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Representation Theory of Weyl Group of Type C_n

Hiroshi NARUSE

University of Tokyo
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Introduction

The irreducible representations of the Weyl group of type C_n, which we denote by $W(C_n)$, is well known. They are constructed for example using the fact that $W(C_n)$ is the semi direct product of the symmetric group $\mathfrak{S}_n$ and an elementary abelian group of even order (cf. [1]). In this paper we take another approach. We use quite similar technique to construct irreducible representations to that of the symmetric group as in [2]. That is, our approach is to use two disjoint subgroups (horizontal and vertical) and their linear characters (Theorem 2). The irreducible characters are then also constructed explicitly using Schur function (Theorem 8). From this construction we deduce the analogue of Nakayama's formula for $W(C_n)$ to calculate the character value using Young diagram (Theorem 9). We also construct multiplicity formula in the induced representation of linear character of subgroup of type $W_B \cong \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_r} \times W(C_{\mu_1}) \times \cdots \times W(C_{\mu_s})$ (Theorem 7). As an application we determine the I-set of W-graph corresponding to the irreducible representation of $W(C_n)$ (Theorem 11). This application seems to show some significance of our method.

§1. The construction of irreducible representations.

1.1 Let $W = W(C_n)$ be the Weyl group of type C_n. To begin with, we realize W as a subgroup of $\mathfrak{S}_\Omega$, the symmetric group on the set Ω where $\Omega = \{1, 2, \cdots, n, -n, \cdots, -2, -1\}$ as follows:

$$W = \{x \in \mathfrak{S}_\Omega | x w_0 = w_0 x\}$$

where $w_0 = (1, -1)(2, -2) \cdots (n, -n) \in \mathfrak{S}_\Omega$. For x in W, we decompose x into cyclic permutations as an element of $\mathfrak{S}_\Omega$. Then we get two kinds

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of cycles in x:

(i) even cycle $(a_1, a_2, \cdots, a_r)$ $|a_i| \neq |a_j|$ if $i \neq j$,

(ii) odd cyle $(a_1, a_2, \cdots, a_r, -a_1, -a_2, \cdots, -a_r)$.

If an even cycle $(a_1, a_2, \cdots, a_r)$ appears in x, then even cycle, "the minus of $(a_1, a_2, \cdots, a_r)$", $(-a_1, -a_2, \cdots, -a_r)$ also appears in x since $xw_0 = w_0x$.

Let α_i be the number of even-cycle-pairs of length i of x, and β_j be the number of odd cycles of length $2j$ of x. Then we define $(\alpha; \beta)_x$ or simply $(\alpha; \beta)$ by

$$(\alpha; \beta) = (1^{\alpha_1}, 2^{\alpha_2}, \cdots; 1^{\beta_1}, 2^{\beta_2}, \cdots)$$

and call it the cycle type of x. Then one has obviously from the definition

$$\sum_i i\alpha_i + \sum_j j\beta_j = n .$$

LEMMA 1. *For x, y in W, x is W-conjugate to y if and only if x and y have the same cycle type.*

PROOF. Easy.

Therefore

PROPOSITION 1. *The number of conjugacy classes in W is*

$$\sum_{0 \leq r \leq n} p(r) \cdot p(n-r)$$

where $p(r)$ is the number of partitions of r.

PROOF. Similar as in the case of $\mathfrak{S}_n$.

NOTATIONS.

$$\mathscr{P}_n = \{(\lambda, \mu) \mid \lambda \text{ and } \mu \text{ are partitions and } |\lambda| + |\mu| = n\} ,$$

$$\mathscr{D}_n = \left\{ D = (D_1, D_2) \;\middle|\; \begin{array}{l} D_1 \text{ is a Young diagram of type } \lambda \\ D_2 \text{ is a Young diagram of type } \mu \\ (\lambda, \mu) \in \mathscr{P}_n \end{array} \right\} ,$$

$$BTab_n^{\pm} = \left\{ B = (B_1, B_2) \;\middle|\; \begin{array}{l} B_1 \text{ is a Young tableau of type } \lambda \\ B_2 \text{ is a Young tableau of type } \mu \\ (\lambda, \mu) \in \mathscr{P}_n \\ \text{and their entries are 1 to } n \text{ up} \\ \text{to sign} \end{array} \right\} .$$

e.g. if $D=\left(\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline & \\ \hline\end{array}\right)$, then $BTab_5^{\pm}$ consists of the following pairs.

$$\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline -3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 4 & 5 \\ \hline\end{array}\right), \left(\begin{array}{|c|c|}\hline -2 & -1 \\ \hline 5 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline -4 & 3 \\ \hline\end{array}\right), \left(\begin{array}{|c|c|}\hline 5 & -2 \\ \hline 4 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 3 & -1 \\ \hline\end{array}\right), \text{ etc.}$$

We define the action of W on $BTab_n^{\pm}$ as follows. For $B=(B_1, B_2)$ in $BTab_n^{\pm}$ and x in W, $xB=(B_1', B_2')$ where B_r and B_r' are the same type and $B_r'(i, j)=x\cdot B_r(i, j)$, $r=1, 2$. For $B=(B_1, B_2)$ in $BTab_n^{\pm}$, we define $\bar{B}$ by $\bar{B}=({}^tB_2, {}^tB_1)$ in $BTab_n^{\pm}$ where ${}^tB_r$ is the transpose of B_r, $r=1, 2$. Then we define $\mathscr{A}_{\lambda_i}$, $\mathscr{C}_{\mu_j}$ for a given B as follows. Let $B=(B_1, B_2)$, where type of B_1 is $=(\lambda_1, \cdots, \lambda_r)$, type of B_2 is $=(\mu_1, \cdots, \mu_s)$. Let $\mathscr{A}_{\lambda_i}$ be the subgroup of W whose elements permute entries of i-th row of B_1 and stabilize other part. Let $\mathscr{C}_{\mu_j}$ be the subgroup of W which is generated by the permutation of entries in the j-th row of B_2 and transpositions $(k, -k)$, k is in the j-th row of B_2. We define a subgroup W_B of W by

$$W_B=\mathscr{A}_{\lambda_1}\times\cdots\times\mathscr{A}_{\lambda_r}\times\mathscr{C}_{\mu_1}\times\cdots\times\mathscr{C}_{\mu_s}\subset W$$

which is isomorphic to $\mathfrak{S}_{\lambda_1}\times\cdots\times\mathfrak{S}_{\lambda_r}\times W(C_{\mu_1})\times\cdots\times W(C_{\mu_s})$. We call W_B the horizontal permutation subgroup associated to B, and $W_{\bar{B}}$ the vertical permutation subgroup associated to B.

1.2 To construct irreducible representations of W, we use following

THEOREM 1 (c.f. [2; Th. 1.23, 1.24]). *Let G be a finite group and H_1, H_2 be two subgroups of G. φ_i is a linear character of H_i over $\boldsymbol{C}$ $i=1, 2$. $R=\boldsymbol{C}[G]$ the group ring of G*

$$e_i=\frac{1}{|H_i|}\sum_{h\in H_i}\varphi_i(h)h\in R$$

$\mathscr{L}_i=Re_i$ the left ideal of R generated by e_i. Suppose $H_1\cap H_2=\{1\}$. Then

$$\langle\mathscr{L}_1, \mathscr{L}_2\rangle_R=\dim \mathrm{Hom}_R(\mathscr{L}_1, \mathscr{L}_2)=1$$

if and only if

(1.1) *for each σ in $G-H_1H_2$ there exists y in $H_2\cap\sigma^{-1}H_1\sigma$ such that $\varphi_2(y)\neq\varphi_1(\sigma y\sigma^{-1})$.*

In this case the common minimal left ideal in $\mathscr{L}_1$ and $\mathscr{L}_2$ is Re_1e_2 ($\cong Re_2e_1$).

N. B. $\mathscr{L}_i=\mathrm{Ind}_{H_i}^{G}\varphi_i^{-1}$ as R-modules.

We now define four linear characters 1, ε, η, ξ of W. First of all we express W as a semi-direct product. Let $H=\langle\sigma_1, \cdots, \sigma_{n-1}\rangle$ and $N=\langle\tau_1, \cdots, \tau_n\rangle$ be the two subgroups of W, where $\sigma_i=(i, i+1)(-i, -(i+1))$ and $\tau_j=(j, -j)$. Then N is a normal subgroup and $W=H\ltimes N$. We define $1, \varepsilon, \eta, \xi$ as

$$1(\sigma_i)=1(\tau_j)=1\ , \qquad \varepsilon(\sigma_i)=\varepsilon(\tau_j)=-1\ ,$$
$$\eta(\sigma_i)=-1\ , \quad \eta(\tau_j)=1\ , \qquad \xi(\sigma_i)=1\ , \quad \xi(\tau_j)=-1\ .$$

We use the same notations as the restrictions of them to the subgroup W_B. Now we can state the main theorems in this note.

THEOREM 2. *For B in $BTab_n^{\pm}$, we have*

1. $\langle \mathrm{Ind}_{W_B}^W 1, \mathrm{Ind}_{W_{\bar{B}}} \varepsilon\rangle_R=1$. *We write the common irreducible component as $\mathfrak{s}_B$.*
2. $\langle \mathrm{Ind}_{W_B}^W \xi, \mathrm{Ind}_{W_{\bar{B}}}^W \eta\rangle_R=1$. *We write the common irreducible component as $\mathfrak{t}_B$.*
3. $\mathfrak{t}_B=\mathfrak{s}_B\otimes\xi$ *and $\mathfrak{s}_B$ and $\mathfrak{t}_B$ are realizable over $\boldsymbol{Q}$.*

THEOREM 3. *For B and B' in $BTab_n^{\pm}$, the following three conditions are equivalent.*

a) *The shape of B is equal to the shape of B'.*
b) *$\mathfrak{s}_B\simeq\mathfrak{s}_{B'}$ as R-modules.*
c) *$\mathfrak{t}_B\simeq\mathfrak{t}_{B'}$ as R-modules.*

From Theorem 3 and Proposition 1 every irreducible representation of W appears in $\{\mathfrak{t}_B\}_{B\in BTab_n^{\pm}}$ and the shapes of B parametrize the equivalence classes.

1.3 To prove Theorem 2, we use

LEMMA 2. *For B in $BTab_n^{\pm}$ and g in W, we have*

1. $gW_Bg^{-1}=W_{gB}$.
2. $W_B\cap W_{\bar{B}}=\{1\}$.
3. $g\in W_BW_{\bar{B}}$ *if and only if* $W_B\cap gW_{\bar{B}}g^{-1}=\{1\}$.
4. *If $g\in W-W_BW_{\bar{B}}$, then $W_B\cap gW_{\bar{B}}g^{-1}$ has an element of the form $(i, -i)$ or $(a, b)(-a, -b)$.*

This lemma can be shown in a quite similar way to [2; Lemma 2.1, 2.2].

PROOF OF THEOREM 2. We apply Theorem 1 for $(H_1, \varphi_1)=(W_B, 1)$ and $(H_2, \varphi_2)=(W_{\bar{B}}, \varepsilon)$. As $(i, -i)=\tau_i$ and $(a, b)(-a, -b)$ is conjugate to σ_j, the condition (1.1) is satisfied by Lemma 2.4. 2. is also proved by setting $(H_1, \varphi_1)=(W_B, \xi)$ and $(H_2, \varphi_2)=(W_{\bar{B}}, \eta)$. 3. is clear by construction.

1.4 To prove Theorem 3, we use

LEMMA 3 (c.f. [2; Th. 1.22]). *G is a finite group and H_i a subgroup of G for $i=1, 2$. φ_i is a linear character of H_i over $\boldsymbol{C}$, and $\ell_i=Re_i$, $e_i=(1/|H_i|)\sum_{h\in H_i}\varphi_i(h)h$ for $i=1, 2$ where $R=\boldsymbol{C}[G]$ is the group ring. Then*

$$\langle \ell_1, \ell_2\rangle_R=\#\{\text{double coset } H_1\sigma H_2 | \forall y\in H_2\cap\sigma^{-1}H_1\sigma \Longrightarrow \varphi_2(y)=\varphi_1(\sigma y\sigma^{-1})\}\ .$$

PROOF OF THEOREM 3. If B and B' have the same shape, then there exists g in W such that $B'=gB$, therefore $\ell_B\simeq\ell_{B'}$. If the shape of B is (λ, μ) and the shape of B' is (λ', μ'), we set

$$e_1=\frac{1}{|W_B|}\sum_{\sigma\in W_B}\sigma\ ,\qquad e_2=\frac{1}{|W_{\bar{B}}|}\sum_{\sigma\in W_{\bar{B}}}\varepsilon(\sigma)\sigma\ ,$$
$$e_1'=\frac{1}{|W_{B'}|}\sum_{\sigma\in W_{B'}}\sigma\ ,\qquad e_2'=\frac{1}{|W_{\bar{B}'}|}\sum_{\sigma\in W_{\bar{B}'}}\varepsilon(\sigma)\sigma\ .$$

Then $\langle Re_1, Re_2'\rangle\geqq\langle \ell_B, \ell_{B'}\rangle$ and $\langle Re_2, Re_1'\rangle\geqq\langle \ell_B, \ell_{B'}\rangle$.

Suppose $(\lambda, \mu)\neq(\lambda', \mu')$. We claim one of the left side of the above inequalities is zero. First of all, suppose $\lambda\neq\lambda'$. Let k be the smallest i such that $\lambda_i\neq\lambda_i'$. We may assume $\lambda_k>\lambda_k'$. For $\sigma\in W$, we set $B''=\sigma B'$. Then the shape of B'' is (λ', μ'). If there is a letter i both in B_2 and B_1'' up to multiple of ±1, then $(i, -i)\in W_{\bar{B}''}\cap W_B$. Otherwise the letters in B_2 and B_2'' are equal up to ±1 and the same for B_1 and B_1''. Therefore $|\lambda|=|\lambda'|$ and $|\mu|=|\mu'|$, and $W_{\bar{B}''}\cap W_B$ is conjugate to a subgroup of $\mathfrak{S}_n$. In this case, there are letters a, b in the same row of B_1 and in the same column of B_1''. For if there aren't such letters, it happens $\lambda_i'\geqq\lambda_i$ for all i and this contradicts the condition $\lambda_k>\lambda_k'$. Then $(a, b)(-a, -b)$ is in $W_{\bar{B}''}\cap W_B$. As $W_{\bar{B}'}\cap\sigma^{-1}W_B\sigma=\sigma^{-1}(W_{B''}\cap W_B)\sigma$ we get $\langle Re_1, Re_2'\rangle=0$ by the Lemma 3.

In the case $\mu\neq\mu'$, we also get by the same way $\langle Re_1, Re_2'\rangle=0$ or $\langle Re_2, Re_1'\rangle=0$. Therefore the equivalence a)$\Leftrightarrow$b) is proved. b)$\Leftrightarrow$c) is clear.

§2. Explicit formula for irreducible characters.

In this section we calculate the character of irreducible representation $\mathfrak{k}_B$.

2.1 $\mathrm{Ind}_{W_B}^W 1$.

LEMMA 4. *G is a finite group and H is a subgroup of G. Let ψ be the character of $\mathrm{Ind}_H^G 1$. Then for a in G,*

$$\psi(a)=\frac{|C_G(a)|\cdot|H\cap\mathfrak{k}_a|}{|H|}$$

where $C_G(a)$ is the centralizer of a and $\mathfrak{k}_a$ is the conjugacy class of a in G.

PROOF. See e.g. [2; Lemma 3.1].

LEMMA 5. *If a in $W=W(C_n)$ has cycle type $(\alpha; \beta)=(1^{\alpha_1}2^{\alpha_2}\cdots; 1^{\beta_1}2^{\beta_2}\cdots)$, then*

$$|\mathfrak{k}_a|=\frac{n!}{1^{\alpha_1}\alpha_1!\,2^{\alpha_2}\alpha_2!\cdots 1^{\beta_1}\beta_1!\,2^{\beta_2}\beta_2!\cdots}\times 2^{n-\Sigma\alpha_i-\Sigma\beta_j}$$

and

$$|C_W(a)|=1^{\alpha_1}\alpha_1!\,2^{\alpha_2}\alpha_2!\cdots 1^{\beta_1}\beta_1!\,2^{\beta_2}\beta_2!\cdots\times 2^{\Sigma\alpha_i+\Sigma\beta_j}\,.$$

We write $N(\alpha; \beta)=|\mathfrak{k}_a|$.

PROOF. Similar with [2].

Take B in $BTab_n^{\pm}$ of shape (λ, μ). Set $H=W_B$. Then

$$H\simeq\mathfrak{S}_{\lambda_1}\times\mathfrak{S}_{\lambda_2}\times\cdots\times W(C_{\mu_1})\times W(C_{\mu_2})\times\cdots\,. \tag{2.1}$$

Therefore $|H|=2^{\Sigma\mu_i}\prod_i(\lambda_i!)\prod_j(\mu_j!)$. There is an onto map Φ according to (2.1)

$$\Phi\colon H\cap\mathfrak{k}_a\longrightarrow\{(\alpha_{ij}, p_{ij}, q_{ij}) \text{ with condition (2.2)}\}\,.$$

$$\left\{\begin{array}{l}\alpha_{ij},\ p_{ij},\ q_{ij} \text{ are non negative integers for } 1\leqq i,\ j\leqq n,\\ \sum_j j\alpha_{ij}=\lambda_i\,, \qquad \sum_j jp_{ij}+\sum_j jq_{ij}=\mu_i\,,\\ \sum_i\alpha_{ij}+\sum_i p_{ij}=\alpha_j\,, \qquad \sum_i q_{ij}=\beta_j\,.\end{array}\right\} \tag{2.2}$$

If $h\in H\cap\mathfrak{k}_a$ and $h=h_1h_2\cdots h_1'h_2'\cdots$ according to (2.1), then h_i has cycle type $1^{\alpha_{i1}}2^{\alpha_{i2}}$ and h_i' has cycle type $(1^{p_{i1}}2^{p_{i2}}\cdots; 1^{q_{i1}}2^{q_{i2}}\cdots)$. More over

$$|\Phi^{-1}(\alpha_{ij}, p_{ij}, q_{ij})|=\prod_{i=1}^{n}P(\alpha_{i\cdot})\cdot\prod_{i=1}^{n}N(p_{i\cdot}, q_{i\cdot})$$

where

$$P(\alpha_{i\cdot})=\frac{(\sum_j j\alpha_{ij})!}{1^{\alpha_{i1}}\alpha_{i1}!\,2^{\alpha_{i2}}\alpha_{i2}!\cdots}\,.$$

Therefore

$$|H\cap\mathfrak{k}_a|=\frac{\lambda_1!\,\lambda_2!\cdots\mu_1!\,\mu_2!\cdots}{1^{\alpha_1+\beta_1}2^{\alpha_2+\beta_2}\cdots}\sum_{(2.2)}\prod_i\frac{1}{\alpha_{i1}!\cdots}\prod_i\frac{2^{\mu_i-\sum_j(p_{ij}+q_{ij})}}{p_{i1}!\cdots q_{i1}!\cdots}\,.$$

Using lemma 4, 5, if we define $\Psi_{\lambda,\mu}=\mathrm{Ind}_{W_B}^{W} 1$,

$$\Psi_{\lambda,\mu}(a)=\sum_{(2.2)}\left(\prod_i \frac{\alpha_i!}{\alpha_{i1}!\,\alpha_{i2}!\cdots}\prod_i \frac{\beta_i!}{p_{i1}!\,p_{i2}!\cdots q_{i1}!\,q_{i2}!\cdots}\right)\times 2^{\sum_{i,j}\alpha_{ij}} .$$

Therefore we get

THEOREM 4. *For a in $W=W(C_n)$ of type $(\alpha;\beta)$,*

$$\prod_i (2x^{(i)}+y^{(i)})^{\alpha_i}\prod_i (y^{(i)})^{\beta_i}=\sum_{\substack{|\lambda|+|\mu|=n\\ \lambda,\mu\in Z_+^n}}\Psi_{\lambda,\mu}(a)x^\lambda y^\mu$$

where

$$x^{(i)}=x_1^i+x_2^i+\cdots+x_n^i ,$$

$$y^{(i)}=y_1^i+y_2^i+\cdots+y_n^i ,$$

$$x^\lambda=x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_n^{\lambda_n} ,$$

$$y^\mu=y_1^{\mu_1}y_2^{\mu_2}\cdots y_n^{\mu_n} ,$$

Z_+ is the set of non negative integers, and $x_1,\cdots,x_n$, $y_1,\cdots,y_n$ are variables.

If we set $\Psi'_{\lambda,\mu}=\mathrm{Ind}_{W_B}^{W}\xi$ then $\Psi'_{\lambda,\mu}=\Psi_{\lambda,\mu}\otimes\xi$ and using $\xi(a)=(-1)^{\sum\beta_i}$ we get

THEOREM 5. *For a in $W=W(C_n)$ of type $(\alpha;\beta)$,*

$$\prod_i (2x^{(i)}+y^{(i)})^{\alpha_i}\prod_i (-y^{(i)})^{\beta_i}=\sum_{\substack{|\lambda|+|\mu|=n\\ \lambda,\mu\in Z_+^n}}\Psi'_{\lambda,\mu}(a)x^\lambda y^\mu .$$

2.2 Schur function and $\chi_{\lambda,\mu}$.

For a partition $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_n)$, $\lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_n\geqq 0$, we consider a polynomial $S_\lambda(x)$ in $x_1, x_2,\cdots,x_n$ defined by

$$S_\lambda(x)=\frac{\Delta(\lambda+\delta)}{\Delta(\delta)}$$

where

$$\Delta(l)=\begin{vmatrix} x_1^{l_1} & x_1^{l_2} & \cdots & x_1^{l_n}\\ x_2^{l_1} & x_2^{l_2} & \cdots & x_2^{l_n}\\ \vdots & \vdots & & \vdots\\ x_n^{l_1} & x_n^{l_2} & \cdots & x_n^{l_n}\end{vmatrix}\qquad \text{for } l=(l_1,l_2,\cdots,l_n)$$

and $\delta=(n-1,n-2,\cdots,0)$.

$S_\lambda(x)$ is a symmetric polynomial in $x_1, \cdots, x_n$ and these $S_\lambda(x)$ form a $\boldsymbol{Z}$-base of the ring of symmetric polynomials with coefficient in $\boldsymbol{Z}$ when λ runs all the partitions.

We define a class function $\chi_{\lambda,\mu}$ on $W=W(C_n)$ as follows. If a in W has cycle type $(\alpha; \beta)$, then

$$\prod_i (x^{(i)}+y^{(i)})^{\alpha_i} \prod_i (x^{(i)}-y^{(i)})^{\beta_i}=\sum \chi_{\lambda,\mu}(a)S_\lambda(x)S_\mu(y) . \tag{2.3}$$

Theorem 6 (Orthogonality). *Let* (λ, μ) *and* (π, ρ) *be in* $\mathscr{P}_n$. *Then*

$$\frac{1}{|W|}\sum_{a\in W} \chi_{\lambda,\mu}(a)\chi_{\pi,\rho}(a)=\begin{cases}1 & \text{if } (\lambda, \mu)=(\pi, \rho)\\ 0 & \text{if } (\lambda, \mu)\neq(\pi, \rho) .\end{cases}$$

Proof. We set $M_n=\{\lambda|\lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_n\geqq 0\}$. It is known c.f. [2] that

$$\begin{cases}\sum_{\lambda\in M_n} S_\lambda(x)S_\lambda(z)=\prod_{m=1}^{\infty} e^{(x^{(m)}z^{(m)}/m)}\\ \sum_{\mu\in M_n} S_\mu(y)S_\mu(w)=\prod_{m=1}^{\infty} e^{(y^{(m)}w^{(m)}/m)} .\end{cases} \tag{2.4}$$

If we set

$$X^{(m)}=\frac{x^{(m)}+y^{(m)}}{2} , \qquad Y^{(m)}=\frac{x^{(m)}-y^{(m)}}{2} ,$$

$$Z^{(m)}=z^{(m)}+w^{(m)} , \qquad W^{(m)}=z^{(m)}-w^{(m)} ,$$

then

$$X^{(m)}Z^{(m)}+Y^{(m)}W^{(m)}=x^{(m)}z^{(m)}+y^{(m)}w^{(m)} . \tag{2.5}$$

Let Θ_n be the n-th homogeneous part of $\sum_{\lambda,\mu\in M_n} S_\lambda(x)S_\lambda(z)S_\mu(y)S_\mu(w)$ with respect to x and y. Using (2.4) and (2.5) we get

$$\Theta_n=\frac{1}{|W|}\sum_{\substack{(\lambda,\mu)\in\mathscr{P}_n\\(\pi,\rho)\in\mathscr{P}_n}}\sum_{a\in W} \chi_{\lambda,\mu}(a)\chi_{\pi,\rho}(a)S_\lambda(x)S_\mu(y)S_\pi(z)S_\rho(w) .$$

Therefore we have Theorem 6.

2.3 The relation between $\Psi'_{\lambda,\mu}$ and $\chi_{\pi,\rho}$.

For a in W of cycle type $(\alpha; \beta)$,

$$\prod (2x^{(i)}+y^{(i)})^{\alpha_i}\prod(-y^{(i)})^{\beta_i}=\sum \Psi'_{\lambda,\mu}(a)x^\lambda y^\mu ,$$

$$\prod (X^{(i)}+Y^{(i)})^{\alpha_i}\prod(X^{(i)}-Y^{(i)})^{\beta_i}=\sum \chi_{\pi,\rho}(a)S_\pi(X)S_\rho(Y) .$$

If we set $X^{(i)}=x^{(i)}$ and $Y^{(i)}=x^{(i)}+y^{(i)}$, then

$$X^{(i)}+Y^{(i)}=2x^{(i)}+y^{(i)} \quad \text{and} \quad X^{(i)}-Y^{(i)}=-y^{(i)} .$$

Therefore

$$\sum \Psi'_{\lambda,\mu}x^{\lambda}y^{\mu}=\sum \chi_{\pi,\rho}S_{\pi}(x)S_{\rho}(x, y) .$$

By [4; I (5.9), (5.7)]

$$S_{\rho}(x, y)=\sum_{\tau\subset\rho} S_{\tau}(x)S_{\rho/\tau}(y) ,$$

$$S_{\pi}(x)S_{\tau}(x)=S_{\pi*\tau}(x) ,$$

where $\pi*\tau$ is the skew diagram as the Diagram 1, and $S_{\rho/\tau}(y)$ and $S_{\pi*\tau}(x)$ are skew Schur polynomials. More over by [4; I (5.13)]

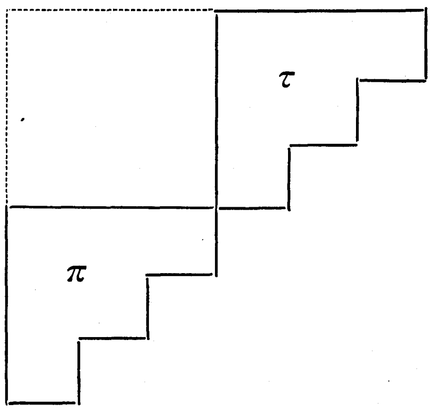


DIAGRAM 1

the coefficient of x^{λ} in $S_{\rho/\tau}(x)$ is $K_{\rho-\tau,\lambda}$.

Therefore we have

THEOREM 7.

$$\Psi'_{\lambda,\mu}=\sum_{\pi,\rho\in\mathscr{P}_n} K^{\lambda,\mu}_{\pi,\rho}\chi_{\pi,\rho}$$

where

$$K^{\lambda,\mu}_{\pi,\rho}=\sum_{\tau\subset\rho} K_{\pi*\tau,\lambda}K_{\rho/\tau,\mu}\geqq 0 .$$

$K^{\lambda,\mu}_{\pi,\rho}$ has following properties. We define linear order $>$ on partitions.

$$\lambda>\mu\Longleftrightarrow\lambda_1=\mu_1, \cdots, \lambda_{k-1}=\mu_{k-1} \quad \text{and} \quad \lambda_k>\mu_k .$$

We also define a linear order relation on the set $\mathscr{P}_n$ of pairs of partitions as follows.

$$(\pi,\rho)\geqq(\lambda,\mu)\Longleftrightarrow\begin{cases}|\pi|<|\lambda| & \\ \text{or}\quad |\pi|=|\lambda| & \text{and}\quad \pi>\lambda \\ \text{or}\quad \pi=\lambda & \text{and}\quad \rho\geqq\mu\,.\end{cases}$$

PROPOSITION 2. 1. *If* $K^{\lambda,\mu}_{\pi,\rho}\neq 0$, then $(\lambda,\mu)\leqq(\pi,\rho)$.
2. $K^{\lambda,\mu}_{\lambda,\mu}=1$.
3. $K^{\lambda,\mu}_{\phi,\rho}=K^{\lambda,\mu}_{\phi,\rho}$, $K^{\lambda,\phi}_{\pi,\rho}=K^{\lambda,\phi}_{\rho,\pi}$.

These are clear from the definition of $K_{\lambda,\mu}$ c.f. [4].

N. B. The formula proved in [1; Theorem III. 5] is essentially equal to Theorem 7. The equality is proved by a combinatorial argument.

THEOREM 8. *$\chi_{\lambda,\mu}$ is the character of $\mathfrak{k}_{\lambda,\mu}$.*

PROOF. From Theorem 6 $\pm\chi_{\lambda,\mu}$ is an irreducible character. But by Theorem 7, $\chi_{\lambda,\mu}$ is a real character. By definition $\mathfrak{k}_{\lambda,\mu}$ appears in $\Psi'_{\lambda,\mu}$. (ϕ,n) is the maximum element in $\mathscr{P}_n$. As $\chi_{\phi,n}=\Psi'_{\phi,n}=\xi$ and $\mathfrak{k}_{\phi,n}=\xi$, it follows that $\chi_{\phi,n}=\mathfrak{k}_{\phi,n}$. Using Theorem 7 and Proposition 2 inductively according to the linear order on $\mathscr{P}_n$, we get $\mathfrak{k}_{\lambda,\mu}=\chi_{\lambda,\mu}$.

2.4 Nakayama's formula for $W=W(C_n)$.

Let a in W have cycle type $(\alpha;\beta)$. Then by definition

$$\prod(x^{(i)}+y^{(i)})^{\alpha_i}\prod(x^{(i)}-y^{(i)})^{\beta_i}=\sum_{(\lambda,\mu)\in\mathscr{P}_n}\chi_{\lambda,\mu}(a)S_\lambda(x)S_\mu(y)\,.$$

If $\alpha_v\geqq 1$, let a' be the element in W taken off an even cycle pair of length v from a. a' has cycle type $(\alpha';\beta)$, where $\alpha'_i=\alpha_i$ for $i\neq v$ and $\alpha'_v=\alpha'_v-1$. As

$$\sum_{(\lambda,\mu)\in\mathscr{P}_n}\chi_{\lambda,\mu}(a)S_\lambda(x)S_\mu(y)=(x^{(v)}+y^{(v)})\sum_{(\pi,\rho)\in\mathscr{P}_{n-v}}\chi_{\pi,\rho}(a')S_\pi(x)S_\rho(y)\,,$$

we have

$$\chi_{\lambda,\mu}(a)=\sum_i\chi_{\lambda-v_i,\mu}(a')+\sum_i\chi_{\lambda,\mu-v_i}(a')$$

where $v_i=(0,0,\cdots,v,0,\cdots)\in \boldsymbol{Z}^n$, v in the i-th factor, and extend $\chi_{\lambda,\mu}$ for λ, μ not partitions as in [2; Th. 3.11]. If $\beta_u\geqq 1$, let a'' be the element in W taken off an odd cycle of length $2u$ from a. a'' has cycle type $(\alpha;\beta')$, where $\beta'_i=\beta_i$ for $i\neq u$ and $\beta'_u=\beta_u-1$. We also have

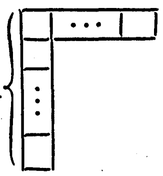
$$\chi_{\lambda,\mu}(a)=\sum_i\chi_{\lambda-u_i,\mu}(a'')-\sum_i\chi_{\lambda,\mu-u_i}(a'')$$

where $u_i=(0,0,\cdots,u,0,\cdots)\in\boldsymbol{Z}^n$, u in the i-th factor. These correspond to Murnaghan's formula. Therefore we get

THEOREM 9.

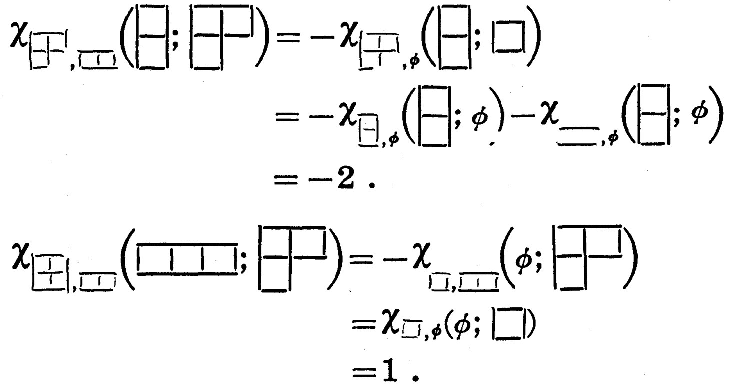
$$\chi_{\lambda,\mu}(\alpha;\beta)=\sum_{\Gamma\in H_v(\lambda)}\operatorname{sign}(\Gamma)\chi_{\lambda-\Gamma,\mu}(\alpha-v;\beta)+\sum_{\Gamma\in H_v(\mu)}\operatorname{sign}(\Gamma)\chi_{\lambda,\mu-\Gamma}(\alpha-v;\beta)$$
$$=\sum_{\Gamma\in H_u(\lambda)}\operatorname{sign}(\Gamma)\chi_{\lambda-\Gamma,\mu}(\alpha;\beta-u)-\sum_{\Gamma\in H_u(\mu)}\operatorname{sign}(\Gamma)\chi_{\lambda,\mu-\Gamma}(\alpha;\beta-u)$$

where $H_v(\lambda)$ is the set of hooks of length v in λ and $\operatorname{sign}(\Gamma)=(-1)^m$ *if*

$\Gamma = m+1\{$ 

and $\lambda-\Gamma$ means the diagram taken off Γ from λ and packed the gap.

EXAMPLES.



for $u=2$

for $u=1$

for $v=3$

for $u=2$

Using Theorem inductively we get

COROLLARY 1.

$$\dim\chi_{\lambda,\mu}=\chi_{\lambda,\mu}(1)=\frac{n!}{|\lambda|!\,|\mu|!}d(\lambda)d(\mu)$$

where $d(\lambda)$ is the dimension of the irreducible representation of $\mathfrak{S}_{|\lambda|}$ corresponding to the partition λ.

§3. The I-set of irreducible representations of $W(C_n)$.

3.1 W-graph.

For a Coxeter group (W, S), the notion of W-graph was introduced in [3]. Consider a triple $\rho=(\Gamma, I, \mu)$ where $\Gamma=(\Gamma^0, \Gamma^1)$ a graph with vertex set Γ^0 and edge set Γ^1, $I\colon \Gamma^0\to\mathscr{P}(S)$ a mapping from vertex set to the powerset of S, $\mu\colon \Gamma^0\times\Gamma^0\to\boldsymbol{Z}$ a mapping from the set of ordered pair of vertices to the set of integers, such that

$$\{x, y\} \in \Gamma^1 \quad \text{if and only if either} \quad \mu(x, y) \neq 0 \quad \text{or} \quad \mu(y, x) \neq 0 .$$

Let q be an indeterminate and $A = \mathbf{Z}[q^{1/2}, q^{-1/2}]$. For x in Γ^0 we define τ_s ($s \in S$) by

$$\tau_s(x) = \begin{cases} -x & s \in I(x) \\ qx + q^{1/2} \sum_{s \in I(y)} \mu(y, x) y & s \notin I(x) . \end{cases}$$

ρ is a W-graph if $\{\tau_s\}$ induce a representation of the Hecke algebra of (W, S) over A. If we specialize $q^{1/2}$ to 1, we get a representation χ_ρ of $\mathbf{Z}[W]$.

DEFINITION. We call $I(x)$ $x \in \Gamma^0$ the I-set of ρ (including multiplicity).

3.2 I-set.

THEOREM 10. *I-set is determined only by the representation χ_ρ. More over it can be given explicitly by $\langle \chi_\rho|_{W_J}, 1_{W_J} \rangle$ (J runs all the subset of S).*

N. B. W_J is the subgroup of W generated by the elements in J.

PROOF. Let $\rho = (\Gamma, I, \mu)$ be a W-graph of a Coxeter group (W, S) and $J \subset S$. We define two numbers $A_\rho(J)$ and $B_\rho(J)$ by

$$A_\rho(J) = \#\{x \in \Gamma^0 | I(x) \subset J\}$$

$$B_\rho(J) = \#\{x \in \Gamma^0 | I(x) = J\} .$$

Then I-set is determined by $B_\rho(J)$. But by definition

$$A_\rho(J) = \sum_{K \subset J} B_\rho(K) .$$

Using inversion formula, we get

$$B_\rho(J) = \sum_{K \subset J} (-1)^{|J-K|} A_\rho(K) .$$

Therefore it is enough to determine $A_\rho(J)$. Now the theorem is proved by the following

LEMMA 6. *$A_\rho(J) = \langle \chi_\rho|_{W_{\hat{J}}}, 1_{W_{\hat{J}}} \rangle$ where $\hat{J} = S - J$.*

The above lemma follows by the fact

$$\langle \chi_\rho, 1 \rangle = \#\{x \in \Gamma^0 | I(x) = \emptyset\} .$$

3.3 Now consider the case $W=W(C_n)$. W is Coxeter group with generator set $S=\{s_1, s_2, \cdots, s_n\}$, where $s_i=(i, i+1)(-i, -(i+1))$ for $i<n$ and $s_n=(n, -n)$.

As stated in §1, 2 the irreducible representations of W are parametrized by the set of double partitions $\mathscr{P}_n$. For (λ, μ) in $\mathscr{P}_n$, we define

$$BTab_{\lambda,\mu}=\left\{D=(D_1, D_2)\;\middle|\;\begin{array}{l} D_1 \text{ is a Young tableau of shape } \lambda \\ D_2 \text{ is a Young tableau of shape } \mu \\ \text{the entries in } D_1 \text{ and } D_2 \text{ are} \\ \text{exactly 1 to } n,\ D_1 \text{ and } D_2 \text{ are} \\ \text{standard} \end{array}\right\}.$$

Then it is easily proved that

$$\# BTab_{\lambda,\mu}=\frac{n!}{|\lambda|!\,|\mu|!}d(\lambda)d(\mu)=\dim \chi_{\lambda,\mu} \quad \text{(c.f. Cor. 1)}\ .$$

So it is natural to set $\Gamma^0=BTab_{\lambda,\mu}$.

For $D=(D_1, D_2)$ in $BTab_{\lambda,\mu}$, we set

$$\begin{aligned} I(D)=\{s_i|i,\ i+1\in D_1 &\text{ and } i \text{ is in the upper rows than } i+1\} \\ \cup\{s_i|i,\ i+1\in D_2 &\text{ and } i \text{ is in the upper rows than } i+1\} \\ \cup\{s_i|i\in D_2 &\text{ and } i+1\in D_1\} \\ \cup\{s_n|n\in D_2&\}\ . \end{aligned}$$

THEOREM 11. *$I(D)$ $D\in BTab_{\lambda,\mu}$ is the I-set of $\chi_{\lambda,\mu}$.*

PROOF. Using a special formula for Theorem 7, Stanley [5; Prop. 6.2] proved by a combinatoiral argument that

$$\langle \chi_{\lambda,\mu}, \operatorname{Ind}_{W_{\hat{J}}}^{W} 1\rangle=\#\{D\in BTab_{\lambda,\mu}|I(D)\subset J\}\ .$$

The left hand side is equal to $A_{\chi_{\lambda,\mu}}(J)$, therefore the theorem follows.

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Present Address:
DEPARTMENT OF MATHEMATICS
FACULTY OF EDUCATION
OKAYAMA UNIVERSITY
TSUSHIMANAKA, OKAYAMA 700

ADDED IN PROOF. This construction of the representation of $W(C_n)$ was written in the author's master thesis (1980). But we found that essentially the same construction technique was used in [7].

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