefore, we have;

THEOREM 5. (I) *If* $\tilde{\mathfrak{g}}-1+(1/2)\sum_{\substack{a|n\\ a\neq n}}(1-n/a)(l_a+m_a)\geqq 0$ *is valid, then the number* $C^*(n; \mathfrak{l}, \mathfrak{m})$ *of elements of* $\mathscr{P}_n^+(\tilde{\mathfrak{g}}, \tilde{l}, \tilde{m}, \tilde{\mathfrak{l}}, \tilde{\mathfrak{m}})$ *is equal to* $C(n; \mathfrak{l}, \mathfrak{m})/2+Q(n; \mathfrak{l}, \mathfrak{m})/2$, *where* $\mathfrak{l}=(l_a)_{\substack{a|n\\ a\neq n}}$ *is a vector of non-negative integers* $l_a=\tilde{l}_a/a$ *and* $\mathfrak{m}=(m_a)_{\substack{a|n\\ a\neq n}}$ *is a vector of non-negative integers* $m_a=\tilde{m}_a/a$.

(II) *If* $\tilde{\mathfrak{g}}-1+(1/2)\sum_{\substack{a|n\\ a\neq n}}(1-n/a)(l_a+m_a)+n=0$ *is valid, then the number of elements of* $\mathscr{P}_n^+(\tilde{\mathfrak{g}}, \tilde{l}, \tilde{m}, \tilde{\mathfrak{l}}, \tilde{\mathfrak{m}})$ *is given by;*

$$C^*(n; \mathfrak{l}, \mathfrak{m})-\sum_{i=1}^{s}C^*\Big(\frac{n}{p_i}; \mathfrak{l}^{(p_i)}, \mathfrak{m}^{(p_i)}\Big)+\sum_{1\leq i<j\leq s}C^*\Big(\frac{n}{p_ip_j}; \mathfrak{l}^{(p_ip_j)}, \mathfrak{m}^{(p_ip_j)}\Big)+\cdots$$
$$+(-1)^t\sum_{1\leq j_1<j_2<\cdots<j_t\leq s}C^*\Big(\frac{n}{p_{j_1}p_{j_2}\cdots p_{j_t}}; \mathfrak{l}^{(p_{j_1}p_{j_2}\cdots p_{j_t})}, \mathfrak{m}^{(p_{j_1}p_{j_2}\cdots p_{j_t})}\Big)+\cdots$$

$$+(-1)^s C^*\left(\frac{n}{p_1 p_2 \cdots p_s}; \mathfrak{l}^{(p_1 p_2 \cdots p_s)}, \mathfrak{m}^{(p_1 p_2 \cdots p_s)}\right)$$

$$\left(=\sum_{q|n} \mu(q) C^*\left(\frac{n}{q}; \mathfrak{l}^{(q)}, \mathfrak{m}^{(q)}\right)\right).$$

Finally we will obtain the following:

THEOREM 6. *There exists an algorithm for determining whether two elements of P_n^+ are equivalent or not.*

Let $(f, M) \in P_n^+(\tilde{g}, \tilde{l}, \tilde{m}, \tilde{\mathfrak{l}}, \tilde{\mathfrak{m}})$, that is, M is a compact orientable surface of genus $\tilde{g}$ with $\tilde{l}$ boundary components $D_1, D_2, \cdots, D_{\tilde{l}}$, and $\mathcal{S}(f)$ consists of $\tilde{m}$ points $\tilde{S}_1, \tilde{S}_2, \cdots, \tilde{S}_{\tilde{m}}$. Let $U(\tilde{S}_j)$ be a sufficiently small neighborhood in M homeomorphic to a disk, and let S_j be the boundary $\partial U(\tilde{S}_j)$. Then S_j is a simple loop in $M-\mathcal{S}$. We give orientations to $D_1, D_2, \cdots, D_{\tilde{l}}, S_1, S_2, \cdots, S_{\tilde{m}}$ such that $D_1+D_2+\cdots+D_{\tilde{l}}+S_1+S_2+\cdots+S_{\tilde{m}}=0$ in $H_1(M-\mathcal{S})$. Without loss of generarity, we may assume that (a) $f^b(D_i)\neq D_j$ if $1\leqq b<n$ and $1\leqq i<j\leqq l$, (b) for each k $(l+1\leqq k\leqq \tilde{l})$, there exist positive integers i and b $(1\leqq i\leqq l, 1\leqq b<n)$ such that $f^b(D_i)=D_k$, (c) $f^b(\tilde{S}_i)\neq \tilde{S}_j$ if $1\leqq b<n$ and $1\leqq i<j\leqq m$, (d) for each k $(m+1\leqq k\leqq \tilde{m})$, there exist positive integers i and b $(1\leqq i\leqq m, 1\leqq b<n)$ such that $f^b(\tilde{S}_i)=\tilde{S}_k$. Then, we will define an integer δ_i for each D_i $(1\leqq i\leqq l)$ and an integer θ_j for each S_j $(1\leqq j\leqq m)$ as follows:

If $D_i \in D(n)$, then we define $\delta_i=0$. If $D_i \in D(a)$ $(a\neq n)$, then we take a point x_i on D_i. Then, a set $A=\{x_i, f^a(x_i), f^{2a}(x_i), \cdots, f^{(n/a-1)a}(x_i)\}$ consists of distinct n/a points on D_i. Starting at x_i and proceeding along D_i in the direction of the orientation of D_i, let $f^{\delta a}(x_i)$ be the point which first encounters A, after leaving x_i. Then, we set $\delta_i=\delta a$. For S_j, we take a point y_j on S_j and in the same way, we define θ_j. Then we have a system of integers $(\boldsymbol{\delta}, \boldsymbol{\theta})=(\delta_1, \delta_2, \cdots, \delta_l, \theta_1, \theta_2, \cdots, \theta_m) \in Z_n(0; l, m)$. Also, for $(f', M') \in P_n^+(\tilde{g}, \tilde{l}, \tilde{m}, \tilde{\mathfrak{l}}, \tilde{\mathfrak{m}})$ we have a system of integers $(\boldsymbol{\delta}', \boldsymbol{\theta}')=(\delta_1', \delta_2', \cdots, \delta_l', \theta_1', \theta_2', \cdots, \theta_m') \in Z_n(0; l, m)$. Hence, by Theorem 1, we determine whether (f, M) and (f', M') are equivalent or not. Hence we have Theorem 6.

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