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The Pseudo Orbit Tracing Properties on the Space of Probability Measures

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Introduction

Let X be a state space of some system and $M(X)$ the space of probability measures on X. The elements of $M(X)$ are viewed as statistical states. The elements of X are imbedded in $M(X)$ as the pure states. Let T be a transformation of X and $\tilde{T}$ the corresponding transformotion of $M(X)$. As against (X, T) is a dynamical system in classical mechanics, $(M(X), \tilde{T})$ can be viewed as one in classical statistical mechanics (cf. [1]).

If X is compact metric and T is continuous, then $M(X)$, provided with the weak topology, is again compact metric and $\tilde{T}$ is continuous. W. Bauer and K. Sigmund studied in [1] the problem of which of the properties of (X, T) (like distality, topologically mixing, expansiveness, etc...) carry over to $(M(X), \tilde{T})$. However they did not treat the pseudo orbit tracing property defined by R. Bowen [2]. The aim of this paper is to study the property of $(M(X), \tilde{T})$ induced by (X, T) which has the pseudo orbit tracing property.

§1. Definitions and results.

Let X be a compact metric space with metric d and $M(X)$ the space of Borel probability measures on X. The Prohorov metric $\tilde{d}$ on $M(X)$ is defined by $\tilde{d}(\mu, \nu)=\inf\{\varepsilon: \mu(A)\leqq\nu(A^{\varepsilon})+\varepsilon$ and $\nu(A)\leqq\mu(A^{\varepsilon})+\varepsilon$ for all Borel sets $A\subset X\}$ for $\mu, \nu\in M(X)$. Here $A^{\varepsilon}=\bigcup_{x\in A}\{y\in X: d(x, y)\leqq\varepsilon\}$. As V. Strassen showed in [7], one has $\tilde{d}(\mu, \nu)=\inf\{\varepsilon: \mu(A)\leqq\nu(A^{\varepsilon})+\varepsilon$ for all Borel sets $A\subset X\}$. The induced topology is just the weak topology for measures. It turns $M(X)$ into a compact space (cf. [5, P. 45]). For $x\in X$, let $\pi(x)\in M(X)$ be a point measure defined by $\pi(x)(A)=1$ if $x\in A$, $=0$ otherwise.

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π is a homeomorphism from X onto a closed subset of $M(X)$. It is obvious that $M(X)$ is convex and the point measures are just the extremal points of $M(X)$. Let $M_n(X)=\{\mu=(1/n)(\pi(x_1)+\cdots+\pi(x_n)),\ x_i\in X$ not necessarily distinct$\}$ for $n=1, 2, \cdots$. Then $M_n(X)$ is a closed subset of $M(X)$ and $\cup_{n\geq 1} M_n(X)$, the measures with finite support, is dense in $M(X)$.

Let T be a homeomorphism of X (i.e. from X onto itself). T induces a map $\tilde{T}: M(X)\to M(X)$ defined by $(\tilde{T}\mu)(A)=\mu(T^{-1}A)$ ($\mu\in M(X), A\subset X$; Borel). It is easy to see that $\tilde{T}$ is a homeomorphism of $M(X)$, sending $\pi(x)$ into $\pi(Tx)$. Clearly $M_n(X)$ $(n\geqq 1)$ is $\tilde{T}$-invariant. We denote the restriction of $\tilde{T}$ to $M_n(X)$ by same symbol $\tilde{T}$. It is well known that the set of T-invariant measures $\{\mu\in M(X)\colon \tilde{T}\mu=\mu\}$, which is just the set of fixed points of $\tilde{T}$, is a nonempty convex closed set ([3, P. 17]). Let $X_n=X\times\cdots\times X$ (n-times) and a metric d_n on X_n define by $d_n(\boldsymbol{x}, \boldsymbol{y})=\max_{1\leq i\leq n} d(x_i, y_i)$ for $\boldsymbol{x}=(x_1, \cdots, x_n)$, $\boldsymbol{y}=(y_1, \cdots, y_n)$ in X_n. T induces a homeomorphism T_n of X_n defined by $T_n(\boldsymbol{x})=(Tx_1, \cdots, Tx_n)$ $(\boldsymbol{x}=(x_1, \cdots, x_n)\in X_n)$.

A sequence $\{x_i\}_{i=a}^{b}$ $(-\infty\leqq a\leqq b\leqq\infty)$ in X is *δ-pseudo orbit* of T if $d(Tx_i, x_{i+1})<\delta$ for $a\leqq i\leqq b-1$. The δ-pseudo orbit $\{x_i\}_{i=a}^{b}$ is said to be *ε-traced* if there is $x\in X$ with $d(T^i x, x_i)\leqq\varepsilon$ $(a\leqq i\leqq b)$. (X, T) has *the pseudo orbit tracing property* (abbrev. P.O.T.P.) if for every $\varepsilon>0$ there is $\delta>0$ such that every δ-pseudo orbit of T is ε-traced. (X, T) is *tracing* if for a sequence $\{x_i\}_{i=0}^{\infty}$ with $\lim_{i\to\infty} d(Tx_i, x_{i+1})=0$ there is $x\in X$ with $\lim_{i\to\infty} d(T^i x, x_i)=0$. In general the tracing does not imply the P.O.T.P.. (X, T) is said to be *T-connected* if for every $x, y\in X$ and every $\alpha>0$ there are α-pseudo orbits $\{x_i\}_{i=0}^{a}$ and $\{y_i\}_{i=0}^{b}$ so that $x_0=x=y_b$ and $y_0=y=x_a$. (X, T) is *topologically mixing* if for every nonempty open sets U and V of X there is $N>0$ such that $U\cap T^{-n}V\neq\emptyset$ for every $n\geqq N$. (X, T) satisfies *specification* if for every $\varepsilon>0$ there is $M=M(\varepsilon)>0$ such that for every $k\geqq 1$ and k points $x_1, \cdots, x_k\in X$, for every set of integers $a_1\leqq b_1<\cdots<a_k\leqq b_k$ with $a_{i+1}-b_i\geqq M$ $(1\leqq i\leqq k-1)$ and for every $p\geqq b_k-a_1+M$ there is $x\in X$ with $d(T^n x, T^n x_i)\leqq\varepsilon$ $(a_i\leqq n\leqq b_i, 1\leqq i\leqq k-1)$ and $T^p x=x$. (X, T) satisfies *weak specification* if (X, T) satisfies the definition of specification except the periodic condition; $T^p x=x$. Our results are following.

THEOREM 1. *Let T be a homeomorphism of a compact metric space X and $\tilde{T}$ an induced homeomorphism of $M(X)$. Then the following holds.*

(1) *If (X, T) has the P.O.T.P., then $(M_n(X), \tilde{T})$ has also the P.O.T.P. for every $n\geqq 1$.*

(2) *If (X, T) is tracing, then $(M_n(X), \tilde{T})$ is also tracing for every* $n \geqq 1$.

THEOREM 2. *Let $\tilde{T}$ be as in Theorem 1. The following holds.*

(1) *If $(M(X), \tilde{T})$ is tracing, then $(M(X), \tilde{T})$ has the P.O.T.P..*

(2) *If $(M(X), \tilde{T})$ has the P.O.T.P., then $(M(X), \tilde{T})$ satisfies specification.*

REMARK. By Theorem 2(2) it follows that if $(M(X), \tilde{T})$ has the P.O.T.P. (resp. tracing), then (X, T) is topologically mixing (cf. Propositions 21.3 and 6.9 in [3]). There is (X, T) which has the P.O.T.P. (resp. tracing) but is not topologically mixing (for example $X=\{0, 1\}$ and $T=id.$), so $(M(X), \tilde{T})$ need not have the P.O.T.P. (resp. tracing) even if (X, T) has the P.O.T.P. (resp. tracing).

§2. Proof of Theorem 1.

For $n \geqq 1$, let us define a map $\varphi_n: X_n \to M_n(X)$ by $\varphi_n(\boldsymbol{x})=(1/n)\sum_{i=1}^{n}\pi(x_i)$ $(\boldsymbol{x}=(x_1, \cdots, x_n) \in X_n)$. Clearly φ_n is a continuous surjection at most $n!$ to one. Moreover φ_n satisfies following;

LEMMA 1. *For every $\boldsymbol{x}, \boldsymbol{y} \in X_n$, $\tilde{d}(\varphi_n(\boldsymbol{x}), \varphi_n(\boldsymbol{y})) \leqq d_n(\boldsymbol{x}, \boldsymbol{y})$.*

PROOF. Let $\boldsymbol{x}=(x_1, \cdots, x_n)$, $\boldsymbol{y}=(y_1, \cdots, y_n) \in X_n$ be given. Put $d_n(\boldsymbol{x}, \boldsymbol{y})=c$, then $d(x_i, y_i) \leqq c$ $(1 \leqq i \leqq n)$. For every Borel set $A \subset X$ if $y_i \in A$ then $x_i \in A^c$. Hence we have $\varphi_n(\boldsymbol{x})(A^c) \geqq \varphi_n(\boldsymbol{y})(A)$, and so by the definition of $\tilde{d}$, $\tilde{d}(\varphi_n(\boldsymbol{x}), \varphi_n(\boldsymbol{y})) \leqq c = d_n(\boldsymbol{x}, \boldsymbol{y})$.

LEMMA 2 (marriage lemma [4]). *Let $B=\{b_1, \cdots, b_n\}$ and $G=\{g_1, \cdots, g_n\}$ be finite sets of cardinal n and $P(G)$ the family of subsets of G. Let Ψ be a map from B into $P(G)$. If Ψ satisfies that $\# \cup_{b \in E}\{\Psi(b)\} \geqq \# E$ for every subset E of B, then there is a permutation $\sigma: \{1, \cdots, n\} \to \{1, \cdots, n\}$ such that $g_{\sigma(i)} \in \Psi(b_i)$ $(1 \leqq i \leqq n)$.*

LEMMA 3. *Let $n \geqq 1$ and $0<\delta<1/n$ be given. For $\mu \in M_n(X)$, take $\boldsymbol{x} \in X_n$ with $\varphi_n(\boldsymbol{x})=\mu$. Then for every $\nu \in M_n(X)$ with $\tilde{d}(\mu, \nu)<\delta$ there is $\boldsymbol{y} \in X_n$ such that $\varphi_n(\boldsymbol{y})=\nu$ and $d_n(\boldsymbol{x}, \boldsymbol{y})<\delta$.*

PROOF. Take $\delta_0>0$ with $\tilde{d}(\mu, \nu)<\delta_0<\delta$ $(<1/n)$. We express $\boldsymbol{x}=(x_1, \cdots, x_n) \in X_n$ and $\nu=(1/n)\sum_{i=1}^{n}\pi(z_i)$. To distinguish between x_i or z_i $(1 \leqq i \leqq n)$ we put $S=\{1, \cdots, n\}$ and $x'_i=(x_i, i)$, $z'_i=(z_i, i) \in X \times S$ $(1 \leqq i \leqq n)$. For $B=\{x'_1, \cdots, x'_n\}$ and $G=\{z'_1, \cdots, z'_n\}$ we define a map $\Psi: B \to P(G)$ by $\Psi(x'_i)=\{z'_j \in G: d(\tau(x'_i), \tau(z'_j)) \leqq \delta_0\}$ $(1 \leqq i \leqq n)$ where $P(G)$ is the family of subsets of G and $\tau: X \times S \to X$ is a natural projection. In order to apply

Lemma 2 we show that $\# \cup_{x'_i \in E} \Psi(x'_i) \geqq E$ for every subset E of B. Indeed, by $\tilde{d}(\mu, \nu)<\delta_0$, it follows that $\nu((\tau E)^{\delta_0}) \geqq \mu(\tau E)-\delta_0$ where $(\tau E)^{\delta_0}=\{x \in X: d(x, \tau E) \leqq \delta_0\}$. Since $\mu(\tau E)=\# E/n$ and $\delta_0<1/n$, we have $\nu((\tau E)^{\delta_0})>(1/n)(\# E-1)$. For every Borel set $A \subset X$, $\nu(A)=(1/n)(\sum_{i=1}^n \pi(z_i))(A)=k/n$ for some $0 \leqq k \leqq n$, so $(1/n)(\sum_{i=1}^n \pi(z_i)(\tau E)^{\delta_0}) \geqq \# E/n$. Since $\# \cup_{x'_i \in E}\{\Psi(x'_i)\}=\#\{z'_j \in G: d(\tau x'_i, \tau z'_j) \leqq \delta_0, x'_i \in E\}=\sum_{i=1}^n \pi(z_i)((\tau E)^{\delta_0})$, we get $\# \cup_{x'_i \in E}\{\Psi(x'_i)\} \geqq \# E$.

Hence by Lemma 2 there is a permutation $\sigma: S \rightarrow S$ with $\Psi(x'_i)=z'_{\sigma(i)}$ $(1 \leqq i \leqq n)$. Put $y_i=z_{\sigma(i)}$ and $\boldsymbol{y}=(y_1, \cdots, y_n) \in X_n$. Then we have $\varphi_n(\boldsymbol{y})=(1/n) \sum_{i=1}^n \pi(y_i)=(1/n) \sum_{i=1}^n \pi(z_{\sigma(i)})=\nu$ and $d_n(\boldsymbol{x}, \boldsymbol{y})=\max_{1 \leqq i \leqq n} d(x_i, z_{\sigma(i)})=\max_{1 \leqq i \leqq n} d(\tau x'_i, \tau z'_{\sigma(i)}) \leqq \delta_0<\delta$.

LEMMA 4. (1) *If (X, T) has the P.O.T.P., then (X_n, T_n) has the P.O.T.P. for every $n \geqq 1$. More precisely, given $\varepsilon>0$ if there is $\delta>0$ such that every δ-pseudo orbit of T is ε-traced, then every δ-pseudo orbit of T_n is ε-traced.*

(2) *If (X, T) is tracing, then so is (X_n, T_n).*

PROOF. (1): Let $\varepsilon>0$ be given and $\delta>0$ be a number decided by the P.O.T.P. of (X, T) corresponding with ε. Let $\{\boldsymbol{x}^k\}_{k=-\infty}^{\infty}$ be a δ-pseudo orbit of (X_n, T_n). If one denotes $\boldsymbol{x}^k=(x_1^k, \cdots, x_n^k)$ $(k \in \boldsymbol{Z})$ then $\{x_i^k\}_{k=-\infty}^{\infty}$ $(1 \leqq i \leqq n)$ is a δ-pseudo orbit of (X, T) because $d(T x_i^k, x_i^{k+1}) \leqq d_n(T_n \boldsymbol{x}^k, \boldsymbol{x}^{k+1})<\delta$ $(1 \leqq i \leqq n, k \in \boldsymbol{Z})$. By assumption, there is $y_i \in X$ $(1 \leqq i \leqq n)$ with $d(T^k y_i, x_i^k) \leqq \varepsilon$ $(k \in \boldsymbol{Z})$. Put $\boldsymbol{y}=(y_1, \cdots, y_n)$ then $d_n(T_n^k \boldsymbol{y}, \boldsymbol{x}^k)=\max_{1 \leqq i \leqq n} d(T^k y_i, x_i^k) \leqq \varepsilon$; i.e. $\boldsymbol{y} \in X_n$ ε-traces $\{\boldsymbol{x}^k\}_{k=-\infty}^{\infty}$.

(2): Let a sequence $\{\boldsymbol{x}^k\}_{k=0}^{\infty}$ of X_n satisfy that $\lim_{k \rightarrow \infty} d_n(T_n \boldsymbol{x}^k, \boldsymbol{x}^{k+1})=0$. If one denotes $\boldsymbol{x}^k=(x_1^k, \cdots, x_n^k)$ $(k \in \boldsymbol{Z})$ then it follows that $\lim_{k \rightarrow \infty} d(T x_i^k, x_i^{k+1}) \leqq \lim_{k \rightarrow \infty} d_n(T_n \boldsymbol{x}^k, \boldsymbol{x}^{k+1})=0$ $(1 \leqq i \leqq n)$, so there is $y_i \in X$ with $\lim_{k \rightarrow \infty} d(T^k y_i, x_i^k)=0$. Putting $\boldsymbol{y}=(y_1, \cdots, y_n)$ we have $d_n(T_n^k \boldsymbol{y}, \boldsymbol{x}^k)=\max_{1 \leqq i \leqq n} d(T^k y_i, x_i^k) \rightarrow 0$ $(k \rightarrow \infty)$. This proves the lemma.

PROOF OF THEOREM 1. (1): Let $n \geqq 1$ and $\varepsilon>0$ be given. By assumption there is $0<\delta<1/n$ such that every δ-pseudo orbit of (X, T) is $\varepsilon/2$-traced. Then, by Lemma 4(1), every δ-pseueo ordit of (X_n, T_n) is $\varepsilon/2$-traced. At first we show that for every $m>0$, every finite δ-pseudo orbit $\{\nu_k\}_{k=0}^m$ of $(M_n(X), \tilde{T})$ is $\varepsilon/2$-traced. Since φ_n is surjective there is $\boldsymbol{x}^0=(x_1^0, \cdots, x_n^0) \in X_n$ with $\varphi_n(\boldsymbol{x}^0)=\nu_0$. Then, since $\tilde{d}(\tilde{T} \nu_0, \nu_1)<\delta$, by Lemma 3 we can find $\boldsymbol{x}^1=(x_1^1, \cdots, x_n^1) \in X_n$ such that $\varphi_n(\boldsymbol{x}^1)=\nu_1$ and $d_n(T_n \boldsymbol{x}^0, \boldsymbol{x}^1)<\delta$. As $\tilde{d}(\tilde{T} \nu_1, \nu_2)<\delta$, by Lemma 3 we have again $\boldsymbol{x}^2=(x_1^2, \cdots, x_n^2) \in X_n$ such that $\varphi_n(\boldsymbol{x}^2)=\nu_2$ and $d_n(T_n \boldsymbol{x}^1, \boldsymbol{x}^2)<\delta$. Repeated this process we get a δ-pseudo orbit $\{\boldsymbol{x}^k\}_{k=0}^m$ of (X_n, T_n) with $\varphi_n(\boldsymbol{x}^k)=\nu_k$ $(0 \leqq k \leqq m)$. Hence there is $\boldsymbol{y}=(y_1, \cdots, y_n) \in X_n$ which $\varepsilon/2$-traces $\{\boldsymbol{x}^k\}_{k=0}^m$. Put $\mu=(1/n) \sum_{i=1}^n \pi(y_i) \in M_n(X)$, then we have

$$\tilde{d}(\tilde{T}^k\mu, \nu_k)=\tilde{d}(\varphi_n(T_n^k y), \varphi_n(x^k))\leqq d_n(T_n^k y, x^k)\leqq \varepsilon/2 \quad \text{(by Lemma 1)};$$

i.e. $\{\nu_k\}_{k=0}^m$ is $\varepsilon/2$-traced by μ.

Now let $\{\nu_k\}_{k=-\infty}^{\infty}$ be a δ-pseudo orbit of $(M_n(X), \tilde{T})$. For every $m>0$ let us put $\nu'_j=\nu_{j-m}$ $(0\leqq j\leqq 2m)$. Then by the above argument there is $\mu'_m\in M_n(X)$ which $\varepsilon/2$-traces $\{\nu'_j\}_{j=0}^{2m}$. Put $\mu_m=\tilde{T}^m\mu'_m$ and take a limit point μ of sequence μ_m. Then, since for every $k\in \boldsymbol{Z}$ there is $m>|k|$ so that $\tilde{d}(\tilde{T}^k\mu_m, \tilde{T}^k\mu)\leqq\varepsilon/2$, we get $\tilde{d}(\tilde{T}^k\mu, \nu_k)\leqq\tilde{d}(\tilde{T}^k\mu, \tilde{T}^k\mu_m)+\tilde{d}(\tilde{T}^{k+m}\mu'_m, \nu'_{k+m})\leqq\varepsilon/2+\varepsilon/2=\varepsilon$. This shows that $(M_n(X), \tilde{T})$ has the P.O.T.P..

(2): Let $n\geqq 1$ be given and $\{\nu_k\}_{k=0}^{\infty}$ be a sequence of $M_n(X)$ with $\lim_{k\to\infty}\tilde{d}(\tilde{T}\nu_k, \nu_{k+1})=0$. Without loss of generality we may assume $\tilde{d}(\tilde{T}\nu_k, \nu_{k+1})<1/n$ $(k\geqq 0)$. By same argument as in the proof of (1) we get a sequence $\{x^k\}_{k=0}^{\infty}$ of X_n such that $\varphi_n(x^k)=\nu_k$ $(k\geqq 0)$ and $\lim_{k\to\infty} d_n(T_n x^k, x^{k+1})=0$. Since (X_n, T_n) is tracing (by Lemma 4(2)), there is $x\in X_n$ with $\lim_{k\to\infty} d_n(T_n^k x, x^k)=0$. Then for $\mu=\varphi_n(x)$ it follows that $\tilde{d}(\tilde{T}^k\mu, \nu_k)=\tilde{d}(\varphi_n(T_n^k x), \varphi_n(x^k))\leqq d_n(T_n^k x, x^k)\to 0$ $(k\to\infty)$. The proof is completed.

§3. Proof of Theorem 2.

We shall prove Theorem 2 by a series of lemmas.

LEMMA 5. *Let μ, ν and μ_i $(i=1, 2)$ belong to $M(X)$.*
(1) $\tilde{d}(\alpha\mu+(1-\alpha)\nu, \beta\mu+(1-\beta)\nu)\leqq\beta-\alpha$ *for* $0\leqq\alpha\leqq\beta\leqq 1$.
(2) $\tilde{d}(\alpha\mu_1+(1-\alpha)\mu_2, \nu)\leqq\max\{\tilde{d}(\mu_1, \nu), \tilde{d}(\mu_2, \nu)\}$ *for* $0\leqq\alpha\leqq 1$.

PROOF. (1): Let a Borel set $A\subset X$ be given. Then we calculate

$$\begin{aligned}&(\beta\mu+(1-\beta)\nu)(A^{\beta-\alpha})+(\beta-\alpha)-(\alpha\mu+(1-\alpha)\nu)(A)\\ &\quad=\alpha(\mu(A^{\beta-\alpha})-\mu(A))+(1-\alpha)(\nu(A^{\beta-\alpha})-\nu(A))\\ &\qquad+(\beta-\alpha)(1+\mu(A^{\beta-\alpha})-\nu(A^{\beta-\alpha}))\geqq 0\end{aligned}$$

where $A^{\beta-\alpha}=\bigcup_{x\in A}\{y\in X: d(x, y)\leqq\beta-\alpha\}$. From this we have conclusion.

(2): Put $c=\max\{\tilde{d}(\mu_1, \nu), \tilde{d}(\mu_2, \nu)\}$ and take $\varepsilon>c$. For every Borel set $A\subset X$, since $\mu_i(A^{\varepsilon})+\varepsilon-\nu(A)\geqq 0$ $(i=1, 2)$, we have

$$\begin{aligned}&(\alpha\mu_1+(1-\alpha)\mu_2)(A^{\varepsilon})+\varepsilon-\nu(A)\\ &\quad=\alpha(\mu_1(A^{\varepsilon})+\varepsilon-\nu(A))+(1-\alpha)(\mu_2(A^{\varepsilon})+\varepsilon-\nu(A))\geqq 0\,.\end{aligned}$$

Therefore $\tilde{d}(\alpha\mu_1+(1-\alpha)\mu_2, \nu)\leqq\inf\{\varepsilon: c<\varepsilon\}=c$, proving the lemma.

LEMMA 6. *Let T be a homeomorphism of X and $\tilde{T}$ an induced homeomorphism of $M(X)$. Then $(M(X), \tilde{T})$ is $\tilde{T}$-connected.*

PROOF. Let $\mu, \nu\in M(X)$ and $\alpha>0$ be given. Take $n\geqq 1$ with $1/n<\alpha$.

For $0 \leqq i \leqq n$ let us define μ_i and ν_i by $\mu_i = (1-(i/n))\tilde{T}^i\mu + (i/n)\tilde{T}^{i-n}\nu$ and $\nu_i = (1-(i/n))\tilde{T}^i\nu + (i/n)\tilde{T}^{i-n}\mu$. Obviously $\mu_0 = \mu = \nu_n$ and $\nu_0 = \nu = \mu_n$. By Lemma 5(1) we have

$$\tilde{d}(\tilde{T}\mu_i, \mu_{i+1}) = \tilde{d}\Big(\Big(1-\frac{i}{n}\Big)\tilde{T}^{i+1}\mu + \frac{i}{n}\tilde{T}^{i+1-n}\nu, \Big(1-\frac{i+1}{n}\Big)\tilde{T}^{i+1}\mu + \frac{i+1}{n}\tilde{T}^{i+1-n}\nu\Big)$$
$$\leqq \frac{i+1}{n} - \frac{i}{n} < \alpha \quad (0 \leqq i \leqq n-1)$$

and similarly $\tilde{d}(\tilde{T}\nu_i, \nu_{i+1}) < \alpha$ $(0 \leqq i \leqq n-1)$. This proves the lemma.

LEMMA 7. *If (X, T) is tracing and T-connected, then (X, T) has the P.O.T.P..*

PROOF. Suppose the lemma is false. Then there is $\varepsilon > 0$ such that for every $k \geqq 1$ there is a $(1/k)$-pseudo orbit $\{x_1^k, \cdots, x_{N_k}^k\}$ of (X, T) such that there is no $z \in X$ with $d(T^j z, x_j^k) \leqq \varepsilon$ $(1 \leqq j \leqq N_k)$. By T-connectedness, there is a $(1/k)$-pseudo orbit $\{z_0^k, \cdots, z_{L_k}^k\}$ with $z_0^k = x_{N_k}^k$ and $z_{L_k}^k = x_1^{k+1}$ $(k \geqq 1)$. Renewing the indices of a sequence

$$\{\cdots, x_1^k, \cdots, x_{N_k}^k, z_1^k, \cdots, z_{N_k-1}^k, x_1^{k+1}, \cdots\},$$

we have a sequence $\{x_n\}_{n=0}^{\infty}$ which satisfies $\lim_{n\to\infty} d(Tx_n, x_{n+1}) = 0$. Since (X, T) is tracing, there is $z \in X$ with $\lim_{n\to\infty} d(T^n z, x_n) = 0$. Therefore for some $k > 0$, $\{x_i^k\}_{i=1}^{N_k}$ is ε-traced. This is a contradiction.

If $(M(X), \tilde{T})$ is tracing, by Lemmas 6 and 7, $(M(X), \tilde{T})$ has the P.O.T.P.. This prove Theorem 2(1). Next we show Theorem 2(2).

LEMMA 8. *Assume (X, T) is T-connected and has the P.O.T.P.. If the set of fixed points under T is nonempty, then (X, T) is topologically mixing.*

PROOF. Let $U, V \subset X$ be nonempty open sets. There are $x \in U$, $y \in V$ and $\varepsilon > 0$ such that $B(x; \varepsilon) \subset U$ and $B(y; \varepsilon) \subset V$, where $B(z; \varepsilon) = \{z' \in X: d(z, z') \leqq \varepsilon\}$. Since (X, T) has the P.O.T.P., there is $\delta > 0$ so that every δ-pseudo orbit of (X, T) is ε-traced. Take a fixed point $p \in X$ under T. By T-connectedness, there are δ-pseudo orbits $\{x_i\}_{i=0}^{a}$ and $\{y_i\}_{i=0}^{b}$ such that $x_0 = x$, $x_a = p = y_0$ and $y_b = y$. Put $N = a + b + 1 \geqq 0$. Given $n \geqq N$, since a sequence

$$\{x_0, x_1, \cdots, x_a, p, \cdots, p \ (n-N \text{ times}), y_0, \cdots, y_b\}$$

is a δ-pseudo orbit, there is $z \in X$ which ε-traces this sequence. As $d(x, z) = d(x_0, z) \leqq \varepsilon$ and $d(y, T^n z) = d(y_b, T^n z) \leqq \varepsilon$, we have $z \in B(x; \varepsilon) \cap T^{-n}B(y; \varepsilon) \subset U \cap T^{-n}V$; i.e. (X, T) is topologically mixing.

LEMMA 9. *If (X, T) is topologically mixing and has the P.O.T.P., then (X, T) satisfies weak specification.*

PROOF. Let $\varepsilon>0$ be given. There is $\delta>0$ such that every δ-pseudo orbit is ε-traced. By topological mixing and the compactness of X, there is $M>0$ such that for every $x, y \in X$, $T^n B(x;\delta)\cap B(y;\delta)\neq\emptyset$ for every $n\geqq M$, where $B(z;\delta)=\{z'\in X: d(z,z')\leqq\delta\}$. For every $k\geqq 1$, let k points $x_1, \cdots, x_k \in X$ and a set of integers $a_1\leqq b_1<\cdots<a_k\leqq b_k$ with $a_{i+1}-b_i\geqq M$ $(1\leqq i\leqq k-1)$ be given. By choice of M, there are $y_i\in B(T^{b_i}x_i;\delta)$ $(1\leqq i\leqq k-1)$ with $T^{a_{i+1}-b_i}y_i\in B(T^{a_{i+1}}x_{i+1};\delta)$. Then a sequence $\{T^{a_1}x_1, \cdots, T^{b_1-1}x_1, y_1, \cdots, T^{a_2-b_1}y_1, T^{a_2}x_2, \cdots, T^{b_k}x_k\}$ is δ-pseudo orbit, so there is $z\in X$ with $d(T^n z, T^n x_i)\leqq\varepsilon$ $(a_i\leqq n\leqq b_i, 1\leqq i\leqq k)$. This proves the Lemma.

We remarked in §1 that the set of fixed points of $(M(X), \tilde{T})$ is non-empty. Hence if $(M(X), \tilde{T})$ has the P.O.T.P., by Lemmas 6, 8 and 9, $(M(X), \tilde{T})$ satisfies weak specification. Therefore the proof of Theorem 2(2) is completed by the following lemma.

LEMMA 10. *If $(M(X), \tilde{T})$ satisfies weak specification, then $(M(X), \tilde{T})$ satisfies specification.*

PROOF. Let $\varepsilon>0$ be given and $M=M(\varepsilon)>0$ an integer determined from the definition of weak specification. For every $k\geqq 1$, let k points $\mu_1, \cdots, \mu_k\in M(X)$ and a set of integers $a_1\leqq b_1<\cdots<a_k\leqq b_k$ with $a_{i+1}-b_i\geqq M$ $(1\leqq i\leqq k-1)$ and $p\geqq b_k-a_1+M$ be given. Using weak specification, we can find a sequence $\nu_j\in M(X)$ $(j=1,2,\cdots)$ such that $\tilde{d}(\tilde{T}^{n+mp}\nu_j, \tilde{T}^n\mu_i)\leqq\varepsilon$ $(a_i\leqq n\leqq b_i, 1\leqq i\leqq k, 0\leqq m\leqq j)$. Take a limit point ν of the sequence ν_j. Then it follows that $\tilde{d}(\tilde{T}^{n+mp}\nu, \tilde{T}^n\mu_j)\leqq\varepsilon$ $(a_i\leqq n\leqq b_i, 1\leqq i\leqq k, m\geqq 0)$. Put $\mu=\lim_{s\to\infty}(1/s)\sum_{m=0}^{s-1}\tilde{T}^{mp}\nu$ for some $\{s\}$. By Lemma 5(2), we have for $a_i\leqq n\leqq b_i$ $(1\leqq i\leqq k)$,

$$\begin{aligned}\tilde{d}(\tilde{T}^n\mu, \tilde{T}^n\mu_i)&=\lim_{s\to\infty}\tilde{d}\Big((1/s)\sum_{m=0}^{s-1}\tilde{T}^{n+mp}\nu, \tilde{T}^n\mu_i\Big)\\&\leqq\lim_{s\to\infty}\max\{\tilde{d}(\tilde{T}^{n+mp}\nu, \tilde{T}^n\mu_i); 0\leqq m\leqq s-1\}\leqq\varepsilon .\end{aligned}$$

Also, by Lemma 5(1), we have $\tilde{d}(\tilde{T}^p\mu, \mu)\leqq\lim_{s\to\infty}\{\tilde{d}(s^{-1}\sum_{m=1}^{s}\tilde{T}^{mp}\nu, (s-1)^{-1}\times\sum_{m=1}^{s-1}\tilde{T}^{mp}\nu)+\tilde{d}((s-1)^{-1}\sum_{m=1}^{s-1}\tilde{T}^{mp}\nu, s^{-1}\sum_{m=0}^{s-1}\tilde{T}^{mp}\nu)\}\leqq\lim_{s\to\infty}2/s=0$, hence $\tilde{T}^p\mu=\mu$. This completes the proof.

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