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Classification of T^2-bundles over T^2

Koichi SAKAMOTO and Shinji FUKUHARA

Tsuda College

By T^2, we mean a two dimensional torus $\boldsymbol{R}^2/\boldsymbol{Z}^2$. The purpose of this paper is to classify fiber bundles which have T^2 as fibers and base spaces. (Simply we call them *T^2-bundles over T^2*.)

In view of bundle isomorphisms, we obtain Theorem 4 which gives necessary and sufficient condition that two bundles are bundle isomorphic. By this theorem, one might determine even by computer whether two bundles are isomorphic or not.

In view of homeomorphism types of total spaces, we obtain Theorem 5 which says that total spaces are homeomorphic if and only if their fundamental groups are isomorphic.

In Theorem 3, we show that any T^2-bundle over T^2 is isomorphic to one of some standard types of bundles.

§ 1. Notations and definitions.

Given $A, B \in GL(2, \boldsymbol{Z})$ such that $AB=BA$, and $m, n \in \boldsymbol{Z}$, we construct a T^2-bundle over T^2 denoted by $\pi: M(A, B; m, n) \to S$, as follows.

Denote by $\begin{bmatrix} x \\ y \end{bmatrix}$ the point of $T^2=\boldsymbol{R}^2/\boldsymbol{Z}^2$ corresponding to $\begin{pmatrix} x \\ y \end{pmatrix} \in \boldsymbol{R}^2$. Let $F=T^2$, $S=T^2$, and we define

$$M(A, B; 0, 0)=F\times \boldsymbol{R}^2/\sim$$

where

$$\left(\begin{bmatrix} s \\ t \end{bmatrix}, \begin{pmatrix} x+1 \\ y \end{pmatrix}\right) \sim \left(\left[A\begin{pmatrix} s \\ t \end{pmatrix}\right], \begin{pmatrix} x \\ y \end{pmatrix}\right),$$

and

$$\left(\begin{bmatrix} s \\ t \end{bmatrix}, \begin{pmatrix} x \\ y+1 \end{pmatrix}\right) \sim \left(\left[B\begin{pmatrix} s \\ t \end{pmatrix}\right], \begin{pmatrix} x \\ y \end{pmatrix}\right).$$

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Denote the point of $M_0=M(A, B; 0, 0)$ which corresponds to

$$\left(\begin{bmatrix} s \\ t \end{bmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right), \quad \text{by} \quad \left[\begin{pmatrix} s \\ t \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right] \quad \text{or} \quad \begin{bmatrix} s, x \\ t, y \end{bmatrix}.$$

Then $\pi: M_0 \to S$ is a T^2-bundle over T^2, where π is defined by $\pi\begin{bmatrix} s, x \\ t, y \end{bmatrix}=\begin{bmatrix} x \\ y \end{bmatrix}$. Let D be a small disk in S centered at $\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ with radius ε, and let

$$M(A, B; m, n)=(M_0-\pi^{-1}(\operatorname{Int} D)) \cup (F \times D)$$

where $F\times\partial D$ is attached to $\pi^{-1}(\partial D)$ by the homeomorphism $h: \pi^{-1}(\partial D) \to F\times\partial D$:

$$h\left[\begin{pmatrix} s \\ t \end{pmatrix}, \varepsilon(\theta)\right]=\left(\left[\begin{pmatrix} s \\ t \end{pmatrix}+(\theta/2\pi)\begin{pmatrix} m \\ n \end{pmatrix}\right], [\varepsilon(\theta)]\right)$$

where $\varepsilon(\theta)=\begin{pmatrix} 1/2+\varepsilon\cos\theta \\ 1/2+\varepsilon\sin\theta \end{pmatrix}$. Define the map $\pi: M(A, B; m, n) \to S$

$$\pi\begin{bmatrix} s, x \\ t, y \end{bmatrix}=\begin{bmatrix} x \\ y \end{bmatrix} \quad \text{if} \quad \begin{bmatrix} x \\ y \end{bmatrix} \notin D, \quad \text{and}$$

$$\pi\left(\begin{bmatrix} s \\ t \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{bmatrix} x \\ y \end{bmatrix} \quad \text{if} \quad \begin{bmatrix} x \\ y \end{bmatrix} \in D.$$

Then this is a T^2-bundle over T^2.

For the homeomorphism group Homeo (T^2) of T^2, it is known that $\pi_0(\text{Homeo}(T^2)) \cong GL(2, \boldsymbol{Z})$ and $\pi_1(\text{Homeo}(T^2))$ is isomorphic to $\boldsymbol{Z}\oplus\boldsymbol{Z}$ under the isomorphism which assigns h to (m, n) (see [2]). Hence every T^2-bundle over T^2 is isomorphic to the form as above, where the pair (A, B) represents the monodromy, and the pair (m, n) represents the obstruction for constructing a cross-section.

Corresponding to a T^2-bundle over T^2, $\pi: M \to S$, there is an exact sequence

$$1 \longrightarrow \pi_1 F \longrightarrow \pi_1 M \longrightarrow \pi_1 S \longrightarrow 1$$

where F is a fiber. We call this the *associated exact sequence*. Finally, we define the loops in $M=M(A, B; m, n)$ as follows:

$$\alpha(u)=\begin{bmatrix} 0, u \\ 0, 0 \end{bmatrix}, \quad \beta(u)=\begin{bmatrix} 0, 0 \\ 0, u \end{bmatrix},$$

$$\sigma(u)=\begin{bmatrix} u, 0 \\ 0, 0 \end{bmatrix}, \quad \tau(u)=\begin{bmatrix} 0, 0 \\ u, 0 \end{bmatrix} \quad (0 \leqq u \leqq 1).$$

We use the same notations for the corresponding elements in $\pi_1 M$. Then $\pi_1 M$ is generated by α, β, σ, τ and generating relations are given by

$$\sigma\tau=\tau\sigma\ , \qquad \alpha^{-1}\sigma\alpha=\sigma^a\tau^c\ , \qquad \alpha^{-1}\tau\alpha=\sigma^b\tau^d\ ,$$
$$\beta^{-1}\sigma\beta=\sigma^p\tau^r\ , \qquad \beta^{-1}\tau\beta=\sigma^q\tau^s\ ,$$
$$\alpha\beta\alpha^{-1}\beta^{-1}=\sigma^m\tau^n$$

where $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B=\begin{pmatrix} p & q \\ r & s \end{pmatrix}$. We call the set $\{\alpha, \beta, \sigma, \tau\}$ the *canonical generators* of $\pi_1(M(A, B; m, n))$.

§ 2. Fundamental lemmas.

Let $M=M(A, B; m, n)$. Since $H_1 M$ is isomorphic to the abelianization of $\pi_1 M$, we obtain from the above presentation of $\pi_1 M$:

PROPOSITION 1. *$H_1(M(A, B; m, n))$ is isomorphic to $\boldsymbol{Z}^2\oplus(\boldsymbol{Z}^2/K)$, where K is the subgroup of $\boldsymbol{Z}^2$ generated by $\begin{pmatrix} m \\ n \end{pmatrix}$ and the column vectors of $A-E$ and $B-E$ $\left(E \text{ stands for } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$.*

In the following proposition, we study about typical bundle isomorphisms.

PROPOSITION 2. *Let $A, B, A', B' \in GL(2, \boldsymbol{Z})$ such that $AB=BA$ and $A'B'=B'A'$. Let $\alpha, \beta, \sigma, \tau$ and $\alpha', \beta', \sigma', \tau'$ are canonical generators of $\pi_1 M$ and $\pi_1 M'$ respectively, where $M=M(A, B; m, n)$ and $M'=M(A', B'; m', n')$.*

(1) *Assume $A'=A^pB^r$, $B'=A^qB^s$ and $\begin{pmatrix} m' \\ n' \end{pmatrix}=\delta\begin{pmatrix} m \\ n \end{pmatrix}$ for some $P=\begin{pmatrix} p & q \\ r & s \end{pmatrix}\in GL(2, \boldsymbol{Z})$ where $\delta=ps-qr=\pm 1$. Then there is a bundle isomorphism $f\colon M'\to M$ such that*

$$f_\sharp(\sigma')=\sigma\ , \qquad f_\sharp(\tau')=\tau \quad \text{and}$$
$$\bar{f}_\sharp(\bar{\alpha}')=\bar{\alpha}^p\bar{\beta}^r\ , \qquad \bar{f}_\sharp(\bar{\beta}')=\bar{\alpha}^q\bar{\beta}^s$$

where $\bar{f}\colon S'\to S$ is a corresponding homeomorphism between the base spaces and $\bar{\alpha}=\pi_\sharp(\alpha)$ etc..

(2) *Assume $A'=P^{-1}AP$, $B'=P^{-1}BP$ and $\begin{pmatrix} m \\ n \end{pmatrix}=P\begin{pmatrix} m' \\ n' \end{pmatrix}$ for some $P=\begin{pmatrix} p & q \\ r & s \end{pmatrix}\in GL(2, \boldsymbol{Z})$. Then there is a bundle isomorphism $f\colon M'\to M$ such that*

$$f_\sharp(\alpha')=\alpha\ , \qquad f_\sharp(\beta')=\beta \quad \text{and}$$
$$f_\sharp(\sigma')=\sigma^p\tau^r\ , \qquad f_\sharp(\tau')=\sigma^q\tau^s\ .$$

(3) *Assume* $A'=A$, $B'=B$ *and* $\binom{m'}{n'}-\binom{m}{n}=(A-E)\binom{p}{q}+(B-E)\binom{k}{l}$ *for some* $p, q, k, l \in \boldsymbol{Z}$. *Then there is a bundle isomorphism* $f: M' \to M$ *such that*

$$f_\sharp(\alpha')=\sigma^{k'}\tau^{l'}\alpha\ , \qquad f_\sharp(\beta')=\sigma^{p'}\tau^{q'}\beta \quad \text{and}$$
$$f_\sharp(\sigma')=\sigma\ , \qquad f_\sharp(\tau')=\tau\ ,$$

where $\binom{k'}{l'}=B\binom{k}{l}$ *and* $\binom{p'}{q'}=A\binom{p}{q}$.

PROOF. Let $R=\binom{1/2}{1/2}\in I^2$ $(I=[0, 1])$. We identify I^2 and the cone over ∂I with the vertex R. Denote $u*X=(1-u)R+uX$ $(0\leqq u\leqq 1,\ X\in\partial I^2)$. Then $u*X\in I^2$, $0*X=R$ and $1*X=X$.

Proof of (1): Since the group $GL(2, \boldsymbol{Z})$ is generated by $\begin{pmatrix}1&0\\1&1\end{pmatrix}$ and $\begin{pmatrix}0&1\\1&0\end{pmatrix}$, it is sufficient to prove (1) in the cases when $P=\begin{pmatrix}p&q\\r&s\end{pmatrix}=\begin{pmatrix}1&0\\1&1\end{pmatrix}$ and $P=\begin{pmatrix}0&1\\1&0\end{pmatrix}$. Assume $P=\begin{pmatrix}1&0\\1&1\end{pmatrix}$. Then there exists a homeomorphism $\bar{f}: S'\to S$ which is isotopic to the map $\begin{bmatrix}x\\y\end{bmatrix}\to\left[P\binom{x}{y}\right]$ such that $\bar{f}\begin{bmatrix}0\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$ and, in some neighbourhood of D, $\bar{f}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\y\end{bmatrix}$. (For example $\bar{f}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\y\end{bmatrix}$ if $0\leqq x\leqq 2/3$ and $\bar{f}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\y+3x-2\end{bmatrix}$ if $2/3\leqq x\leqq 1$.)

Define the map $f: M'\to M$ as follows:

$$f\begin{bmatrix}s, x\\t, y\end{bmatrix}=\left[\binom{s}{t}\ ,\ \bar{f}\binom{x}{y}\right] \quad \text{on} \quad M'-\pi^{-1}(\operatorname{Int} D) \quad \text{and}$$
$$f\left(\begin{bmatrix}s\\t\end{bmatrix},\begin{bmatrix}x\\y\end{bmatrix}\right)=\left(\begin{bmatrix}s\\t\end{bmatrix},\begin{bmatrix}x\\y\end{bmatrix}\right) \quad \text{on} \quad \pi^{-1}(D)\ .$$

The map f satisfies the desired condition. Next assume $P=\begin{pmatrix}0&1\\1&0\end{pmatrix}$. Define $f: M'\to M$ as follows:

$$f\begin{bmatrix}s, x\\t, y\end{bmatrix}=\begin{bmatrix}s, y\\t, x\end{bmatrix} \quad \text{on} \quad M'-\pi^{-1}(\operatorname{Int} D) \quad \text{and}$$
$$f\left(\begin{bmatrix}s\\t\end{bmatrix},\begin{bmatrix}x\\y\end{bmatrix}\right)=\left(\left[\binom{s}{t}+(1/4)\binom{m}{n}\right],\begin{bmatrix}y\\x\end{bmatrix}\right) \quad \text{on} \quad \pi^{-1}(D)\ .$$

Then f is well defined and a desired map.

Proof of (2): Define $f: M'-\pi^{-1}(\operatorname{Int} D)\to M-\pi^{-1}(\operatorname{Int} D)$ by

$$f\left[\binom{s}{t}\ ,\ u*X\right]=\left[P\binom{s}{t}\ ,\ u*X\right].$$

Let $h, h': \pi^{-1}(\partial D) \to F \times \partial D$ be the attaching maps in the definitions of M and M'. Then the map $h \circ f \circ (h')^{-1}: F \times \partial D \to F \times \partial D$ is as follows:

$$h \circ f \circ (h')^{-1}\left(\begin{bmatrix} s \\ t \end{bmatrix}, [\varepsilon(\theta)]\right) = \left(\begin{bmatrix} s' \\ t' \end{bmatrix}, [\varepsilon(\theta)]\right),$$

where $\binom{s'}{t'} = P\left(\binom{s}{t} - (\theta/2\pi)\binom{m'}{n'}\right) + (\theta/2\pi)\binom{m}{n} = P\binom{s}{t}$. Therefore, it can be extended to a bundle map $f: F \times D^2 \to F \times D^2$ by

$$f\left(\begin{bmatrix} s \\ t \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\right) = \left(\begin{bmatrix} P\begin{pmatrix} s \\ t \end{pmatrix} \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\right).$$

Then f is a desired map.

Proof of (3): Define $f: M' - \pi^{-1}(\mathrm{Int}\, D) \to M - \pi^{-1}(\mathrm{Int}\, D)$ by

$$f\left[\begin{pmatrix} s \\ t \end{pmatrix}, u * X\right] = \begin{cases} \left[\begin{pmatrix} s \\ t \end{pmatrix} + xB\begin{pmatrix} k \\ l \end{pmatrix}, u * X\right] & \text{if } X = \begin{pmatrix} x \\ 0 \end{pmatrix}, \\ \left[\begin{pmatrix} s \\ t \end{pmatrix} + x\begin{pmatrix} k \\ l \end{pmatrix}, u * X\right] & \text{if } X = \begin{pmatrix} x \\ 1 \end{pmatrix}, \\ \left[\begin{pmatrix} s \\ t \end{pmatrix} - yA\begin{pmatrix} p \\ q \end{pmatrix}, u * X\right] & \text{if } X = \begin{pmatrix} 0 \\ y \end{pmatrix}, \\ \left[\begin{pmatrix} s \\ t \end{pmatrix} - y\begin{pmatrix} p \\ q \end{pmatrix}, u * X\right] & \text{if } X = \begin{pmatrix} 1 \\ y \end{pmatrix}. \end{cases}$$

This is well defined, and its restriction to $\pi^{-1}(\partial D)$ is expressed as follows:

$$f\left[\begin{pmatrix} s \\ t \end{pmatrix}, \varepsilon(\theta)\right] = \left[\begin{pmatrix} s \\ t \end{pmatrix} + \phi(\theta), \varepsilon(\theta)\right],$$

where $\phi: \boldsymbol{R} \to \boldsymbol{R}^2$ is a continuous map such that

$$\phi(\theta + 2\pi) - \phi(\theta) = (A - E)\begin{pmatrix} p \\ q \end{pmatrix} + (B - E)\begin{pmatrix} k \\ l \end{pmatrix}.$$

So, the map $h \circ f \circ (h')^{-1}: F \times \partial D \to F \times \partial D$ as in the proof of (2) becomes:

$$\left(\begin{bmatrix} s \\ t \end{bmatrix}, [\varepsilon(\theta)]\right) \longrightarrow \left(\left[\begin{pmatrix} s \\ t \end{pmatrix} + \phi'(\theta)\right], [\varepsilon(\theta)]\right),$$

where $\phi'(\theta) = -(\theta/2\pi)\left(\binom{m'}{n'} - \binom{m}{n}\right) + \phi(\theta)$. Since $\phi'(\theta + 2\pi) = \phi'(\theta)$, f can be extended to a bundle map $M' \to M$. And it is a desired map.

REMARK. The last result (3) of the above proposition corresponds to the fact that the obstruction class to constructing a cross section lies in $H^2(S, \tilde{\pi}_1(F))$ ($\tilde{\pi}_1(F)$ is the locally constant sheaf whose stalk at $x \in S$ is naturally isomorphic to $\pi_1 F_x$, where $F_x = \pi^{-1}(x)$), and that $H^2(S, \tilde{\pi}_1(F))$ is isomorphic to the quotient group $Z^2/\langle A-E, B-E\rangle$ where $\langle A-E, B-E\rangle$ is the subgroup generated by the column vector of $A-E$ and $B-E$.

When we apply Proposition 2, we need some lemmas about the group $GL(2, Z)$.

DEFINITION. Assume A is a matrix in $GL(2, Z)$. A is called *exceptional* when one of the following conditions is satisfied:

(1) $\det A = 1$ and $|\operatorname{trace} A| \leqq 2$,

(2) $\det A = -1$ and $\operatorname{trace} A = 0$.

Otherwise, A is called *general*.

LEMMA 1. *If A is exceptional, then it is conjugate to one and only one of the following matrices:*

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \pm\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad \pm\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \quad (n \geqq 0),$$
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

LEMMA 2. (1) *If A is general, it is conjugate to a following type of matrix:*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad or \quad -\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a \geqq b \geqq d \geqq 0$, $a \geqq c \geqq d$ and $ad - bc = \pm 1$.

(2) *There is a one-to-one correspondence between the finite sequences $(l_1, \cdots, l_m)$ of positive integers and the above matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, as follows:*

$$\begin{pmatrix} l_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} l_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} l_m & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and conversely $a/c = [l_1, l_2, \cdots, l_m]$ such that $(-1)^m = ad - bc$ (see Theorem 177, [3]).

(3) *Two matrices as in (2) are conjugate to each other, if and only if the corresponding sequences coincide up to a cyclic permutation.*

(4) *The class of A^{-1} corresponds to the sequence $(l_m, \cdots, l_2, l_1)$. (For the notations of the continued fractions, see Chapt,* X, [3].)

LEMMA 3. *Let $A \in GL(2, \boldsymbol{Z})$ and $A \neq E$. Then there exists $C \in GL(2, \boldsymbol{Z})$ as follows*:

(i) *$A = \pm C^p$ for some integer p, and*

(ii) *if $BA = AB$ where $B \in GL(2, \boldsymbol{Z})$ then $B = \pm C^q$ for some integer q.*

Proofs and more precise statements are given in § 5.

§ 3. Main results.

The problem of bundle isomorphisms is reduces to the group theory of the associated exact sequences by the following theorem.

THEOREM 1. *Let $\pi: M \to S$ and $\pi': M' \to S'$ be T^2-bundles over T^2. Then the following statements are equivalent.*

(1) *They are bundle isomorphic to each other.*

(2) *Their associated exact sequences are isomorphic to each other, that is, there exist isomorphisms of groups $\psi: \pi_1 M' \to \pi_1 M$ and $\bar{\psi}: \pi_1 S' \to \pi_1 S$ such that $\pi_\# \circ \psi = \bar{\psi} \circ (\pi')_\#$.*

PROOF. We have only to prove that (2) implies (1). Let $M = M(A, B; m, n)$ and $M' = M(A', B'; m', n')$, and let α, β, σ, τ and α', β', σ', τ' are canonical generators of $\pi_1 M$ and $\pi_1 M'$. Denote $\bar{\alpha} = \pi_\#(\alpha)$ etc.. Since $\bar{\psi}$ is an isomorphism and $\pi_1(S) \cong \boldsymbol{Z}^2$, there exists $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in GL(2, \boldsymbol{Z})$ such that

$$\bar{\psi}_\#(\bar{\alpha}') = \bar{\alpha}^p \bar{\beta}^r , \qquad \bar{\psi}_\#(\bar{\beta}') = \bar{\alpha}^q \bar{\beta}^s .$$

From Proposition 2-(1), it follows that there exists a bundle isomorphism $f: M' \to M''$ such that $\bar{f}_\#(\bar{\alpha}') = \bar{\alpha}''^p \bar{\beta}''^r$ and $\bar{f}_\#(\bar{\beta}') = \bar{\alpha}''^q \bar{\beta}''^s$, where $M'' = M(A'', B''; m'', n'')$, $A' = (A'')^p (B'')^r$, $B' = (A'')^q (B'')^r$ and $\begin{pmatrix} m'' \\ n'' \end{pmatrix} = \delta \begin{pmatrix} m' \\ n' \end{pmatrix}$ $(\delta = ps - qr = \pm 1)$. Then $\psi' = \psi \circ f_\#^{-1}: \pi_1 M'' \to \pi_1 M$ induces an isomorphism of associated exact sequences, and $\bar{\psi}'(\bar{\alpha}'') = \bar{\alpha}$, $\bar{\psi}'(\bar{\beta}'') = \bar{\beta}$.

ψ' induces an isomorphism $\pi_1 F'' \to \pi_1 F \cong \boldsymbol{Z}^2$. So, by the similar way above, using Proposition 2-(2), we have that there is an isomorphism of associated exact sequences $\psi'': \pi_1 M''' \to \pi_1 M$ such that M''' is bundle isomorphic to M', and $\bar{\psi}''(\bar{\alpha}''') = \bar{\alpha}$, $\bar{\psi}''(\bar{\beta}''') = \bar{\beta}$, $\psi''(\sigma''') = \sigma$ and $\psi''(\tau''') = \tau$, where $M''' = M(A''', B'''; m''', n''')$, $A'' = PA'''P^{-1}$, $B'' = PA'''P^{-1}$ and $\begin{pmatrix} m'' \\ n'' \end{pmatrix} = P \begin{pmatrix} m''' \\ n''' \end{pmatrix}$ for some $P \in GL(2, \boldsymbol{Z})$. As $\bar{\psi}''(\bar{\alpha}''') = \bar{\alpha}$ and $\bar{\psi}''(\bar{\beta}''') = \bar{\beta}$, $\psi''(\alpha''') = \sigma^{k'} \tau^{l'} \alpha$, $\psi''(\beta''') = \sigma^{p'} \tau^{q'} \beta$ for some $p', q', k', l' \in \boldsymbol{Z}$.

Again, by the same way as above using Proposition 2-(3), we obtain isomorphism of associated sequences $\psi_0: \pi_1 M_0 \to \pi_1 M$ such that M_0 is bundle isomorphic to M', and $\psi_0(\alpha_0) = \alpha$, $\psi_0(\beta_0) = \beta$, $\psi_0(\sigma_0) = \sigma$ and $\psi_0(\tau_0) = \tau$, where

$M_0=M(A_0, B_0; m_0, n_0)$, $A_0=A'''$, $B_0=B'''$ and $\binom{m'''}{n'''}-\binom{m_0}{n_0}=(A-E)\binom{p}{q}+(B-E)\binom{k}{l}$ for some $p, q, k, l\in \boldsymbol{Z}$.

Comparing the presentations of $\pi_1 M$ and $\pi_1 M_0$ w.r.t. the canonical generators, we see that $A_0=A$, $B_0=B$, $m_0=m$ and $n_0=n$, hence $M=M_0$.

Q.E.D.

From the above proof, substituting P by δP^{-1} ($\delta=\det P$), we have:

COROLLARY. *Two fibrations $M(A, B; m, n)$ and $M(A', B'; m', n')$ are isomorphic if and only if there exist $\begin{pmatrix} p & q \\ r & s\end{pmatrix}$ and $P\in GL(2, \boldsymbol{Z})$ as follows:*

$$A^p B^r = PA'P^{-1}\,,\qquad A^q B^s = PB'P^{-1} \quad \textit{and}$$
$$P\begin{pmatrix} m' \\ n'\end{pmatrix}-\begin{pmatrix} m \\ n\end{pmatrix}\in \langle A-E,\ B-E\rangle\,,$$

where $\langle A-E, B-E\rangle$ is the subgroup of $\boldsymbol{Z}^2$ generated by the column vectors of $A-E$ and $B-E$.

THEOREM 2. *Let $\pi\colon M\to S$ and $\pi'\colon M'\to S'$ be T^2-bundles over T^2.*

(1) $\operatorname{rank}(H_1M)=4$ *if and only if $M=M(E, E; 0, 0)$, which is a 4-dimensional torus.*

(2) *Assume* $\operatorname{rank}(H_1M)\neq 3$. *Then the above fibrations are isomorphic if and only if $\pi_1 M$ and $\pi_1 M'$ are isomorphic.*

PROOF. (1) follows from Proposition 1. Hence we have only to prove (2) when $\operatorname{rank}(H_1M)=2$. Since $\pi_1 S\cong \boldsymbol{Z}^2$ is abelian, the homomorphism $\pi_\sharp\colon \pi_1 M\to\pi_1 S$ is factored by the Hurewitz homomorphism $\pi_1 M\to H_1 M$. Therefore if $\operatorname{rank}(H_1M)=2$, the homomorphism $\pi_\sharp$ is identified with $\pi_1 M\to H_1 M\to H_1 M/\text{Torsion}$ in the group theoretical meaning, since both homomorphisms are surjective. And so, $\pi_1 M\cong \pi_1 M'$ if and only if the associated exact sequences are isomorphic. In view of Theorem 1, this completes the proof.

Any fibration has a simple expression as follows:

THEOREM 3. *Any T^2-bundle over T^2 is isomorphic to one of the following types:*

$$M(A, B; m, n) \quad \textit{where} \quad B=\pm E\,.$$

Furthermore, we may assume that A satisfies the following conditions:

(1) *if* $\det A=-1$, *then* $\operatorname{trace} A\geq 0$,

(2) *if* $\det A=1$ *and* $B=-E$, *then* $\operatorname{trace} A\geq 2$ *and* $A\neq E$,

(3) *A is one of the matrices as in Lemma* 1 *or* 2.

PROOF. Let $M=M(A, B; m, n)$. By Lemma 3, we see that $A=\pm C^s$ and $B=\pm C^{-q}$ for some integers s and q. We may assume that s and q are relatively prime, i.e., $ps-qr=1$ for some integers p and r. So, applying Proposition 2-(1), M is bundle isomorphic to $M(\pm C, \pm E; m, n)$. This proves the first part.

(1) if $\det A=-1$, $M(A, \pm E; m, n)$ is isomorphic to $M(A^{-1}, \pm E; -m, -n)$ by Proposition 2-(1). Since $\operatorname{trace}(A^{-1})=-\operatorname{trace} A$, we may assume $\operatorname{trace} A\geqq 0$.

(2) Assume $\det A=1$. Then, by Proposition 2-(1), $M(A, -E; m, n)\cong M(A(-E), -E; m, n)=M(-A, -E; m, n)$. So, we may assume that $\operatorname{trace} A\geqq 0$. And if $\operatorname{trace} A=0$, $M(A, -E; m, n)\cong M(A, A^2(-E); m, n)=M(A, E; m, n)$, and if $\operatorname{trace} A=1$, $M(A, -E; m, n)\cong M(-A, -E; m, n)\cong M(-A, (-A)^3(-E); m, n)=M(-A, E; m, n)$.

(3) follows from Proposition 2-(2). Q.E.D.

REMARK. Under the above assumption (2), $B=E$ if and only if the subgroup of $GL(2, \boldsymbol{Z})$ generated by A and B is a cyclic group. The conjugacy class of this group in $GL(2, \boldsymbol{Z})$ is an invariant of the associated exact sequence. In fact, if $\rho: \pi_1 S\to \operatorname{Aut}(\pi_1 F)$ is the homomorphism defined by

$$\rho(\pi_\sharp(x))(y)=x^{-1}yx \qquad (x\in\pi_1 M,\ y\in\pi_1 F\subset\pi_1 M)\ ,$$

then $\operatorname{Im}\rho$ is mapped onto the above group by a global isomorphism from $\operatorname{Aut}(\pi_1 F)$ to $GL(2, \boldsymbol{Z})$.

THEOREM 4. *Assume* $M=M(A, B; m, n)$ *and* $M'=M(A', B'; m', n')$ *satisfy the condition of Theorem* 3. *Denote by* $\langle A-E\rangle$ *the subgroup of* $\boldsymbol{Z}^2$ *generated by the column vectors of* $A-E$, *and similarly for* $\langle A-E, 2E\rangle$.

(0) *If* M *and* M' *are bundle isomorphic to each other, then* $B=B'$.

(1) *Assume,* $B=B'=E$. *Then* M *is bundle isomorphic to* M', *if and only if there exists a matrix* $P\in GL(2, \boldsymbol{Z})$ *such that*

(i) $PA'P^{-1}=A$ *or* $PA'P^{-1}=A^{-1}$ *and*

(ii) $\begin{pmatrix} m\\ n\end{pmatrix}-P\begin{pmatrix} m'\\ n'\end{pmatrix}\in\langle A-E\rangle$.

(2) *Assume,* $B=B'=-E$. *Then* M *is bundle isomorphic to* M', *if and only if there exists a matrix* $P\in GL(2, \boldsymbol{Z})$ *such that*

(i) $PA'P^{-1}=\pm A$ *or* $PA'P^{-1}=\pm A^{-1}$ *and*

(ii) $\begin{pmatrix} m\\ n\end{pmatrix}-P\begin{pmatrix} m'\\ n'\end{pmatrix}\in\langle A-E, 2E\rangle$.

PROOF. (0) follows from the above remark.

(1) By the corollary to Theorem 1, M and M' are isomorphic if and only if there exists $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$, $P \in GL(2, \mathbf{Z})$ such that

$$PA'P^{-1} = A^p , \qquad E = A^q \quad \text{and} \quad P\begin{pmatrix} m' \\ n' \end{pmatrix} - \begin{pmatrix} m \\ n \end{pmatrix} \in \langle A - E \rangle .$$

From Lemma 1, we see that the order of matrix in $GL(2, \mathbf{Z})$ is infinite or 1, 2, 3, 4 or 6. Hence q is a multiple of 0, 1, 2, 3, 4 or 6, respectively. And so, since p is relatively prime to q, $p = \pm 1$ or $p \equiv \pm 1$ modulo the order of A. Therefore $A^p = A^{\pm 1}$, this proves the only if part of (1).

The if part follows from the corollary to Theorem 1.

(2) is proved by the same way as (1).

§ 4. Homeomorphism types.

Let $\pi: M \to S$ be a T^2-bundle over T^2. If rank $(H_1 M) \neq 3$, the bundle isomorphism type is determined by $\pi_1 M$ (Theorem 2). Now we consider the case when rank $(H_1 M) = 3$. According to Proposition 1, rank $(H_1(M(A, B; m, n))) = 3$ if and only if the rank of the 2×5 matrix $\left(A - E, B - E, \begin{pmatrix} m \\ n \end{pmatrix}\right)$ is equal to 1. Hence in view of Theorem 3, M is isomorphic to one of the following forms:

(1) $M\left(\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, E; m, 0\right)$ $(k \geqq 0)$

(2) $M\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E; 0, n\right)$ or

(3) $M\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, E; m, -m\right)$.

Furthermore, we have:

PROPOSITION 3. *If* rank $(H_1 M) = 3$, *M is homeomorphic to one and only one of the following forms*:

(1) $M\left(\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}, E; 0, 0\right)$ $(d > 0)$

(2) $M\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E; 0, n\right)$ ($n = 0$ *or* 1).

Before proving this, it is convenient to introduce another isomorphic description for $M(A, E; m, n)$. Let $M = F \times \mathbf{R}^2 / \sim$ where

$$\left(\begin{bmatrix} s \\ t \end{bmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) \sim \left(\begin{bmatrix} s \\ t \end{bmatrix}, \begin{pmatrix} x \\ y+1 \end{pmatrix}\right) \sim \left(\left[A^{-1}\begin{pmatrix} s \\ t \end{pmatrix} + y\begin{pmatrix} m \\ n \end{pmatrix}\right], \begin{pmatrix} x+1 \\ y \end{pmatrix}\right),$$

and let $\begin{Bmatrix} s, & x \\ t, & y \end{Bmatrix}$ or $\left\{\begin{pmatrix} s \\ t \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right\}$ denote the point of M corresponding to

$\left(\begin{bmatrix} s \\ t \end{bmatrix}, \begin{pmatrix} x \\ y+1 \end{pmatrix}\right)$. Define $\pi: M \to S$ by $\pi\begin{Bmatrix} s, x \\ t, y \end{Bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$. Clearly, $\pi: M \to S$ is a T^2-bundle over T^2. Let α, β, σ, τ be a loop in M defined as follows:

$$\alpha(u) = \begin{Bmatrix} 0, u \\ 0, 0 \end{Bmatrix}, \quad \beta(u) = \begin{Bmatrix} 0, 0 \\ 0, u \end{Bmatrix},$$
$$\sigma(u) = \begin{Bmatrix} u, 0 \\ 0, 0 \end{Bmatrix} \quad \text{and} \quad \tau(u) = \begin{Bmatrix} 0, 0 \\ u, 0 \end{Bmatrix} \quad (0 \leqq u \leqq 1).$$

We use the same notations α, β, σ, τ for the corresponding elements of $\pi_1 M$. Note that σ and τ generate $\pi_1 F$, and $\pi_\sharp(\alpha)$ and $\pi_\sharp(\beta)$ generate $\pi_1 S$, and so α, β, σ, τ generate $\pi_1 M$. We can see easily that generating relations for them are the same as that of $\pi_1(M(A, E; m, n))$ for the canonical generators. These facts imply that the associated exact sequences of M and $M(A, E; m, n)$ are isomorphic to each other. From now on, we identify them.

PROOF OF PROPOSITION 3. If M is orientable, then we may assume: $M = M\left(\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, E; m, 0\right)$. Then it is homeomorphic to the following spaces:

$$M_1 = M\left(\begin{pmatrix} 1 & -m \\ 0 & 1 \end{pmatrix}, E; -k, 0\right),$$
$$M_2 = M\left(\begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}, E; -m, 0\right),$$
$$M_3 = M\left(\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, E; m+lk, 0\right) \quad \text{for any integer } l.$$

In fact, $f\begin{Bmatrix} s, x \\ t, y \end{Bmatrix} = \begin{Bmatrix} s, x \\ y, t \end{Bmatrix}$ induces a well defined homeomorphism from M to M_1. And, applying Theorem 4 in the case $P = -E$ and $P = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$, we see that M is bundle isomorphic to M_2 and M_3. Therefore, by the Euclid's algorithm, M is homeomorphic to $M\left(\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}, E; 0, 0\right)$ where $d = \text{g.c.d.}\ (k, m)$. Since Torsion $(H_1 M) \cong \boldsymbol{Z}_d$, this proves the assertion in the orientable case.

Next consider the non-orientable case. According to Theorem 4 in the case $P = E$, $M\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E; 0, n\right)$ is bundle isomorphic to $M_k = M\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E; 0, k\right)$ where $k = 0$ or 1, and $M\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, E; m, -m\right)$ is isomorphic to $M_2 = M\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, E; 0, 0\right)$. Note that Torsion $(H_1 M_0) \cong \boldsymbol{Z}_2$ and Torsion $(H_1 M_1) \cong$ Torsion $(H_1 M_2) \cong 0$. So we have only to prove that M_1 is homeomorphic to M_2. In fact, a homeomorphism $f\colon M_1 \to M_2$ is given by $f\begin{Bmatrix} s, x \\ t, y \end{Bmatrix} = \begin{Bmatrix} -t+y, x \\ t \quad , s \end{Bmatrix}$. Q.E.D.

As a corollary to the above proof, we have:

COROLLARY. *Let $\pi\colon M \to S$ and $\pi'\colon M' \to S'$ be T^2-bundles over T^2. Assume that M and M' are both orientable or both non-orientable, and* rank $H_1M=$ rank $H_1M'=3$. *Then M is homeomorphic to M' if and only if $H_1M \cong H_1M'$.*

REMARK. The orientability of M is an invariant of π_1M. In fact, let $\rho\colon H_1M \to \mathrm{Aut}([\pi_1, \pi_1])$, where $[\pi_1, \pi_1]$ is the commutator subgroup of π_1M and ρ is the homomorphism which is defined similarly to the remark to Theorem 3. When rank $H_1M=3$, by the above proposition, we see that ρ is a trivial map if and only if M is orientable.

This remark and Theorem 2 imply:

THEOREM 5. *Let $\pi\colon M \to S$ and $\pi'\colon M' \to S'$ be T^2-bundles over T^2. Then M is homeomorphic to M' if and only if π_1M is isomorphic to π_1M'.*

§ 5. Proof of Lemmas 1, 2, 3.

Let $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbf{Z})$. If none of the eigenvalues of A is ± 1, then $|\mathrm{trace}\, A|-1 \neq \det A$ and the eigenvalues must be irrational. For any complex number ξ, we define $A(\xi)=(a\xi+b)/(c\xi+d)$, and define the number $\omega(A)$ such that $A(\omega(A))=\omega(A)$ as follows:

$$\omega(A)=(a-d+\sqrt{D})/2c\ , \quad \text{where} \quad D=(\mathrm{trace}\, A)^2-4(\det A)\ .$$

(When $D<0$, we choose the value of $\sqrt{D}$ such that $\mathrm{Im}(\sqrt{D})>0$.) Note that, by the above assumption, we have that $c \neq 0$ and $\omega(A)$ is not a rational number since

$$(\mathrm{trace}\, A)^2-4 \leq D \leq (\mathrm{trace}\, A)^2+4\ .$$

LEMMA 4. *Let $A, B, P \in GL(2, \mathbf{Z})$, and assume that* $\det A=\det B$, trace $A=$ trace B *and that any of the eigenvalues of A and B are not ± 1. Then the following conditions are equivalent:*

(i) $PAP^{-1}=B$

(ii) $P(\omega(A))=\omega(B)$.

PROOF. If $P=\begin{pmatrix}1&1\\0&1\end{pmatrix}$ or $\begin{pmatrix}0&1\\1&0\end{pmatrix}$, then it is easy to prove that $\omega(PAP^{-1})=P(\omega(A))$ by a direct calculation. It follows that (i) implies (ii), because $(PQ)(\xi)=P(Q(\xi))$, and the above matrices generate $GL(2, \boldsymbol{Z})$. Conversely assume that $P(\omega(A))=\omega(B)$. Then, by the above argument, $\omega(B)=\omega(PAP^{-1})$, hence we have only to prove that $A=B$ if $\omega(A)=\omega(B)$.

Let $A=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ and $B=\begin{pmatrix}a'&b'\\c'&d'\end{pmatrix}$, and assume $\omega(A)=\omega(B)$, i.e., $(a-d+\sqrt{D})/2c=(a'-d'+\sqrt{D'})/2c'$. By the assumption of Lemma, $D=D'$ and $\sqrt{D}$ is irrational, hence $c=c'$ $(\neq 0)$, $a-d=a'-d'$. On the other hand, $a+d=a'+d'$ and $ad-bc=a'd'-b'c'=\pm 1$. Therefore $A=B$.

According to this lemma, A is conjugate to B if and only if $\omega(A)$ is *equivalent* to $\omega(B)$, that is, $\omega(A)=P(\omega(B))$ for some $P\in GL(2, \boldsymbol{Z})$.

PROOF OF LEMMA 1. (1) If the eigenvalue of A are ± 1, A is conjugate to $\pm\begin{pmatrix}1&n\\0&1\end{pmatrix}$ or $\begin{pmatrix}1&n\\0&-1\end{pmatrix}$ for some integer n. Let $P=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ and $Q=\begin{pmatrix}1&1\\0&1\end{pmatrix}$, then $P\begin{pmatrix}1&n\\0&1\end{pmatrix}P^{-1}=\begin{pmatrix}1&-n\\0&1\end{pmatrix}$ and $Q\begin{pmatrix}1&n\\0&-1\end{pmatrix}Q^{-1}=\begin{pmatrix}1&n-2\\0&-1\end{pmatrix}$. Hence, A is conjugate to $\pm\begin{pmatrix}1&n\\0&1\end{pmatrix}$ $(n\geqq 0)$ or $\begin{pmatrix}1&n\\0&-1\end{pmatrix}$ $(n=0$ or $1)$. If $A=\pm\begin{pmatrix}1&n\\0&1\end{pmatrix}$ $(n\geqq 0)$ then n is the greatest common devisor of the elements of $A\mp E$, and so, n is determined by the conjugate class of A. Similarly, $\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ is not conjugate to $\begin{pmatrix}1&1\\0&-1\end{pmatrix}$, and $\begin{pmatrix}1&1\\0&-1\end{pmatrix}$ is conjugate to $\begin{pmatrix}0&1\\1&0\end{pmatrix}$ as $\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&1\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}^{-1}$.

(2) If $\det A=1$ and $\operatorname{trace} A=0$ or ± 1, then $\omega(A)$ is not a real number. As is well known, for any $\xi\in C$ which is not real, there exists a matrix $P\in GL(2, \boldsymbol{Z})$ as follows (see, for example, p. 107 of [1]). Let $\eta=P(\xi)$, then $\operatorname{Im}\eta>0$, $|\eta|\geqq 1$ and $-1/2\leqq \operatorname{Re}\eta<1/2$, where if $|\eta|=1$ then $-1/2\leqq\operatorname{Re}\eta\leqq 0$. Hence by Lemma 4, A is conjugate to a matrix B such that $\omega(B)$ satisfies the above conditions for η. Let $B=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ then $\omega(B)=(a-d+\sqrt{D})/2c$ where $D=(\operatorname{trace} B)^2-4(\det B)=-4$ or -3. By the above condition, $c>0$, $(a-d)^2+|D|\geqq 4c^2$ and $-c\leqq a-d<c$. Hence $4c^2\leqq c^2+|D|$, and so, $3c^2\leqq|D|\leqq 4$. Therefore, $c=1$ and $a-d=0$ or -1. If $a+d=0$ or 1, it follows that $a=0$ and $B=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ or $\begin{pmatrix}0&-1\\1&1\end{pmatrix}$. And therefore, if $\det A=1$ and $\operatorname{trace} A=-1$ then $-A$ is conjugate to $\begin{pmatrix}0&-1\\1&1\end{pmatrix}$, as desired.

The above two lemmas are also proved in [4]. Lemma 3 for an exceptional A can be shown by a direct calculation using Lemma 1. So, we assume that A is a general matrix in the rest of this section. We assume Lemma 2-(2) which can be proved by an easy induction on n.

Since $\omega(A)$ is a real quadratic irrational number, it is expressed as a continued cyclic fraction: $\omega(A)=[l_1, \cdots, l_q, \dot{k}_1, \cdots, \dot{k}_r]$, where l_j and k_j are positive integers except l_1 (see Theorem 177, [3]). Then $\omega(P^{-1}AP)=P^{-1}(\omega(A))$ by Lemma 4, and $P^{-1}(\omega(A))=[\dot{k}_1, \cdots, \dot{k}_r]$ where $P=\begin{pmatrix} l_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} l_q & 1 \\ 1 & 0 \end{pmatrix}$. Now, Lemma 3 follows from:

LEMMA 5. *Let* $\omega=\omega(A)=[\dot{k}_1, \cdots, \dot{k}_r]$ (r *is the minimal period.*) *and* $C=\begin{pmatrix} k_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} k_r & 1 \\ 1 & 0 \end{pmatrix}$.

(i) *If* $BA=AB$ ($B\in GL(2, \boldsymbol{Z})$), *then* $B=\pm C^q$ *for some integer* q.

(ii) *If* trace $A>0$, $A=C^p$ *for some positive integer* p.

(iii) *If* trace $A<0$, $A=-C^p$ *for some positive integer* p.

PROOF. (i) Since $BAB^{-1}=A$, $B(\omega)=\omega$ by Lemma 4. Note that $C(\omega)=\omega$, and let $C^n=\begin{pmatrix} p_n & q_n \\ r_n & s_n \end{pmatrix}$. Applying Lemma 2-(2) for C^n, it follows that when $n\to\infty$, $p_n/r_n\to\omega$, and also $q_n/s_n\to\omega$ since $p_n/r_n-q_n/s_n=\pm 1/q_ns_n\to 0$, as q_n and s_n are monotone increasing. By the same way, since ${}^tC=\begin{pmatrix} k_n & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} k_1 & 1 \\ 1 & 0 \end{pmatrix}$ and $({}^tC)^n=\begin{pmatrix} p_n & r_n \\ q_n & s_n \end{pmatrix}$, we have $r_n/s_n\to\omega({}^tC)=[\dot{k}_r, \cdots, \dot{k}_1]>1$.

Let $B=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $D=\begin{pmatrix} x & y \\ z & w \end{pmatrix}=BC^n$. Then $D(\omega)=\omega$, and for large n, $z/w=(r_n/s_n)\times(c(p_n/r_n)+d)/(c(q_n/s_n)+d)>1$. Substituting D by $-D$ if necessary, we may assume that $z>w>1$. Thus, by Theorem 172 of [3], $D=C^k$ for some positive integer k. Thus $B=\pm C^q$ for some q.

(ii) It follows from (i) that $A=\pm C^q$ for some integer q. Let $n=|q|$. Then trace $(-C^n)=$ trace $(-\delta C^{-n})=-$trace $C^n<0$, where $\delta=\det C^n$, so that A is neither conjugate to $-C^n$ nor to $-\delta C^{-n}$. On the other hand, $\delta C^{-n}=\begin{pmatrix} s & -q \\ -r & p \end{pmatrix}$ where $C^n=\begin{pmatrix} p & q \\ r & s \end{pmatrix}$, hence $\omega(\delta C^{-n})=(s-p+\sqrt{(s-p)^2+4qr})/(-2r)<0$. But $\omega(A)\geqq k_1>0$ so that $A\neq\delta C^{-n}$, hence $A=C^n$.

(iii) follows from (ii).

Finally, we prove Lemma 2. (1) follows from (2) and Lemma 5. (3) follows from Lemmas 4, 5 and the fact that $\omega(C^n)=\omega(C)=[\dot{k}_1, \cdots, \dot{k}_p]$ for any positive integer n. If $A=\pm C^n$ as above, $\delta A^{-1}=P\,{}^tAP^{-1}=\pm P({}^tC)^nP^{-1}$ where $\delta=\det A$ and $P=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Since ${}^tC=\begin{pmatrix} k_p & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} k_1 & 1 \\ 1 & 0 \end{pmatrix}$, (4) follows.

§ 6. Examples.

Given $M(A, B; m, n)$, by the arguments of § 5, we can easily obtain an isomorphic form as in Theorem 3. And so, when we compare two fibrations, we are led to the situation of Theorem 4. Hereafter, the notations in Theorem 4 are assumed. Again by the arguments of § 5, we can decide whether A' is conjugate to A (or A^{-1}, or $\pm A^{-1}$) or not, and, if they are conjugate, we can obtain a matrix P satisfying the conditions (1)-(i) or (2)-(i) of Theorem 4. So we consider the case when $A=A'$ and $B=B'$ $(=\pm E)$.

Then the set of matrices P satisfying the conditions (1)-(i) or (2)-(i) forms a subgroup G of $GL(2, \boldsymbol{Z})$. Applying Theorem 4 in the case when $P=E$, we see that:

REMARK 1. If $\begin{pmatrix} m \\ n \end{pmatrix} \equiv \begin{pmatrix} m' \\ n' \end{pmatrix} \bmod \langle A-E, B-E \rangle$, then M is isomorphic to M'.

As an immediate corollary of this remark:

REMARK 2. $M(A, -E; m, n)$ is isomorphic to $M(A, -E; i, j)$, where

$$(i, j)=(0, 0),\ (0, 1),\ (1, 0) \quad \text{or} \quad (1, 1)\ .$$

On the other hand, if $\begin{pmatrix} m \\ n \end{pmatrix} \in \langle A-E, B-E \rangle$ and $P \in G$, then $P\begin{pmatrix} m \\ n \end{pmatrix} \in \langle A-E, B-E \rangle$. Hence we can regard $\begin{pmatrix} m \\ n \end{pmatrix}$ as an element of $\boldsymbol{Z}^2/\langle A-E, B-E \rangle$, and furthermore:

REMARK 3. The group G operates on $\boldsymbol{Z}^2/\langle A-E, B-E \rangle$ as a group of automorphisms.

For the group G, it is easy to see:

REMARK 4. If $A=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, G is generated by $-E$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

REMARK 5. Assume $A \neq E$. Let C be a matrix as in Lemma 3 (or Lemma 5). If A is not conjugate to A^{-1}, or if $\det A=-1$ and $B=E$, then the group G is generated by $-E$ and C. Otherwise $-E$ and C generate a subgroup of G of index 2.

Furthermore, since $A=C^p$ (or $-C^p$), C^p (or C^{2p}) operates trivially on $\boldsymbol{Z}^2/\langle A-E, B-E \rangle$.

REMARK 6. Assume A is general, $\omega(A)=[\dot{k}_1, \cdots, \dot{k}_p]$ and A is conjugate to δA^{-1} $(\delta=\det A)$. Then by Lemma 4, for some l $(1 \leqq l \leqq p)$ $k_i=k_j$ when $i+j \equiv l+1 \pmod p$. Then $P^{-1}CP=\varepsilon C^{-1}$ where $\varepsilon=\det C$, and so, $P^{-1}AP=\delta A^{-1}$, where $P=\begin{pmatrix} k_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} k_l & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Finally, we give some examples.

EXAMPLE 1. Let $M=M(A, E; m, n)$.

(i) If $A=\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}$, $Z^2/\langle A-E\rangle=0$. Hence M is isomorphic to $M(A, E; 0, 0)$.

(ii) If $A=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$, $Z^2/\langle A-E\rangle\cong Z_2$. Hence M is isomorphic to $M(A, E; 0, 0)$ or $M(A, E; 1, 0)$.

(iii) If $A=\begin{pmatrix}0 & 1\\ -1 & -1\end{pmatrix}$, $Z^2/\langle A-E\rangle\cong Z_3$. Hence M is isomorphic to $M(A, E; 0, 0)$ or $M(A, E; \pm 1, 0)$. Note that $-E\in G$ and $(-E)\begin{pmatrix}1\\ 0\end{pmatrix}=\begin{pmatrix}-1\\ 0\end{pmatrix}$. So, $M(A, E; 1, 0)$ is isomorphic to $M(A, E; -1, 0)$.

(iv) If $A=E$, $G=GL(2, Z)$. Hence M is isomorphic to $M(A, E; d, 0)$ where $d=$g.c.d. (m, n).

(v) If $A=-E$, $Z^2/\langle A-E\rangle\cong Z_2\oplus Z_2$ and $G=GL(2, Z)$. Hence M is isomorphic to $M(A, E; 0, 0)$ or $M(A, E; 1, 0)$.

Note that according to Proposition 1, in each of the above cases, the bundle isomorphism types are characterized by H_1M.

EXAMPLE 2. Let $M=M(A, E; m, n)$, where $A=\begin{pmatrix}25 & 18\\ 18 & 13\end{pmatrix}$. Since $\langle A-E\rangle=\left\langle\begin{pmatrix}24 & 18\\ 18 & 12\end{pmatrix}\right\rangle=\left\langle\begin{pmatrix}6 & 0\\ 0 & 6\end{pmatrix}\right\rangle$, we may assume that $0\leqq m, n<6$. Since $\det A=1$ and $25/18=[1, 2, 1, 1, 2, 1]$, $C=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}2 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}4 & 3\\ 3 & 2\end{pmatrix}$ and $A=C^2$. Let $P=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ then $PAP^{-1}=A^{-1}$. Therefore $G=\{\pm C^k, \pm C^kP\}$. And so, since C^2 acts identically on $Z^2/\langle A-E\rangle$, the number of the elements of any orbit by the action of G is a devisor of 8.

In fact, there are 10 types of bundle isomorphism classes with the form $M(A, E; m, n)$, which corresponds to the following orbit of $\begin{pmatrix}m\\ n\end{pmatrix}$ in $Z^2/\langle A-E\rangle$.

(1) $\left\{\begin{pmatrix}0\\ 0\end{pmatrix}\right\}$

(2) $\left\{\begin{pmatrix}3\\ 3\end{pmatrix}\right\}$

(3) $\left\{\begin{pmatrix}3\\ 0\end{pmatrix}, \begin{pmatrix}0\\ 3\end{pmatrix}\right\}$

(4) $\left\{\begin{pmatrix}1\\ 3\end{pmatrix}, \begin{pmatrix}5\\ 3\end{pmatrix}, \begin{pmatrix}3\\ 1\end{pmatrix}, \begin{pmatrix}3\\ 5\end{pmatrix}\right\}$

(5) $\left\{\begin{pmatrix}2\\ 0\end{pmatrix}, \begin{pmatrix}4\\ 0\end{pmatrix}, \begin{pmatrix}0\\ 2\end{pmatrix}, \begin{pmatrix}0\\ 4\end{pmatrix}\right\}$

(6) $\left\{\begin{pmatrix}1\\ 1\end{pmatrix}, \begin{pmatrix}1\\ 5\end{pmatrix}, \begin{pmatrix}5\\ 5\end{pmatrix}, \begin{pmatrix}5\\ 1\end{pmatrix}\right\}$

(7) $\left\{\binom{2}{2}, \binom{2}{4}, \binom{4}{4}, \binom{4}{2}\right\}$

(8) $\left\{\binom{1}{2}, \binom{4}{1}, \binom{5}{4}, \binom{2}{5}\right\}$

(9) $\left\{\binom{2}{1}, \binom{5}{2}, \binom{4}{5}, \binom{1}{4}\right\}$

(10) $\left\{\binom{1}{0}, \binom{4}{3}, \binom{5}{0}, \binom{2}{3}, \binom{0}{1}, \binom{3}{2}, \binom{0}{5}, \binom{3}{4}\right\}$.

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Present Address:
DEPARTMENT OF MATHEMATICS
TSUDA COLLEGE
TSUDA-MACHI, KODAIRA, TOKYO 187