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A Construction of the Fundamental Solution for the Relativistic Wave Equation, I.

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A construction of the fundamental solution for the Dirac equation in an external electro-magnetic field is discussed. The fundamental solution is obtained in the form of infinite product of some oscillatory integral operators without using the classical mechanics.

Introduction

When R. P. Feynman gave his new formulation of the non-relativistic quantum mechanics [1], he appropriated the notion of Markov process for a description of the dynamical structure. Since his discussions involve integrations with respect to a $\boldsymbol{C}$-valued unbounded measure on some path space (Feynman integral), it is extremely difficult to construct regorously its mathematical theory. Among various approaches by many authers (for example [2] [3] [4] [5] [6] [7]), Fujiwara [7] may be most faithful to the Feynman's original version. He constructed the fundamental solution of the Schrödinger equation as an infinite product of operators. Kitada, who discussed a generalization [8] of [7], treated some pseudo-differential operators near to the Schrödinger's one with the same method.

In this paper we pursue an analogy of [7] and [8] in the case of Dirac equation:

$$(0.1)\qquad D_0\varphi(x)=(\vec{\alpha}(\vec{D}+\vec{A}(x))+m\beta+A_0(x))\varphi(x)\equiv H(\vec{D}, x)\varphi(x)\ ,$$

where

$$x=(x^0, \vec{x})\in \boldsymbol{R}\times \boldsymbol{R}^3\ ,$$

$$D_\mu=\frac{1}{i}\frac{\partial}{\partial x^\mu}\quad (\mu=0, 1, 2 \text{ and } 3)\ ,$$

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$m>0$: the mass of an electron ,

$\alpha_k (k=1, 2 \text{ and } 3)$ and β: 4×4 hermitian constant

matrices with the following relations:

$$\begin{cases} \alpha_k\alpha_j+\alpha_j\alpha_k=2\delta_{kj} \\ \alpha_k\beta+\beta\alpha_k=0,\ \beta^2=1\ , \end{cases}$$

and $A_\mu(\mu=0, 1, 2 \text{ and } 3)$ denote the electro-magnetic potentials which are real valued functions on $\boldsymbol{R}^4$, and we chose a suitable system of units so that the Planck constant $h=1$. In what follows we shall show, in accordance with [7], the convergence of an infinite product of some oscillatory integral operators, which are approximate solutions of (0.1). They are obtained by the following observation and not through any classical mechanical consideration.

We may naturally expect that the following pseudo-differential operator

$$\Gamma(x^0, y^0)\varphi(\vec{x})=\int d\vec{\xi}\int d\vec{y}e^{i\vec{\xi}(\vec{x}-\vec{y})}e^{i(x^0-y^0)H(\vec{\xi},x)}\varphi(\vec{y}) \tag{0.2}$$

gives an approximate evolution from y^0 to x^0 of the solution of (0.1) for initial state φ. Now note that the Dirac operator $H(\vec{D}, x)$ can be written as

$$H(\vec{D}, x)=H_0(\vec{D}+\vec{A}(x))+A_0(x)$$

with

$$H_0(\vec{\xi})\equiv\vec{\alpha}\vec{\xi}+m\beta\ ,$$

which satisfies the relation:

$$H_0(\vec{\xi})^2=\langle\vec{\xi}\rangle^2 \quad (\langle\vec{\xi}\rangle\equiv\sqrt{|\vec{\xi}|^2+m^2})\ . \tag{0.3}$$

Then we can rewrite (0.2), translating $\vec{\xi}$ into $\vec{\xi}+\vec{A}(x)$, as

$$\begin{aligned}\Gamma(x^0, y^0)\varphi(\vec{x})&=\int d\vec{\xi}\int d\vec{y}e^{i(\vec{\xi}+\vec{A}(x))(\vec{x}-\vec{y})+i(x^0-y^0)A_0(x)}e^{i(x^0-y^0)H_0(\vec{\xi})\langle\vec{\xi}\rangle^{-1}\cdot\langle\vec{\xi}\rangle}\varphi(\vec{y}) \\ &=\int d\vec{\xi}\int d\vec{y}\sum_{\pm}\frac{1}{2}\{1\pm H_0(\vec{\xi})\langle\vec{\xi}\rangle^{-1}\}e^{i(\xi+A(x))(x-y)}\varphi(\vec{y})\ ,\end{aligned}$$

where

$$\begin{cases}(\xi+A(x))(x-y)=\sum_{\mu=0}^{3}(\xi_\mu+A_\mu(x))(x^\mu-y^\mu)\ , \\ \xi_0=\pm\langle\vec{\xi}\rangle\ .\end{cases}$$

Making use of this expression for $\Gamma(x^0, y^0)$, we obtain the following theorem which will be proved in §2.

THEOREM. *Assume the following condition for* A_μ:

(A) $A_\mu \in C^\infty(\boldsymbol{R}^4)$ *and each one of their first derivatives belongs to* $\mathscr{B}(\boldsymbol{R}^4)$.

Then there exists a family of unitary operators $U(x^0, y^0)(x^0$ *and* $y^0 \in \boldsymbol{R})$ *on* $\mathscr{H} \equiv L^2(\boldsymbol{R}^3; \boldsymbol{C}^4)$ *defined by the equality*:

$$U(x^0, y^0) = \lim_{|\varDelta| \to 0} \Gamma(x^0, x^0_{N-1}) \cdots \Gamma(x^0_2, x^0_1) \Gamma(x^0_1, y^0) \ . \tag{0.4}$$

Here $\varDelta \equiv (x^0, x^0_{N-1}, \cdots, x^0_2, x^0_1, y^0)$ *is a subdivision of time interval* $[x^0, y^0] \subset \boldsymbol{R}$ *with the norm*

$$|\varDelta| = \max_{j=0,\cdots,N-1} |x^0_{j+1} - x^0_j| \quad (x^0_N = x^0,\ x^0_0 = y^0) \ ,$$

and the limit in (0.4) *is taken in the sense of operator convergence on* $\mathscr{H}$.

Let us give some remarks on the theorem.

(1) Dirac operator $H(\vec{D}, x)$ is of the symmetric hyperbolic type. It is well known that the initial value problem for (0.1) has the unique solution [9]. In fact $U(t, s)$ in the theorem coincides with the fundamental solution of (0.1). At the same time we should notice that our approximating sequence given by the theorem converges to the exact solution $U(x^0, y^0)\varphi$ uniformly in the initial value φ as is the case in [7].

(2) In order to obtain the solution of the initial value problem for (0.1), one can also apply a general scheme for constructing the fundamental solution associated with evolution equation of hyperbolic type using oscillatory integral [10]. With this method Yajima [11] discussed under the same assumption (A) the quasi-classical limit for the fundamental solution of (0.1). In our case we have to abandon the argument about the quasi-classical limit $(h \to 0)$ because the limit in (0.4) is not uniform in $h \in (0, 1]$. This would be a general feature of the product integral.

(3) The amplitude functions $(1/2)(1 \pm H_0(\vec{\xi})\langle\vec{\xi}\rangle^{-1})$ of $\Gamma(x^0, y^0)$ can be interpreted as the projection matrices to positive and negative energy states because of (0.3), while the phase function $(\xi + A(x))(x-y)$ is obtained by translating the free phase function $\xi(x-y)$ on cotangent space.

(4) Since the support of the distribution $\varDelta_{z^0}(\vec{z})$ defined by

$$\varDelta_{z^0}(\vec{z}) = \int d\!\!{}^-\vec{\xi} \sum_{\xi_0 = \pm\langle\vec{\xi}\rangle} e^{iz\xi} \xi_0$$

is contained in $\{\vec{z}\in \boldsymbol{R}^3;\ z^{02}-|z|^2\geqq 0\}$, we find that $\Gamma(x^0, y^0)$ has the finite propagation speed and so confirm the relativistic causality.

§ 1. Analysis of $\Gamma(x^0, y^0)$.

In this section we shall study properties of integral operators like $\Gamma(x^0, y^0)$ in slightly more general context.

DEFINITION 1.1 For $A(x)=(A_0(x), A_1(x), A_2(x), A_3(x))$ satisfying the assumption (A) of the theorem stated in the introduction, we put

$$\phi^{\pm}(x, \vec{\xi}, y)=(\xi+A(x))(x-y)\ , \tag{1.1}$$

where

$$\xi=(\xi_0, \vec{\xi})\ ,$$
$$\xi_0=\pm\langle\vec{\xi}\rangle\equiv\pm\sqrt{|\vec{\xi}|^2+m^2}\quad (m>0)\ .$$

Let $\mathscr{B}(\boldsymbol{R}^4\times\boldsymbol{R}^3\times\boldsymbol{R}^4; M_4(\boldsymbol{C}))$ be the space of 4×4 matrix-valued functions whose components are the elements of $\mathscr{B}(\boldsymbol{R}^4\times\boldsymbol{R}^3\times\boldsymbol{R}^4)$ and $\mathscr{S}\equiv\mathscr{S}(\boldsymbol{R}^4; \boldsymbol{C}^4)$ be the space of rapidly decreasing $\boldsymbol{C}^4$-valued functions.

DEFINITION 1.2. For $a(x, \vec{\xi}, y)\in\mathscr{B}(\boldsymbol{R}^4\times\boldsymbol{R}^3\times\boldsymbol{R}^4; M_4(\boldsymbol{C}))$, we define the operators $J^{\pm}(x^0, y^0)$ on $\mathscr{S}$ by

$$J^{\pm}(x^0, y^0)\varphi(\vec{x})=\int d\vec{\xi}\int d\vec{y}\ a(x, \vec{\xi}, y)e^{i\phi^{\pm}(x,\vec{\xi},y)}\varphi(\vec{y})\ . \tag{1.2}$$

PROPOSITION 1.3. (i) *For any x^0 and $y^0\in\boldsymbol{R}$, $J^{\pm}(x^0, y^0)$ maps $\mathscr{S}$ into itself.*

(ii) *For $\varphi\in\mathscr{S}$, each derivative of $J^{\pm}(x^0, y^0)\varphi(\vec{x})$ as a function of $x^0, \vec{x}$ and y^0 is given by the equality:*

$$D^{\alpha}J^{\pm}(x^0, y^0)\varphi(\vec{x})=\int d\vec{\xi}\int d\vec{y}D^{\alpha}\{a(x, \vec{\xi}, y)e^{i\phi^{\pm}(x,\vec{\xi},y)}\}\varphi(\vec{y})\ , \tag{1.3}$$

where

$$D^{\alpha}=D_{x^0}^{\alpha_0}\cdots D_{x^3}^{\alpha_3}D_{y^0}^{\alpha_4}\quad (\alpha_j=0, 1, \cdots)\ .$$

PROOF. Let $\varphi\in\mathscr{S}$ and put

$$\Psi^{\pm}(x, \xi)=\int d\vec{y}\ e^{i\phi^{\pm}(x,\vec{\xi},y)}\varphi(\vec{y})\ .$$

Let L and L^* be the differential operators defined by

(1.4) $$L=\langle\vec{\xi}\rangle^{-2}(m^2-\vec{\xi}\vec{D}_y)\ ,\quad L^*=\langle\vec{\xi}\rangle^{-2}(m^2+\vec{\xi}\vec{D}_y)$$

respectively. Note that L satisfies

(1.4)′ $$Le^{i\xi(x-y)}=e^{i\xi(x-y)}\ .$$

Then, for any integer $l\geq 0$, partial integration gives that

(1.5) $$\begin{aligned}\Psi^{\pm}(x,\vec{\xi})&=\int d\vec{y}(L^l e^{i\xi(x-y)})e^{iA(x)(x-y)}\varphi(\vec{y})\\&=\int d\vec{y}\ e^{i\xi(x-y)}L^{*l}(e^{iA(x)(x-y)}\varphi(\vec{y}))\\&=\sum_{\substack{|\alpha|\leq l\\ \beta\leq\alpha}}\int d\vec{y}\binom{l}{\alpha}\binom{\alpha}{\beta}m^{2(l-|\alpha|)}\vec{\xi}^{\alpha}\langle\vec{\xi}\rangle^{-2l}\\&\quad\times(-\vec{A}(x))^{\beta}e^{i\phi^{\pm}}\vec{D}_y^{\alpha-\beta}\varphi(\vec{y})\ .\end{aligned}$$

By taking $l=4$, we find $\Psi^{\pm}(x,\vec{\xi})$ integrable with respect to $\vec{\xi}$. This makes $J^{\pm}(x^0,y^0)\varphi(\vec{x})$ well-defined. Moreover, taking l large enough, we can easily show that $J^{\pm}(x^0,y^0)\varphi(\vec{x})$ is a smooth function of x^0, y^0 and $\vec{x}$, and its derivatives are given by (1.3).

Next, applying Lemma 3.2 in the Appendix to the term $\vec{A}(x)^{\beta}$ appeared in (1.5), we obtain the following expression:

(1.6) $$\begin{aligned}J^{\pm}(x^0,y^0)\varphi(\vec{x})=&\sum_{\substack{|\alpha|\leq l\\ |\beta|+|\gamma|\leq|\alpha|}}\int d\vec{\xi}\int d\vec{y}B^{l\pm}_{\alpha\beta\gamma}(x,\vec{\xi},y)\\&\times(\vec{x}-\vec{y})^{\gamma}e^{i\phi^{\pm}(x,\vec{\xi},y)}A(x^0,\vec{y})^{\beta}\vec{D}_y^{\alpha}\varphi(\vec{y})\ ,\end{aligned}$$

where $B^{l\pm}_{\alpha\beta\gamma}$ are smooth functions obeying the estimate:

$$|\vec{D}_x^{\lambda}\vec{D}_{\xi}^{\mu}\vec{D}_y^{\nu}B^{l\pm}_{\alpha\beta\gamma}(x,\vec{\xi},y)|\leq C(\lambda,\mu,\nu;l)\langle\vec{\xi}\rangle^{-l}$$

for all λ, μ, ν, and l with a positive constant $C(\lambda,\mu,\nu;l)$. Now, observing that

(1.7) $$(\vec{x}-\vec{y})^{\gamma}e^{i\phi^{\pm}}=\vec{D}_{\xi}^{\gamma}e^{i(\vec{\xi}+\vec{A}(x))(\vec{x}-\vec{y})}\times e^{i(\xi_0+A_0(x))(x^0-y^0)}\ ,$$

we can rewrite (1.6) as

$$\begin{aligned}J^{\pm}(x^0,y^0)\varphi(\vec{x})=&\sum_{\substack{|\alpha|\leq l\\ |\beta|\leq|\alpha|}}\int d\vec{\xi}\int d\vec{y}B^{l\pm}_{\alpha\beta}(x,\vec{\xi},y)\\&\times e^{i\phi^{\pm}}\cdot\vec{A}(x^0,\vec{y})^{\beta}\vec{D}_y^{\alpha}\varphi(\vec{y})\ ,\end{aligned}$$

where $B^{l\pm}_{\alpha\beta}$ are smooth functions satisfying the estimate:

$$|D_x^{\lambda}D_{\xi}^{\mu}D_y^{\nu}B^{l\pm}_{\alpha\beta}(x,\vec{\xi},y)|\leq\tilde{C}(\lambda,\mu,\nu;l)\langle\vec{\xi}\rangle^{-l}$$

for all λ, μ, ν and l with a positive constant $\tilde{C}(\lambda, \mu, \nu; l)$. Therefore, for any multi-indices γ and δ, we have

$$|\vec{x}^{\gamma}\vec{D}_x^{\delta}J^{\pm}(x^0, y^0)\varphi(\vec{x})| \leqq C_{\gamma\delta} .$$

LEMMA 1.4. (i) *For* $\varphi \in \mathscr{S}$ *and* $k=1, 2, 3$, *it holds that*

$$\int d\vec{\xi}\int d\vec{y}e^{i\phi\pm}(x^k-y^k)a(x, \vec{\xi}, y)\varphi(\vec{y})$$
$$=\int d\vec{\xi}\int d\vec{y}e^{i\phi\pm}\left\{-(x^0-y^0)\frac{\xi_k}{\xi_0}-D_{\xi_k}\right\}a(x, \vec{\xi}, y)\varphi(\vec{y}) .$$

(ii) *There exists a function* $\tilde{a}(x, \vec{\xi}, y) \in \mathscr{B}(R^4\times R^3\times R^4;\ M_4(C))$ *such that*

$$\int d\vec{\xi}\int d\vec{y}e^{i\phi\pm}\xi_0 a(x, \vec{\xi}, y)\varphi(\vec{y})=\int d\vec{\xi}\int d\vec{y}e^{i\phi\pm}\Big[\tilde{a}(x, \vec{\xi}, y)$$
$$+\frac{\xi_k}{\xi_0}a(x, \vec{\xi}, y)(D_{y^k}-A_k(x^0, \vec{y}))\Big]\varphi(\vec{y}) .$$

PROOF. (i) is clear by virtue of (1.7).

(ii) Let L be as in (1.4). Making use of (1.4)', we can easily show by partial integration as in (1.5) that

$$\int d\vec{\xi}\int d\vec{y}e^{i\phi\pm}\xi_0 a(x, \vec{\xi}, y)\varphi(\vec{y})$$
$$=\int d\vec{\xi}\int d\vec{y}e^{i\phi\pm}\left[\frac{m^2+\vec{\xi}\vec{D}_y}{\xi_0}-\frac{\vec{\xi}}{\xi_0}\vec{A}(x)\right]a(x, \vec{\xi}, y)\varphi(\vec{y}) .$$

Moreover, since Lemma 3.1 in the Appendix implies

$$A_k(x^0, \vec{x})=\sum_j B_{kj}(x^0, \vec{x}, \vec{y})(x^j-y^j)+A_k(x^0, \vec{y}) \quad (k=1, 2 \text{ and } 3)$$

for suitable $B_{kj}(x, \vec{y}) \in \mathscr{B}(R^4\times R^3)(k, j=1, 2$ and $3)$, we may find that (i) implies (ii), where

$$\tilde{a}(x, \vec{\xi}, y)=\left[\frac{m^2}{\xi_0}-B_{kj}(x, \vec{y})\left\{(x^0-y^0)\frac{\xi_k\xi_j}{\xi_0^2}+\frac{\delta_{kj}}{\xi_0}-\frac{\xi_k\xi_j}{\xi_0^3}\right\}\right]a(x, \vec{\xi}, y)$$
$$+\left[-B_{kj}(x, \vec{y})\frac{\xi_k}{\xi_0}D_{\xi_k}+\frac{\xi_j}{\xi_0}D_{y^k}\right]a(x, \vec{\xi}, y) .$$

PROPOSITION 1.5. *There exists a positive constant* K_1 *(depending only on* $|a|_l$*) with the following property*:

If $|x^0-y^0|\leqq K_1$, $J^{\pm}(x^0, y^0)$ *has a bounded extention on* $\mathscr{H}=L^2(R^3;\ C^4)$.

PROOF. We apply Theorem 3.3 of the Appendix to the right hand side of (1.2). Since

$$\frac{\partial^2\phi^\pm}{\partial x^i\partial y^j}=-A_{j,i}(x)\ , \quad \frac{\partial^2\phi^\pm}{\partial\xi_i\partial y^j}=-\delta_{ij}\ ,$$
$$\frac{\partial^2\phi^\pm}{\partial x^i\partial\xi_j}=\delta_{ij}\ , \quad \frac{\partial^2\phi^\pm}{\partial\xi_i\partial\xi_j}=(x^0-y^0)\Big(\frac{\delta_{ij}}{\xi_0}-\frac{\xi_i\xi_j}{\xi_0^3}\Big)\ , \quad (i,\ j=1, 2 \ \text{and}\ 3)\ ,$$

there exists a constant M such that

$$\left|\det\begin{bmatrix}\dfrac{\partial^2\phi^\pm}{\partial x\partial y} & \dfrac{\partial^2\phi^\pm}{\partial\xi\partial y}\\ \dfrac{\partial^2\phi^\pm}{\partial x\partial\xi} & \dfrac{\partial^2\phi^\pm}{\partial\xi\partial\xi}\end{bmatrix}\right|\geqq 1-M|x^0-y^0|$$

because of the assumption (A). Therefore if $|x^0-y^0|\leqq 1/(2M)$, $\phi^\pm$ satisfies the assumption (ii) of Theorem 3.3.

PROPOSITION 1.6. *For every $\varphi\in\mathscr{S}$, the mapping:* $(x^0, y^0)\mapsto J^\pm(x^0, y^0)\varphi$ *of* $\{(x^0, y^0)\in \boldsymbol{R}^2;\ |x^0-y^0|\leqq K_1\}$ *to* $\mathscr{H}$ *is strongly continuous.*

PROOF. Denote $x'=(t', \vec{x})$ and $x=(t, \vec{x})$. Then we have

$$(1.8)\quad J^\pm(t', y^0)\varphi(\vec{x})-J^\pm(t, y^0)\varphi(\vec{x})$$
$$=\int d\!\!\!{}^-\vec{\xi}\int d\vec{y}[a(x', \vec{\xi}, y)-a(x, \vec{\xi}, y)]e^{i\phi^\pm(x',\vec{\xi},y)}\varphi(\vec{y})$$
$$+\int d\!\!\!{}^-\vec{\xi}\int d\vec{y}a(x, \vec{\xi}, y)[e^{i\phi^\pm(x',\vec{\xi},y)-i\phi^\pm(x,\vec{\xi},y)}-1]e^{i\phi^\pm(x,\vec{\xi},y)}\varphi(\vec{y})\ .$$

Clearly, $a(x', \vec{\xi}, y)$ tends to $a(x, \vec{\xi}, y)$ in the space $\mathscr{B}(\boldsymbol{R}^4\times\boldsymbol{R}^3\times\boldsymbol{R}^4;\ M_4(\boldsymbol{C}))$ as t' goes to t. Therefore, Theorem 3.3 implies that the first term in the right hand side of (1.8) converges to zero in $\mathscr{H}$. On the other hand, Lemma 3.1 in the Appendix implies the existence of the functions $B_\lambda(t', t, \vec{x})\in\mathscr{B}(\boldsymbol{R}\times\boldsymbol{R}\times\boldsymbol{R}^3)(\lambda=0, 1, 2$ and $3)$ such that

$$A_\lambda(t', \vec{x})-A_\lambda(t, \vec{x})=B_\lambda(t', t, \vec{x})(t'-t)\ .$$

So we have

$$\phi^\pm(x', \vec{\xi}, y)-\phi^\pm(x, \vec{\xi}, y)$$
$$=(t'-t)\theta^\pm(\vec{\xi}, \vec{x};\ t, y^0)+(t'-t)B_k(t', t, \vec{x})(x^k-y^k)\ ,$$

where

$$\theta^\pm(\vec{\xi}, \vec{x};\ t, y^0)=\xi_0+A_0(x')+(t-y^0)B_0(t', t, \vec{x})\ .$$

Thus we obtain the equalities

$$
\begin{aligned}
e^{i\phi_\pm(x',\vec{\xi},y)-i\phi_\pm(x,\vec{\xi},y)}-1 \\
&=e^{i(t'-t)\theta_\pm(\vec{\xi},\vec{x};t,y^0)}[e^{i(t'-t)B_k(t',t,\vec{x})(x^k-y^k)}-1] \\
&\quad+e^{i(t'-t)\theta_\pm(\vec{\xi},\vec{x};t,y^0)}-1 \\
&=e^{i(t'-t)\theta_\pm(\vec{\xi},\vec{x};t,y^0)}F_l(\vec{x},\vec{y};t',t)\langle\vec{x}-\vec{y}\rangle^l \\
&\quad+F^\pm(\vec{\xi},\vec{x};t',t,y^0)(\langle\vec{\xi}\rangle+\langle\vec{x}\rangle)\ ,
\end{aligned}
$$

where we put

$$
F_l(\vec{x},\vec{y};t',t)=\frac{e^{i(t'-t)B_k(t',t,\vec{x})(x^k-y^k)}-1}{\langle\vec{x}-\vec{y}\rangle}
$$

$$
F^\pm(\vec{\xi},\vec{x};t',t,y^0)=\frac{e^{i(t'-t)\theta_\pm(\vec{\xi},\vec{x};t,y^0)}-1}{\langle\vec{\xi}\rangle+\langle\vec{x}\rangle}\ .
$$

Moreover, taking account of Lemma 3.1 again, we define $h_k(\vec{z})\in\mathscr{B}(\boldsymbol{R}^3)$ $(k=1, 2$ and $3)$ by the following equality:

$$
\langle\vec{z}\rangle=h_k(\vec{z})z^k+m\ .
$$

Then, we obtain

$$
\begin{aligned}
e^{i\phi_\pm(x',\vec{\xi},y)-i\phi_\pm(x,\vec{\xi},y)}-1 \\
&=e^{i(t'-t)\theta_\pm(\vec{\xi},\vec{x};t,y^0)}F_l(\vec{x},\vec{y};t',t)[h_k(\vec{x}-\vec{y})(x^k-y^k)+m]^l \\
&\quad+F^\pm(\vec{\xi},\vec{x};t',t,y^0)[\langle\vec{\xi}\rangle+m+h_k(\vec{x})(x^k-y^k)+h_k(\vec{x})y^k]\ , \\
&\qquad(l=1,2,\cdots)\ .
\end{aligned}
$$

Noting that $e^{i(t'-t)\theta_\pm(\vec{\xi},\vec{x};t,y^0)}F_l(\vec{x},\vec{y};t',t)\in\mathscr{B}^l(\boldsymbol{R}^3_{\vec{x}}\times\boldsymbol{R}^3_{\vec{\xi}}\times\boldsymbol{R}^3_{\vec{y}})$ and $F^\pm(\vec{\xi},\vec{x};t',t,y^0)\in\mathscr{B}(\boldsymbol{R}^3_{\vec{x}}\times\boldsymbol{R}^3_{\vec{\xi}}\times\boldsymbol{R}^3_{\vec{y}})$, and using Theorem 3.3 in the Appendix for l large enough, we can conclude that the second term of (1.8) converges to zero in $\mathscr{H}$ as $t'\to t$. The strong continuity with respect to y^0 can be shown similarly.

PROPOSITION 1.7. *For every* $\varphi\in\mathscr{S}$, *the mapping*: $(x^0,y^0)\mapsto J^\pm(x^0,y^0)\varphi$ *of* $\{(x^0,y^0)\in\boldsymbol{R}^2;\ |x^0-y^0|\leqq K_1\}$ *to* $\mathscr{H}$ *is strongly differentiable.*

PROOF. Let $\varphi\in\mathscr{S}$. The strong differentiability of $J^\pm(x^0,y^0)\varphi$ with respect to x^0 follows from the strong continuity of $(\partial/\partial x^0)J^\pm(x^0,y^0)\varphi$ in x^0, since

$$
J^\pm(\tilde{x}^0,y^0)\varphi(\vec{x})-J^\pm(x^0,y^0)\varphi(\vec{x})=\int_{x^0}^{\tilde{x}^0}dt\frac{\partial}{\partial t}J^\pm(t,y^0)\varphi(\vec{x})\ .
$$

Now Proposition 1.3 (ii) yields

$$\frac{1}{i}\frac{\partial}{\partial x^0}J^{\pm}(x^0, y^0)\varphi(\vec{x})$$
$$=\int d\vec{\xi}\int d\vec{y}e^{i\phi\pm}\left[\frac{1}{i}\frac{\partial}{\partial x^0}+\xi_0+A_{\lambda,0}(x)(x^\lambda-y^\lambda)\right]a(x, \vec{\xi}, y)\varphi(\vec{y}) .$$

We can rewrite the right hand side of this by means of Lemma 1.4 as

$$\frac{1}{i}\frac{\partial}{\partial x^0}J^{\pm}(x^0, y^0)\varphi(\vec{x})=\int d\vec{\xi}\int d\vec{y}e^{i\phi\pm}\left[\frac{1}{i}\frac{\partial}{\partial x^0}+A_{0,0}(x)(x^0-y^0)\right.$$
$$\left.+A_{k,0}(x)\left\{-(x^0-y^0)\frac{\xi_k}{\xi_0}-D_{\xi_k}\right\}\right]a(x, \vec{\xi}, y)\varphi(\vec{y})$$
$$+\int d\vec{\xi}\int d\vec{y}e^{i\phi\pm}\tilde{a}(x, \vec{\xi}, y)\varphi(\vec{y})$$
$$+\int d\vec{\xi}\int d\vec{y}e^{i\phi\pm}\frac{\xi_k}{\xi_0}a(x, \vec{\xi}, y)(D_{y_k}-A_k(x^0, \vec{y}))\varphi(\vec{y}) .$$

Here $[(1/i)(\partial/\partial x^0) + A_{0,0}(x)(x^0 - y^0) + A_{k,0}(x)\{-(x^0 - y^0)(\xi_k/\xi_0) - D_{\xi_k}\}]a(x, \vec{\xi}, y)$, $\tilde{a}(x, \xi, y)$ and $(\xi_k/\xi_0)a(x, \vec{\xi}, y)$ belong to $\mathscr{B}$, and $(D_{\xi_k}-A_k(x^0, \vec{y}))\varphi(\vec{y})$ belongs to $\mathscr{S}$. Applying Proposition 1.6, we can show that the mapping:

$$x^0 \longmapsto \frac{\partial}{\partial x^0}J^{\pm}(x^0, y^0)\varphi(\vec{x})$$

is strongly continuous. The strong continuity with respect to y^0 can be shown similarly.

PROPOSITION 1.8. *Let $\tilde{\phi}^{\pm}(x, \vec{\xi}, y, t)$ and $\tilde{J}^{\pm}(x^0, y^0, t)$ be a function and an operator defined by*

(1.9) $$\tilde{\phi}^{\pm}(x, \vec{\xi}, y, t)=(t-y^0)\xi_0+(x^0-y^0)A_0(x)+(\vec{x}-\vec{y})(\vec{\xi}+\vec{A}(x))$$

and

(1.10) $$\tilde{J}^{\pm}(x^0, y^0, t)\varphi(\vec{x})=\int d\vec{\xi}\int d\vec{y}e^{i\tilde{\phi}\pm(x,\vec{\xi},y,t)}a(x, \vec{\xi}, y)\varphi(\vec{y}) ,$$

respectively, where

$$a(x, \vec{\xi}, y)\in\mathscr{B}(\boldsymbol{R}^4\times\boldsymbol{R}^3\times\boldsymbol{R}^4) \quad \textit{and} \quad \varphi\in\mathscr{S} .$$

Then the following statements hold:

(i) *$\tilde{J}^{\pm}(x^0, y^0, t)$ maps $\mathscr{S}$ to $\mathscr{S}$ for any x^0, y^0 and $t \in \boldsymbol{R}$.*

(ii) *If $|x^0-y^0|<K_1$ and $|t-y^0|\leqq K_1$, $\tilde{J}^{\pm}(x^0, y^0, t)$ has a bounded extention on $\mathscr{H}$.*

(iii) *For every $\varphi \in \mathscr{S}$, the mapping:*

$$(x^0, y^0, t) \longmapsto \tilde{J}^{\pm}(x^0, y^0, t)\varphi$$

of $\{(x^0, y^0, t) \in \boldsymbol{R}^3;\ |x^0-t|\leqq K_1 \text{ and } |t-y^0|\leqq K_1\}$ to $\mathscr{H}$ is strongly differentiable.

PROOF. We can prove (i) in the same way as shown in the proof of Proposition 1.3. If $|x^0-t|$ and $|y^0-t|$ are sufficiently small, we may easily check that

$$\left|\det\begin{bmatrix} \dfrac{\partial^2\tilde{\phi}^{\pm}}{\partial x\partial y} & \dfrac{\partial^2\tilde{\phi}^{\pm}}{\partial \xi\partial y} \\ \dfrac{\partial^2\tilde{\phi}^{\pm}}{\partial x\partial \xi} & \dfrac{\partial^2\tilde{\phi}^{\pm}}{\partial \xi\partial \xi} \end{bmatrix}\right| \geqq 1-M|x^0-y^0|$$

for some positive constant M. So all the arguments in Proposition 1.5, 1.6 and 1.7 remain valid for $\tilde{J}^{\pm}(x^0, y^0, t)$.

§ 2. Construction of the fundamental solution.

We are in a position to prove the theorem stated in the introduction. Let us begin with the following definition.

DEFINITION 2.1. For $\varphi \in \mathscr{S}$ and $|x^0-y^0|\leqq K_1$, we define as

$$\Gamma^{\pm}(x^0, y^0)\varphi(\vec{x}) = \int d\vec{\xi}\int d\vec{y}\, a^{\pm}(\vec{\xi})e^{i\phi^{\pm}(x,\vec{\xi},y)}\varphi(\vec{y})$$

$$\Gamma(x^0, y^0)\varphi(\vec{x}) = \Gamma^{+}(x^0, y^0)\varphi(\vec{x}) + \Gamma^{-}(x^0, y^0)\varphi(\vec{x})\ ,$$

where

$$a^{\pm}(\vec{\xi}) = \frac{1}{2}\left(1+\frac{\vec{\xi}\vec{\alpha}+m\beta}{\xi_0}\right)$$

$$\xi_0 = \pm\langle\vec{\xi}\rangle\ .$$

PROPOSITION 2.2. (i) *If $|x^0-y^0|\leqq K_1$, $\Gamma(x^0, y^0)$ maps $\mathscr{S}$ to $\mathscr{S}$ and has a unique bounded extention on $\mathscr{H}$.*

(ii) *$\Gamma(x^0, x^0)$ coincides with the identity operator.*

(iii) *For every $\varphi \in \mathscr{S}$, the mapping*

$$(x^0, y^0) \longmapsto \Gamma^{\pm}(x^0, y^0)\varphi$$

of $\{(x^0, y^0) \in \boldsymbol{R}^2;\ |x^0-y^0| \leqq K_1\}$ *to* $\mathscr{H}$ *is strongly differentiable.*

PROOF. (i) and (iii). Since $a^{\pm}(\vec{\xi}) \in \mathscr{B}(\boldsymbol{R}^3)$, we can apply Proposition 1.3, 1.5 and 1.7 in the previous section to $\Gamma(x^0, y^0)$. (ii) follows from the fact that $a^+(\vec{\xi})+a^-(\vec{\xi})=1$.

PROPOSITION 2.3. *For* $\varphi \in \mathscr{S}$ *and* $|x^0-y^0| \leqq K_1$, *there exists a positive constant* K_2 *such that*

$$\|D_{x^0}\Gamma(x^0, y^0)\varphi - H(x^0)\Gamma(x^0, y^0)\varphi\| \leqq K_2|x^0-y^0|\ \|\varphi\| \tag{2.1}$$

$$\|D_{y^0}\Gamma(x^0, y^0)\varphi + \Gamma(x^0, y^0)H(y^0)\varphi\| \leqq K_2|x^0-y^0|\ \|\varphi\|\ , \tag{2.2}$$

where

$$H(t)=\vec{\alpha}\cdot(\vec{D}-\vec{A}(t, \vec{x}))+m\beta+A_0(t, \vec{x})\ .$$

PROOF. As $a^{\pm}(\vec{\xi})\cdot a^{\mp}(\vec{\xi})=0$, we have

$$\begin{aligned} &D_{x^0}\Gamma(x^0, y^0)\varphi(\vec{x}) - H(x^0)\Gamma(x^0, y^0)\varphi(\vec{x}) \\ &\quad =\sum_{\pm} (A_{\lambda,0}(x)-\alpha_k A_{\lambda,k}(x))\int d\vec{\xi}\int d\vec{y} e^{i\phi^{\pm}(x,\vec{\xi},y)}(x^\lambda-y^\lambda)a^{\pm}(\vec{\xi})\varphi(\vec{y})\ . \\ &\quad =(A_{\lambda,0}(x)-\alpha_k A_{\lambda,k}(x))\sum_{\pm} \Psi^{\pm,\lambda}_{x^0,y^0}(\vec{x})\ , \end{aligned}$$

where

$$\Psi^{\pm,\lambda}_{x^0,y^0}(\vec{x})=\int d\vec{\xi}\int d\vec{y}\ e^{i\phi^{\pm}(x,\vec{\xi},y)}(x^\lambda-y^\lambda)a^{\pm}(\vec{\xi})\varphi(\vec{y})\ .$$

Since $A_{\lambda,0}(x)-\alpha_k A_{\lambda,k}(x) \in \mathscr{B}$, we have only to estimate the norm of $\Psi^{\pm,k}_{x^0,y^0}$ to establish (2.1).

Let us rewrite $\Psi^{\pm,k}_{x^0,y^0}(\vec{x})$ by Lemma 1.4 as

$$\begin{aligned} \Psi^{\pm,k}_{x^0,y^0}(\vec{x}) &=\sum_{\pm} \int d\vec{\xi}\int d\vec{y} e^{i\phi^{\pm}(x,\vec{\xi},y)}(x^0-y^0)\Big(-\frac{\xi_k}{\xi_0}\Big)a^{\pm}(\vec{\xi})\varphi(\vec{y}) \\ &\quad -\sum_{\pm} \int d\vec{\xi}\int d\vec{y} e^{\phi^{\pm}(x,\vec{\xi},y)} D_{\xi_k}a^{\pm}(\vec{\xi})\varphi(\vec{y})\ . \end{aligned} \tag{2.3}$$

We know that the norm of the first term of the right hand side is estimated by $C_1|x^0-y^0|\ \|\varphi\|$ for some positive constant C_1. In order to estimate the second term, we put

$$\Psi_t(\vec{x})=\sum_{\pm} \int d\vec{\xi}\int d\vec{y} e^{i\tilde{\phi}^{\pm}(x,,\vec{\xi},y;t)} D_{\xi_k}a^{\pm}(\vec{\xi})\varphi(\vec{y})$$

for $|t-y^0|\leqq K_1$, where $\tilde{\phi}^{\pm}$ is defined by (1.9). Then we claim the followings:

(i) If $t=y^0$, then Ψ_t vanishes.

(ii) If $t=x^0$, then Ψ_t coincides with the second term of (2.3).

(iii) Ψ_t is an element of $\mathscr{S}$ and strongly differentiable with the following estimates:

$$\left\|\frac{d}{dt}\Psi_t\right\|\leqq C_2\|\varphi\|$$

$$\|\Psi_t\|\leqq C_2|t-y^0|\,\|\varphi\|\ .$$

In fact, (i) and (ii) is clear while (iii) follows from Lemma 1.8. Therefore we can estimate the second term of (2.3) by $C_2|x^0-y^0|\,\|\varphi\|$ and obtain (2.1).

On the other hand, we have

$$D_{y^0}\Gamma(x^0, y^0)\varphi(\vec{x})+\Gamma(x^0, y^0)H(y^0)\varphi(\vec{x})$$

$$=\sum_{\pm}\int d\vec{\xi}\int d\vec{y}[-A_0(x)+A_0(y)+\vec{\alpha}(\vec{A}(x)-\vec{A}(y))]e^{i\phi^{\pm}(x,\vec{\xi},y)}a^{\pm}(\vec{\xi})\varphi(\vec{y})\ .$$

Applying Lemma 3.1 in the Appendix, we can find functions $B_{\lambda\mu}(x, y)\in\mathscr{B}(\boldsymbol{R}^4\times\boldsymbol{R}^4)$ such that

$$A_\lambda(x)-A_\lambda(y)=B_{\lambda\mu}(x^\mu-y^\mu)\ ,\qquad (\lambda=0, 1, 2 \text{ and } 3)\ .$$

Therefore we can prove (2.2) in the same way as in the proof of (2.1).

The following propositions give the approximate evolution property of $\Gamma(x^0, y^0)$ which will help our construction of the fundamental solution.

PROPOSITION 2.4. *There exists a positive constant K_8 such that*

$$\|\Gamma(x^0, \tau)\Gamma(\tau, y^0)-\Gamma(x^0, y^0)\|\leqq K_8(|x^0-\tau|^2+|\tau-y^0|^2)\ , \tag{2.4}$$

when $|x^0-y^0|\leqq K_1$.

PROOF. Let $\varphi\in C_0^\infty$. By Proposition 2.2, we have

$$\Gamma(x^0, \tau)\Gamma(\tau, y^0)\varphi(\vec{x})-\Gamma(x^0, y^0)\varphi(\vec{x})$$
$$=\int_{y^0}^{\tau}dt\frac{\partial}{\partial t}\Gamma(x^0, t)\Gamma(t, y^0)\varphi(\vec{x})\ , \tag{2.5}$$

where the right hand side is taken as a Bochner integral in $\mathscr{H}$. The direct computation shows that

$$\frac{1}{i}\frac{\partial}{\partial t}\Gamma(x^0, t)\Gamma(t, y^0)\varphi(\vec{x})=\left[\frac{1}{i}\frac{\partial}{\partial t}\Gamma(x^0, t)+\Gamma(x^0, t)H(t)\right]$$

$$\times \Gamma(t, y^0)\varphi(\vec{x}) + \Gamma(x^0, t)\left[\frac{1}{i}\frac{\partial}{\partial t}\Gamma(t, y^0) - H(t)\Gamma(t, y^0)\right]\varphi(\vec{x}) .$$

Applying Proposition 2.3 and 2.2 (i) to right hand side of this, we obtain

$$\left\|\frac{1}{i}\frac{\partial}{\partial t}\Gamma(x^0, t)\Gamma(t, y^0)\varphi\right\| \leqq K_2(|x^0 - t| + |t - y^0|)\|\varphi\| .$$

Then making use of (2.5), we may complete the proof of (2.4).

PROPOSITION 2.5. *If* $|x^0 - y^0| \leqq K_1$, *we have the following estimates.*

$$\|\Gamma(x^0, \tau)^* \Gamma(x^0, y^0) - \Gamma(\tau, y^0)\| \leqq K_3(|x^0 - \tau|^2 + |\tau - y^0|^2) \tag{2.6}$$

$$\|\Gamma(x^0, y^0)\| \leqq \exp K_3 |x^0 - y^0| , \tag{2.7}$$

where $\Gamma(x^0, y^0)^*$ *denotes the adjoint of* $\Gamma(x^0, y^0)$.

PROOF. Let $\varphi, \psi \in C_0^\infty$. By Proposition 2.2, we have

$$(\Gamma(x^0, \tau)^* \Gamma(x^0, y^0)\varphi - \Gamma(\tau, y^0)\varphi, \psi) = \int_\tau^{x^0} dt \frac{d}{dt}(\Gamma(t, y^0)\varphi, \Gamma(t, \tau)\psi) .$$

Since $H(t)$ is a symmetric operator,

$$\frac{d}{dt}(\Gamma(t, y^0)\varphi, \Gamma(t, \tau)\psi) = \left(\frac{d}{dt}\Gamma(t, y^0)\varphi - iH(t)\Gamma(t, y^0)\varphi, \Gamma(t, \tau)\varphi\right)$$
$$+ \left(\Gamma(t, y^0)\varphi, \frac{d}{dt}\Gamma(t, \tau)\psi - iH(t)\Gamma(t, \tau)\psi\right) .$$

So, (2.6) is a direct consequence of Proposition 2.3. Putting $\tau = y^0$ in (2.6), we have (2.7).

PROPOSITION 2.6. *There exists a positive constant* δ *depending on* K_1 *such that* $\Gamma(x^0, y^0)$ *has the bounded inverse if* $|x^0 - y^0| \leqq \delta$. *Besides, there exists s positive constant* K_4 *depending on* δ *such that the following estimates hold:*

$$\|\Gamma(x^0, y^0)^{-1} - \Gamma(x^0, y^0)^*\| \leqq K_4 |x^0 - y^0|^2 \tag{2.8}$$

$$\|\Gamma(x^0, y^0)^{-1} - \Gamma(y^0, x^0)\| \leqq K_4 |x^0 - y^0|^2 , \tag{2.9}$$

for $|x^0 - y^0| \leqq \delta$.

PROOF. Put $\tau = y^0$ in (2.6). Then we have

$$\|\Gamma(x^0, y^0)^* \Gamma(x^0, y^0) - I\| \leqq K_3 |x^0 - y^0|^2 . \tag{2.10}$$

Similarly (2.4) implies

$$\|\Gamma(y^0, x^0)\Gamma(x^0, y^0) - I\| \leqq K_8|x^0 - y^0|^2 . \tag{2.11}$$

If $|x^0 - y^0|$ is sufficiently small, (2.10) and (2.11) imply that the inverse of $\Gamma(x^0, y^0)$ exists. (2.8) can be obtained by (2.4) and the fact that

$$\Gamma(x^0, y^0)^{-1} = (\Gamma(x^0, y^0)^* \Gamma(x^0, y^0))^{-1} \Gamma(x^0, y^0)^* .$$

We can prove (2.9) in the same manner.

Now we are ready to construct the fundamental solution of the Dirac equation. Let

$$\Delta\colon y^0 = \tau^0 < \tau^1 < \cdots < \tau^N = x^0$$

be an arbitrary subdivision of the interval $[y^0, x^0]$, and let

$$|\Delta| = \max_{1 \leq j \leq N} |\tau^j - \tau^{j-1}| .$$

Then we put

$$\Gamma_\Delta(x^0, y^0) = \Gamma(x^0, \tau^{N-1})\Gamma(\tau^{N-1}, \tau^{N-2}) \cdots \Gamma(\tau^1, y^0)$$
$$\Gamma_\Delta(y^0, x^0) = \Gamma(y^0, \tau^1)\Gamma(\tau^1, \tau^2) \cdots \Gamma(\tau^{N-1}, x^0) ,$$

and discuss the convergence of $\Gamma_\Delta(x^0, y^0)$ and $\Gamma_\Delta(y^0, x^0)$ when $|\Delta| \to 0$.

THEOREM 1. *Under the assumption* (A), *there exist bounded operators* $U(x^0, y^0)$ *and* $U(y^0, x^0)$ *in* $\mathscr{H}$ *such that*

$$\|\Gamma_\Delta(x^0, y^0) - U(x^0, y^0)\| \longrightarrow 0 \quad (|\Delta| \longrightarrow 0)$$
$$\|\Gamma_\Delta(y^0, x^0) - U(y^0, x^0)\| \longrightarrow 0 \quad (|\Delta| \longrightarrow 0) .$$

Moreover, $U(x^0, y^0)$ *and* $U(y^0, x^0)$ *satisfy the following properties.*

(i) $U(x^0, y^0)$ *and* $U(y^0, x^0)$ *are unitary on* $\mathscr{H}$.

(ii) $U(\tau, \tau) = I$ *on* $\mathscr{H}$, *for* $\tau \in [y^0, x^0]$.

(iii) $U(x^0, \tau)U(\tau, y^0) = U(x^0, y^0)$.

(iv) $U(x^0, y^0)^{-1} = U(y^0, x^0)$.

The proof of this theorem is based on the estimates from (2.4) to (2.9), and it can be obtained in the same way as shown in Fujiwara [1].

We define a subset W of $\mathscr{H}$:

$$W = \{\varphi \in \mathscr{H} \mid \langle \vec{x} \rangle \varphi,\ D\varphi \in \mathscr{H}\} ,$$

which is a Hilbert space with the norm

$$\|\varphi\|_W^2=\int d\vec{x}\langle\vec{x}\rangle^2|\varphi(\vec{x})|^2+\sum_{|\alpha|=1}\int d\vec{x}|D^\alpha\varphi(\vec{x})|^2 .$$

PROPOSITION 2.7. *$U(x^0, y^0)$ maps the space W into itself, and the following estimate holds:*

$$\|U(x^0, y^0)\varphi\|_W\leqq\|\varphi\|_W \exp K_3|x^0-y^0| . \tag{2.12}$$

PROOF. Using (1.3) and applying Proposition 1.5, we obtain the estimates:

$$\|[D_j, \Gamma(x^0, y^0)]\varphi\|_W\leqq K_1|x^0-y^0|\ \|\varphi\|_W$$
$$\|[x_j, \Gamma(x^0, y^0)]\varphi\|_W\leqq K_1|x^0-y^0|\ \|\varphi\|_W$$

for $\varphi\in W$. By means of (2.7), we also have the following:

$$\|\Gamma_\Delta(x^0, y^0)\varphi\|_W\leqq\|\varphi\|_W \exp K_3|x^0-y^0| . \tag{2.13}$$

Consequently, the set $\{\Gamma_\Delta(x^0, y^0)\varphi\}$ remains bounded in W when $|\Delta|\to 0$. On the other hand, Theorem 1 proves that $\Gamma_\Delta(x^0, y^0)\varphi\to U(x^0, y^0)\varphi$ strongly in $\mathscr{H}$, and so we find that $U(x^0, y^0)\varphi\in W$ because W is continuously embedded in $\mathscr{H}$. Taking $|\Delta|\to 0$ in (2.13), we obtain (2.12).

Next, Theorem 2 shows that $U(x^0, y^0)$ is the fundamental solution of the Dirac equation.

THEOREM 2. *For $\varphi\in W$, we have*

$$U(x^0, y^0)\varphi-\varphi=i\int_{y^0}^{x^0}d\tau H(\tau)U(\tau, y^0)\varphi , \tag{2.14}$$

where the right hand side is taken as a Bochner integral, and

$$\frac{1}{i}\frac{d}{dx}U(x^0, y^0)\varphi=H(x^0)U(x^0, y^0)\varphi \tag{2.15}$$

$$\frac{1}{i}\frac{d}{dy^0}U(x^0, y^0)\varphi=-U(x^0, y^0)H(y^0)\varphi . \tag{2.16}$$

PROOF. Let $\varphi\in W$. Using (iii) of Theorem 1, we have

$$\begin{aligned}U(t+\tau, y^0)\varphi-U(t, y^0)\varphi&=(\Gamma(t+\tau, t)-I)U(t, y^0)\varphi\\&\quad+(U(t+\tau, t)-\Gamma(t+\tau, t))U(t, y^0)\varphi .\end{aligned} \tag{2.17}$$

By Proposition 2.3 and Theorem 1, the mapping: $t\mapsto U(t, y^0)\varphi$ is strongly continuous. Hence for $\psi\in C_0^\infty$, we have

$$(2.18)\qquad (U(t, y^0)\varphi, \psi)-(\varphi, \psi)=\int_{y^0}^{t} d\tau \frac{d}{d\tau}(U(\tau, y^0)\varphi, \psi) .$$

Let

$$\Delta: y^0=t^0<t^1<\cdots<t^n=t$$

be a subdivition of $[y^0, t]$ into subintervals of equal length, and we put

$$\Delta': y^0=t^0<t^1<\cdots<t^{n-1}$$

$$G(\tau, \sigma)=\left[\frac{1}{i}\frac{\partial}{\partial\tau}-H(\tau)\right]\Gamma(\tau, \sigma) .$$

Then we have

$$\frac{1}{i}\frac{d}{dt}\Gamma_\Delta(t, y^0)=H(t)\Gamma_\Delta(t, y^0)+G(t, t^{n-1})\Gamma_\Delta(t^{n-1}, y^0) ,$$

and by Proposition 2.3, we obtain the estimate

$$\left|\frac{1}{i}\frac{d}{dt}(\Gamma_\Delta(t, y^0)\varphi, \psi)-(H(t)\Gamma_\Delta(t, y^0)\varphi, \psi)\right| \leqq K_2|\Delta| \exp|t-y^0| \,\|\varphi\|\,\|\psi\| .$$

Letting $|\Delta|\to 0$, we have

$$(2.19)\qquad \frac{1}{i}\frac{d}{dt}(U(t, y^0)\varphi, \psi)=(H(t)U(t, y^0)\varphi, \psi)$$

by Theorem 1. Therefore, combining (2.18) and (2.19), we obtain

$$(U(x^0, y^0)\varphi, \psi)-(\varphi, \psi)=\int_{y^0}^{x^0} d\tau (H(\tau)U(\tau, y^0)\varphi, \psi) .$$

Since C_0^∞ is dense in $\mathscr{H}$, (2.14) is proved.

THEOREM 3. *For $\varphi\in W$, we have*

$$(2.20)\qquad [D_j, U(x^0, y^0)]\varphi=\int_{y^0}^{x^0} d\tau\, U(x^0, \tau)[D_j, H(\tau)]U(\tau, y^0)\varphi$$

$$(2.21)\qquad [x_j, U(x^0, y^0)]\varphi=\int_{y^0}^{x^0} d\tau\, U(x^0, \tau)[x_j, H(\tau)]U(\tau, y^0)\varphi ,$$

and that $U(x^0, y^0)$ is an isomorphism on $\mathscr{S}$.

PROOF. Let $\varphi\in C_0^\infty$. Then we have the equality

$$([D_j, U(x^0, y^0)]\varphi, \psi)=\int_{y^0}^{x^0} d\tau \frac{d}{d\tau}(U(x^0, \tau)D_j U(\tau, y^0)\varphi, \psi) .$$

Using (2.15), we obtain (2.20). We may derive (2.21) in the same manner as in (2.20) from (2.16). If we take $\varphi \in \mathscr{S}$, it is obvious from (2.20) and (2.21) that $D_j U(x^0, y^0)\varphi \in \mathscr{H}$ and $x_j U(x^0, y^0)\varphi \in \mathscr{H}$ because $[D_j, H(\tau)]$ and $[x_j, H(\tau)]$ are the operators from $\mathscr{H}$ into itself. Similar arguments will give $x^\alpha D^\beta U(x^0, y^0)\varphi \in \mathscr{H}$, for any pair of multi-indices α and β. So, $U(x^0, y^0)\varphi \in \mathscr{S}$ by virtue of the Sobolev embedding theorem. Also we can prove that $U(x^0, y^0)$ is a continuous map from $\mathscr{S}$ into itself by the closed graph theorem. Since $U(x^0, y^0)^{-1} = U(y^0, x^0)$, $U(x^0, y^0)$ is an isomorphism on $\mathscr{S}$.

§ 3. Appendix.

LEMMA 3.1. *Let $f(x)$ be smooth function on $\boldsymbol{R}^d$ with all the first derivatives in $\mathscr{B}(\boldsymbol{R}^d)$. Then there exists a family of functions $f_k(x, y) \in \mathscr{B}(\boldsymbol{R}^d \times \boldsymbol{R}^d)(k=1, 2, \cdots, d)$ such that*

$$f(x) = f(y) + \sum_{k=1}^{d} f_k(x, y)(x^k - y^k)$$

for $x, y \in \boldsymbol{R}^d$. Moreover the following estimate holds:

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha \left(\frac{\partial}{\partial y}\right)^\beta f_k(x, y)\right| \leqq \sup_{\substack{z \in \boldsymbol{R}^d \\ |\gamma| = |\alpha| + |\beta| + 1}} \left|\left(\frac{\partial}{\partial z}\right)^\gamma f(z)\right| \quad (x, y \in \boldsymbol{R}^d) .$$

LEMMA 3.2. *Let n be a positive integer and $f_k(x)(k=1, 2, \cdots, n)$ be smooth functions on $\boldsymbol{R}^d$ whose derivatives of the first order belong to $\mathscr{B}(\boldsymbol{R}^d)$. Then there exists a family of functions $f_{S,\nu}(x, y) \in \mathscr{B}(\boldsymbol{R}^d \times \boldsymbol{R}^d)$ $(S \subset \{1, 2, \cdots, n\}, |\nu| = \# S)$ such that*

$$f_1(x) \cdots f_n(x) = \sum_{\substack{S \subset \{1, \cdots, n\} \\ |\nu| = \# S}} f_{S,\nu}(x, y)(x-y)^\nu \prod_{k \in S} f_k(y)$$

for $x, y \in \boldsymbol{R}^d$.

THEOREM 3.3. (Asada-Fujiwara [12]) *Assume that* (i), (ii), (iii) *and* (iv) *hold.*

(i) *ϕ is a real valued C^∞ function of $(x, \xi, y) \in \boldsymbol{R}^n \times \boldsymbol{R}^m \times \boldsymbol{R}^n$.*

(ii) *There exists a positive constant δ such that*

$$|\det D(\phi)(x, \xi, y)| \geqq \delta ,$$

where $D(\phi)$ is the $(n+m)\times(n+m)$-matrix of the form

$$D(\phi)(x, \xi, y)=\begin{bmatrix} \frac{\partial^2}{\partial x \partial y}\phi(x, \xi, y) & \frac{\partial^2}{\partial x \partial \xi}\phi(x, \xi, y) \\ \frac{\partial^2}{\partial \xi \partial y}\phi(x, \xi, y) & \frac{\partial^2}{\partial \xi \partial \xi}\phi(x, \xi, y) \end{bmatrix}.$$

(iii) *For any multi-indices α, β, γ, there exists a positive constant $C_{\alpha\beta\gamma}$ such that*

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha}\left(\frac{\partial}{\partial \xi}\right)^{\beta}\left(\frac{\partial}{\partial y}\right)^{\gamma} d(\phi)(x, \xi, y)\right| \leqq C_{\alpha\beta\gamma}$$

where $d(\phi)(x, \xi, y)$ denotes each entries of $D(\phi)(x, \xi, y)$.

(iv) *$a(x, \xi, y)$ belongs to the space $\mathscr{B}^l(\boldsymbol{R}^n \times \boldsymbol{R}^m \times \boldsymbol{R}^n)$, where $l=l(n, m)$.*

Then, for $\varphi \in C_0^\infty$, the integral operator of the following type:

$$A\varphi(x)=\iint_{\boldsymbol{R}^n \times \boldsymbol{R}^m} a(x, \xi, y)e^{i\phi(x,\xi,y)}\varphi(y)dyd\xi$$

is L^2-bounded, i.e. there exists a positive constant K depending on $|a|_l$ such that

$$||A\varphi||_{L^2} \leqq k||\varphi||_{L^2} .$$

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References

[1] R. P. FEYNMAN, Space time approach to nonrelativistic quantum mechanics, Rev. Modern Phys., **20** (1948), 367-387.
[2] I. M. GELFAND and A. M. YAGLOM, Integrals in functional spaces and its applications in quantum physics, J. Math. Phys., **1** (1960), 48-69.
[3] R. H. CAMERON, A family of integrals serving to connect the Wiener and Feynman integrals, J. Mathematical and Physical Sci., **39** (1960), 126-140.
[4] E. NELSON, Feynman path integrals and Schrödinger's equation, J. Math. Phys., **5** (1964), 332-343.
[5] S. A. ALBEVERIO and R. J. HOEGH KROHN, Mathematical Theory of Feynman Path Integrals, Springer Lecture Noter of Math., **523**, Springer, Berlin, 1976.
[6] K. ITO, Generalized uniform complex measures in the Hilbertian metric space with their application to the Feynman path integral, Proc. 5th Berkeley Symposiun on Math., Statistics and Probability, **2**, Part 1 (1967), 145-161.
[7] D. FUJIWARA, A construction of the fundamental solution for the Schrödinger equation, J. Analyse Math., **35** (1979), 41-96.
[8] H. KITADA, On a construction of the fundamental solution for the Schrödinger equation, J. Fac. Sci. Univ. Tokyo Sect IA, **27** (1980), 193-226.
[9] K. O. FRIEDRICHS, Symmetric hyperbolic system of linear differential equations, Comm. Pure Appl. Math., **7** (1954).

[10] H. KUMANOGO, A calculus of Fourier integral operators on $\boldsymbol{R}^n$ and the fundamental solution for an operator of hyperbolic type, Comm. Partial Differential Equations, **1** (1976), 1–44.

[11] K. YAJIMA, The quasi-classical approximation to the Dirac equation I, J. Fac. Sci. Univ. Tokyo Sect IA, **29** (1982), 161–194.

[12] K. ASADA and D. FUJIWARA, On some oscillatory integral transformation in $L^2(\boldsymbol{R}^n)$, Japan J. Math., **4** (1978), 299–361.

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