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On the Decay of Correlation for Piecewise Monotonic Mappings I

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Introduction

In this paper, we determine the decay rate of correlation for a certain class of piecewise linear mappings explicitly (for more general cases, we will mention in [11]) and apply it to the critical phenomena in dynamical system. The decay of correlation is already pointed out to be determined in terms of the Fredholm determinant on the physical level in [12]. However, since the Perron-Frobenius operator is generally not of trace class ([15]), it has not been proved except for Markov piecewise linear mappings from mathematical point of view. One of our aims is to give the proof to this assertion for the class of mappings F with constant slope λ which satisfies the conditions given below.

We will consider a power series Φ, called the Fredholm determinant (the reciprocal of the Artin-Mazur-Ruelle zeta function), associated with piecewise linear mapping F (whose definition will be given in §2) and the roots of $\Phi(1/\gamma)=0$ will be called Fredholm eigenvalues. By γ_1, γ_2, we denote the Fredholm eigenvalues which are the first and the second greatest in modulus (in fact, γ_1 equals the slope λ of the mapping F). Our main theorem is stated as follows:

THEOREM 0.1. i) *Suppose that $\gamma_1>\eta$. Then the following two statements are equivalent*:

1) *There exists an absolutely continuous invariant measure with which the dynamical system $((0, 1), \mu, F)$ is mixing.*

2) $$\rho(x)=-\lambda(\Phi'(1/\lambda))^{-1}\chi(1/\lambda; x)\geqq 0 ,$$

for any $x\in[0, 1]$, where the definition of $\chi(z; x)$ will be given in §2.

ii) *If the statements of* i) *hold, then the density function of μ equals*

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ρ, and for any pair of a function of bounded variation $f\in BV$ and a bounded function $g\in L^{\infty}$ and for any $\varepsilon>0$,

$$\lim((\eta+\varepsilon)/\lambda)^{-n}\left\{\int f(x)g(F^{(n)}(x))d\mu-\int fd\mu\int gd\mu\right\}=0\ , \tag{0.1}$$

where

$$\eta=\begin{cases}\lambda & \text{if } \Phi'(1/\lambda)=0\ ,\\ \max\{|\gamma_2|,1\} & \text{otherwise}\ ,\end{cases}$$

and $F^{(i)}$ is the i-fold iterate of F:

$$F^{(i)}(x)=\begin{cases}F(x) & i=1\ ,\\ F(F^{(i-1)}(x)) & i\geqq 2\ .\end{cases} \tag{0.2}$$

On this problem, some related results also can be found in [2] and [6].

As an application of Theorem 0.1, we show two critical phenomena in dynamical systems. One is the case when $\lambda\downarrow 1$ for β-transformations

$$F_{\beta,\lambda}(x)=\lambda x\ (\mathrm{mod.}\ 1)\ .$$

This is a model of a critical phenomenon in the transition from chaos to ordered motion which is first observed in [9] but only along a special sequence of mappings such that the Fredholm determinants are polynominals. The other is the case when $\lambda\downarrow\sqrt{2}$ for unimodal linear transformations

$$F_{*,\lambda}(x)=\begin{cases}\lambda x-\lambda+2 & 0\leqq x\leqq 1-1/\lambda\ ,\\ -\lambda x+\lambda & 1-1/\lambda<x\leqq 1\ .\end{cases}$$

In this case, if $\lambda>\sqrt{2}$, $F_{*,\lambda}$ is mixing, while, if $1<\lambda\leqq\sqrt{2}$, it is ergodic but not mixing. This case is observed in [14] again along a special sequence of mappings. Our second main theorem (Theorem 4.1) can be summarized as follows:

THEOREM 0.2. i) *(β-transformations) The Fredholm eigenvalues around 1, say,*

$$e^{\alpha+i\beta}\ ,$$

have the following asymptotics of α and β:

$$\alpha=\frac{1}{N}\{\log N+\textit{small order}\}\ , \tag{0.3}$$

(0.4) $$\beta=\frac{1}{N}\{n\pi+\textit{small order}\} \qquad n=0, \pm 1, \pm 2, \cdots$$

as $\lambda\downarrow 1$, *where* N *is the first return time of the point* 1 *to the interval* $(1/\lambda, 1]$:

$$F_{\beta,\lambda}^{(i)}(1) \notin (1/\lambda, 1] \qquad \text{for} \quad 1\leqq i\leqq N-1 ,$$
$$F_{\beta,\lambda}^{(N)}(1) \in (1/\lambda, 1] .$$

The second greatest Fredholm eigenvalue is the case $n=\pm 1$, *and so we may say that the decay of the correlation*

(0.5) $$\eta/\lambda=\exp\left\{-\frac{1}{2N}\left(\frac{\pi}{\log N}\right)^2+\textit{small order}\right\} ,$$

as $\lambda\downarrow 1$.

ii) (*unimodal linear transformations*) *The argument of the second greatest Fredholm eigenvalue equals* π *and the decay rate of the correlation*

(0.6) $$\eta/\lambda=2\lambda^{-2}+\textit{small order} ,$$

as $\lambda\downarrow\sqrt{2}$.

This theorem shows:

i) (β-transformations) For λ which is sufficiently close to 1 (for sufficiently large N), the envelope of the series of the correlation functions for $f\in BV$ and $g\in L^\infty$, by Theorem 0.1,

$$\int f(x)g(F^{(n)}(x))d\mu-\int fd\mu\int gd\mu$$

decreases approximately in the order $\exp\{-(1/2N)(\pi/\log N)^2\}$ by (0.5) and one can observe the modulation with the frequency approximately N in virtue of (0.4). This generalizes the results in [9] for the special series F_N such that

$$F_N^{(i)}(1) \notin (1/\lambda_N, 1] \qquad \text{for} \quad 1\leqq i\leqq N-1 ,$$
$$F_N^{(N)}(1)=1 ,$$

where λ_N is the slope of F_N.

ii) (unimodal linear transformations) For λ which is sufficiently close to $\sqrt{2}$, the envelope of the series of the correlation functions decreases appoximately in the order $2\lambda^{-2}$ and the frequency of the modulation equals 2. This generalizes the results shown in [14] near the first band-splitting point.

Finally, let us state the conditions imposed on the mapping F. The mappings F which we consider below are of the unit interval into itself; we always assume that there exists a partition $0=c_0<c_1<\cdots<c_{k+1}=1$ such that

i) F is continuous and strictly monotone on each subintervals (c_i, c_{i+1}), $0\leqq i\leqq k$,

ii) $\lim_{x\downarrow c_i} F(x)$ and $\lim_{x\uparrow c_i} F(x)$ are either 0 or 1, $1\leqq i\leqq k$,

iii) By technical reason, we restrict ourselves to the following two cases:

type 1) $\lim_{x\downarrow 0} F(x)$ is either 0 or 1 and $F(x)$ is monotone increasing on $(c_k, 1)$,

type 2) $\lim_{x\uparrow 1} F(x)$ is either 0 or 1 and $F(x)$ is monotone increasing on $(0, c_1)$,

β-transformations are the examples of type 1 and unimodal linear transformations are the examples of type 2. Any mapping F of type 2 is conjugate to $G=I\circ F\circ I$ which is of type 1 under the conjugacy $I(x)=1-x$. Thus, hereafter, we only treat mappings of type 1. For a mapping F of type 1, we call

i) F is a piecewise linear mapping if $|F'(x)|=\text{constant } \lambda$ $(x\neq c_i$, $0\leqq i\leqq k+1)$, and we call this constant λ the slope of the mapping F.

ii) F is of periodic type if there exists n such that $F^{(n)}(1)=1$. We define the period N of a mapping F of periodic type (N-periodic) to be the number determined by

$$N=\min\{n: F^{(n)}(1)=1 \text{ and } F^{(n)}(x) \text{ is monotone increasing in some neighborhood of 1 in } (0,1)\}\ .$$

Note that the values $F(c_i)$ $(0\leqq i\leqq k+1)$ do not play an essential role in our consideration by the assumptions above. Thus we always assume that

$$F(c_i)=\lim_{x\downarrow c_i} F(x) \quad (0\leqq i\leqq k)$$

and

$$F(1)=\lim_{x\uparrow 1} F(x)\ .$$

§1. Alphabets, words and sentences.

In this section, we will prepare several notations. Put $A=\{0, 1, \cdots, k\}$, and we call each element of the set A an alphabet. For $a\in A$, we define

$$(a)=\begin{cases}[c_a, c_{a+1}) & 0\leqq a\leqq k-1 ,\\ [c_k, 1] & a=k ,\end{cases}$$

$$\operatorname{sgn} a=\begin{cases}1 & \text{if } F \text{ is monotone increasing on } (a) ,\\ -1 & \text{otherwise} .\end{cases}$$

We call a finite sequence of alphabets $w=a_1\cdots a_i$ a word and we define

$$(w)=\{x\in[0, 1]: F^{(j-1)}(x)\in(a_j), 1\leqq j\leqq i\}$$
$$|w|=i$$

and

$$\operatorname{sgn} w=\prod_{j=1}^{i}\operatorname{sgn} a_j .$$

We consider a formal symbol ϕ which we call an empty word and define

$$(\phi)=[0, 1]$$
$$|\phi|=0$$

and

$$\operatorname{sgn}\phi=1 .$$

A set of words we denote by W. For $x\in[0, 1]$, let (a_i^x) be the interval that contains $F^{(i-1)}(x)$ and we call the infinite sequence of alphabets $a_1^x a_2^x\cdots$ the expansion of x. We usually identify x with its expansion. The expansion of 1 plays an essential role throughout this paper. Let

$$K=\{a_1\cdots a_i: \text{there exists } j\ (1\leqq j\leqq i-1) \text{ such that}$$
$$a_1=\cdots=a_j=k \text{ and } a_{j+1}, \cdots, a_i\neq k\}\cup\{\phi\} ,$$

and we call each element of the set K a k-word. Let

$$S=\{w_1\cdots w_i: w_j\in K\quad(1\leqq j\leqq i-1), w_i\in K \text{ or } w_i=k\cdots k\} ,$$

and we call each element of the set S a sentence, where we define

$$w\phi=\phi w=w \qquad \text{for any} \quad w\in W .$$

Of course, we can regard a sentence as a word. Using the expansion of 1 we define k-words $w_i^1=a_{i,1}\cdots a_{i,j(i)}\in K$ by

$$w_1^1 w_2^1 \cdots = a_{1,1} \cdots a_{1,j(1)} a_{2,1} \cdots a_{2,j(2)} \cdots = a_1^1 a_2^1 \cdots ,$$

and we define

$$\theta_p = w_1^1 \cdots w_p^1 \in S .$$

DEFINITION. A word w is called of type (p, q) if

$$p = r - q$$

$$q = \begin{cases} \max\{n : a(r-n+1, r) = a(1, n) \text{ and } \operatorname{sgn} a(1, r-n) = -1\} , \\ 0 \quad \text{if there exists no such } n , \end{cases}$$

where

$$a(i, j) = a_i^1 \cdots a_j^1$$

and

$$r = \begin{cases} \max\{n : w = w' a(1, n) \text{ for some word } w' \in W\} , \\ 0 \quad \text{if there exists no such } n. \end{cases}$$

Now we define orders on A, W and K. On the set A, we consider the natural order, that is, $0<1<\cdots<k$. For words $w=a_1\cdots a_n$ and $w'=a'_1\cdots a'_m$, we say that $w<w'$ if there exists i $(1\leqq i\leqq n, m)$ such that $a_1=a'_1, \cdots, a_{i-1}=a'_{i-1}$ and one of the following holds:

i) $a_i<a'_i$ and $\operatorname{sgn} a_1\cdots a_{i-1}=1$,

ii) $a_i>a'_i$ and $\operatorname{sgn} a_1\cdots a_{i-1}=-1$.

For $w, w' \in K$, we define $w \ll w'$ if $wk<w'k$. For infinite sequences of alphabets, we can consider order as above.

DEFINITION. i) We call a word w admissible if $(w)\neq\emptyset$, and denote

$$W(F) = \{w \in W : w \text{ is admissible}\} ,$$
$$K(F) = \{w \in K : w \text{ is admissible}\} ,$$
$$S(F) = \{s \in S : s \text{ is admissible}\} .$$

An infinite sequence of alphabets $a_1a_2\cdots$ is called admissible if $a_1\cdots a_n \in W(F)$ for any n. On the admissible words, see [16].

ii) We call a word w complete if w is of type $(0, 0)$. For convenience, we also call the empty word ϕ complete.

LEMMA 1.1 i) *A word $w=a_1\cdots a_n$ is admissible if and only if $a_i\cdots a_n\leqq a(1, n-i+1)$ for $1\leqq i\leqq n$.*

ii) *For admissible words $w, w' \in W(F)$, $ww' \in W(F)$ if and only if*

$$a(1,p)w' \leqq a(1,p+|w'|)$$
$$a(1,q)w' \leqq a(1,q+|w'|)\ ,$$

where w is of type (p,q) and we define $a(1,0)=\phi$.

iii) *if a word w is complete, then*

$$F^{(|w|)}((w)) \supset (0,1)\ ,$$

in other words, wx is admissible for all $x \in (0,1)$.

LEMMA 1.2. *Suppose that θ_r is of type $(|\theta_p|, |\theta_q|)$. Then θ_{r+1} is either of type $(|\theta_p|, |\theta_{q+1}|)$ or $(|\theta_{r+1}|, 0)$. Moreover, θ_q is of type $(|\theta_q|, 0)$.*

The proofs of Lemma 1.1 and Lemma 1.2 are almost the same as in [10], thus we omit them.

REMARK. For a mapping F of type 2, let $a_1^0 a_2^0 \cdots$ be the expansion of 0, then $\bar{a}_1^1 \bar{a}_2^1 \cdots$ is the expansion of 1 of the mapping $G=I\circ F\circ I$, where $\bar{a}_i^1 = k - a_i^0$ $(i \geqq 1)$.

§2. Fredholm determinant.

Let us introduce several notations and definitions, and we will construct a renewal equation in terms of the Fredholm determinant which is one of the main tool in this paper.

DEFINITION. For a point $x \in [0,1]$ or ϕ, define:

i) $s(n,x)=$ the number of sentences $s \in S(F)$ such that $|s|=n$ and sx is admissible.

$w(n,x)=$ the number of words $w \in W(F)$ such that $|w|=n$ and wx is admissible.

Note that $w(n,\phi)$ equals the number of subintervals in the partition $\bigvee_{j=0}^{n-1} \{F^{-j}((i)) : 0 \leqq i \leqq k\}$.

$$\text{ii)}\quad \chi(n,x)=\begin{cases} 1 & \text{if } \operatorname{sgn} a(1,n)=1 \text{ and } a(1,n)x \text{ is admissible}\ , \\ -1 & \text{if } \operatorname{sgn} a(1,n)=-1 \text{ and } a(1,n)x \text{ is not admissible}\ , \\ 0 & \text{otherwise}\ . \end{cases}$$

iii) To state a renewal equation for $w(n,x)$, we need the following notations.

$k(p,j)=$ the number of k-words $w \in K(F)$ such that $w \ll w_{p+1}^1$ and $|w|=j$.

$$r(p, n, x)=\begin{cases}\sum_{j\geq 1} k(p, j)s(n-|\theta_p|-j, x) & \text{if} \quad \operatorname{sgn}\theta_p=1\ , \\ s(n-|\theta_p|, x)-\sum_{j\geq 1} k(p, j)s(n-|\theta_p|-j, x) & \\ \qquad -s(n-|\theta_{p+1}|, x) & \text{if} \quad \operatorname{sgn}\theta_p=-1\ .\end{cases}$$

LEMMA 2.1. *For* $x \in [0, 1]$ *or* ϕ

$$s(n, x)=\begin{cases}1 & n=0\ , \\ \sum_{p\geq 0} r(p, n, x)+\chi(n, x) & n\geqq 1\ .\end{cases} \tag{2.1}$$

PROOF. We shall classify the sentences enumerated by $s(n, x)$ according to the place where they differ for the first time from the expansion of 1. For this purpose, let

$$\bar{r}(p, n, x)=\text{the number of sentences } s=w_1\cdots w_q \in S(F) \quad (q\geqq p)$$

such that $|s|=n$, $w_1\cdots w_p=\theta_p$, $w_{p+1}\not\approx w^1_{p+1}$ and sx is admissible. Then by Lemma 1.1,

$$\bar{r}(p, n, x)=r(p, n, x)\ ,$$

if θ_p is of type $(|\theta_p|, 0)$. In case θ_p is of type $(|\theta_u|, |\theta_v|)$ $(v>0)$ two possibility can occur. One possibility occurs when θ_{p+1} is of type $(|\theta_u|, |\theta_{v+1}|)$, and in this case

$$\bar{r}(p, n, x)=\begin{cases}1 & \text{if } a(1, n)x \text{ is admissible and } |\theta_p|\leqq n<|\theta_{p+1}|\ , \\ 0 & \text{otherwise}\ .\end{cases}$$

The other possibility is when θ_{p+1} is of type $(|\theta_{p+1}|, 0)$, and we get in this case, if $\operatorname{sgn}\theta_p=1$,

$$\begin{aligned}\bar{r}(p, n, x)=&\{\text{the number of } k\text{-words } w \text{ such that } |w|=n-|\theta_p| \text{ and } w\ll w^1_{p+1}\} \\ &-\{\text{the number of } k\text{-words } w \text{ such that } |w|=n-|\theta_p| \text{ and } w\ll w^1_{v+1}\}\ .\end{aligned}$$

We get similar results in case of $\operatorname{sgn}\theta_p=-1$ also, and combining these facts we obtain easily the proof of Lemma 2.1 (cf. [10]).

DEFINITION. Let

$$b_p=\begin{cases}a^1_p & \text{if} \quad \operatorname{sgn} a^1_p=\operatorname{sgn} a(1, p-1)=1\ , \\ -a^1_p & \text{if} \quad \operatorname{sgn} a^1_p=1 \text{ and } \operatorname{sgn} a(1, p-1)=-1\ , \\ 1+a^1_p & \text{if} \quad \operatorname{sgn} a^1_p=-1 \text{ and } \operatorname{sgn} a(1, p-1)=1\ , \\ -1-a^1_p & \text{if} \quad \operatorname{sgn} a^1_p=\operatorname{sgn} a(1, p-1)=-1\ .\end{cases}$$

LEMMA 2.2.

$$k(p, j)-kk(p, j-1)=\begin{cases} 0 & \textit{if} \quad j=1 \\ b_{|\theta_p|+j} & \textit{if} \quad 2 \leqq j \leqq |w_{p+1}^1|-1 , \\ b_{|\theta_{p+1}|} & \textit{if} \quad j=|w_{p+1}^1| \textit{ and } \operatorname{sgn} w_{p+1}^1=1 , \\ b_{|\theta_{p+1}|-1} & \textit{if} \quad j=|w_{p+1}^1| \textit{ and } \operatorname{sgn} w_{p+1}^1=-1 , \\ k & \textit{if} \quad j=|w_{p+1}^1|+1 \textit{ and } \operatorname{sgn} w_{p+1}^1=1 , \\ 0 & \textit{otherwise} . \end{cases} \tag{2.2}$$

This lemma can be shown by elementary calculations. Combining the above two lemmas, we get:

THEOREM 2.3.

$$s(n, x)=\sum_{j=1}^{n} b_j s(n-j, x)+\chi(n, x)-k\chi(n-1, x) , \tag{2.3}$$

where $\chi(0, x)=1$.

DEFINITION.

i) The Fredholm determinant Φ associated with the mapping F is defined by

$$\Phi(z)=\begin{cases} 1-\sum_{j=1}^{\infty} b_j z^j & \text{if } F \text{ is aperiodic} , \\ 1-\sum_{j=1}^{N} b_j z^j - z^N & \text{if } F \text{ is } N\text{-periodic} . \end{cases}$$

We call z which satisfy $\Phi(1/z)=0$ a Fredholm eigenvalue of the mapping F. Note that our definition of Fredholm determinant is slightly different from that of [15].

ii) We denote generating functions of $w(n, x)$ and $\chi(n, x)$ by

$$w(z; x)=\sum_{n=0}^{\infty} z^n w(n, x)$$

and

$$\chi(z; x)=\begin{cases} \sum_{n=0}^{\infty} z^n \chi(n, x) & \text{if } F \text{ is aperiodic} , \\ \sum_{n=1}^{N-1} z^n \chi(n, x) & \text{if } F \text{ is } N\text{-periodic} , \end{cases}$$

respectively.

THEOREM 2.4.

$$w(z;x)=\chi(z;x)/\Phi(z)\ . \tag{2.4}$$

PROOF. Suppose that F is aperiodic. Since

$$w(m,x)=\sum_{i=0}^{m}k^{i}s(m-i,x)\ , \tag{2.5}$$

we get by Theorem 2.3

$$\begin{aligned}\sum_{j=1}^{n}b_jw(n-j,x)&=\sum_{j=1}^{n}b_j\sum_{i=0}^{n-j}k^{i}s(n-i-j,x)\\&=\sum_{i=0}^{n-1}k^{i}\sum_{j=1}^{n-i}b_js(n-i-j,x)\\&=\sum_{i=0}^{n-1}k^{i}\{s(n-i,x)-\chi(n-i+1,x)+k\chi(n-i,x)\}\\&=w(n,x)-\chi(n,x)\ .\end{aligned} \tag{2.6}$$

Therefore

$$\begin{aligned}w(z;x)&=\sum_{n=0}^{\infty}z^{n}w(n,x)\\&=\sum_{n=0}^{\infty}z^{n}\sum_{j=1}^{n}b_jw(n-j,x)+\chi(z;x)\\&=\sum_{n=0}^{\infty}b_jz^{j}w(z;x)+\chi(z;x)\ ,\end{aligned}$$

and this completes the proof. The proof for the periodic case is almost the same.

LEMMA 2.5. *For a word w, let $w(w,m,n,x)$ be the number of words v which satisfy $|v|=n-|w|$, wvx is admissible and $wvx<a^1_{m+1}a^1_{m+2}\cdots$. Then we can define $b_j(w,m)$ so that*

$$\left|w(w,m,n,x)-\sum_{j=1}^{n-|w|}b_j(w,m)w(n-|w|-j,x)\right|\le 2\ ; \tag{2.7}$$

an explicit expression for $b_j(w,m)$ is given in the proof.

PROOF. Let $w(m,n,x)$ be the number of admissible words $v=a_1\cdots a_n$ such that $a_i=a^1_i$ $(1\le i\le m)$. Then by elementary calculations as in Lemma 2.1 and Theorem 2.3, we get

$$(2.8)\qquad w(m, n, x)=\begin{cases}\sum_{j=p'+1}^{n} b_j w(n-j, x)+\chi(n, x) & \text{if}\quad \operatorname{sgn} a(1, p')=1\ ,\\ \sum_{j=p'+1}^{n} b_j w(n-j, x)+\chi(n, x)+w(n-p', x) & \\ & \text{if}\quad \operatorname{sgn} a(1, p')=-1\ ,\end{cases}$$

where $a(1, m)$ is of type (p', q'). Now for a word w, let

$$w_m^*=\begin{cases}\phi & \text{if}\quad w<a(m+1, m+|w|)\ ,\\ a(1, m) & \text{otherwise}\ ,\end{cases}$$

$$r=\max\{r^*;\ w_m^* w=w'a(1, r^*) \text{ for some } w'\}\ ,$$

$$q=\begin{cases}\max\{q^*\leqq|w|:\ a(r-q^*+1, r)=a(1, q^*) \text{ and } \operatorname{sgn} a(1, r-q^*)=-1\} & \\ & \text{if}\quad \operatorname{sgn} w_m^*=1\ ,\\ 0 & \text{otherwise}\ ,\end{cases}$$

$$p=r-q\ ,$$

$$\Delta_m(w)=\begin{cases}1 & \text{if}\quad \operatorname{sgn} w_m^*=1 \text{ and } \operatorname{sgn} a(1, p)=-1, \text{ or if}\\ & \operatorname{sgn} w_m^*=\operatorname{sgn} w=-1\ ,\\ 0 & \text{otherwise}\ .\end{cases}$$

Then

$$(2.9)\qquad w(w, m, n, x)=\begin{cases}w(p, n-|w|+p, x) & \text{if}\quad w<a(m+1, m+|w|)\ ,\\ 0 & \text{if}\quad w>a(m+1, m+|w|)\ ,\end{cases}$$

where w is of type (p, q). Now it remains to consider the case $w=a(m+1, m+|w|)$. In this case, we get

$$(2.10)\qquad w(w, m, n, x)=\begin{cases}w(m+|w|, n+m, x) & \text{if}\quad \operatorname{sgn} a(1, m)=1\ ,\\ w(m+|w|-p, m+n-p, x)-w(m+|w|, m+n, x) & \\ & \text{if}\quad \operatorname{sgn} a(1, m)=-1\ .\end{cases}$$

Thus by substituting (2.8) into (2.9) and (2.10), we obtain

$$(2.11)\qquad w(w, m, n, x)=\begin{cases}\operatorname{sgn} w_m^*\{\sum_{j\geqq p+1} b_j w(n-|w|+p+q-j, x) & \\ \quad +\chi(n-|w|+p+q, x)\}+\Delta_m(w)w(n-|w|+q, x) & \\ & \text{if}\quad w\leqq a(m+1, m+|w|)\ ,\\ 0 & \text{otherwise}\ .\end{cases}$$

On the other hand, if $q>0$ then $\Delta_m(w)=1$ and $b_{p+j}=-b_j$ for $1\leqq j\leqq q$.

Thus

$$\sum_{j\geq p+1} b_j w(n-|w|+p+q-j, x)+\Delta_m(w)w(n-|w|+q, x) \tag{2.12}$$

$$=\sum_{j=p+1}^{p+q} b_j w(n-|w|+p+q-j, x)+\sum_{j\geq p+q+1} b_j w(n-|w|+p+q-j, x)$$

$$+w(n-|w|+q, x)$$

$$=-\sum_{j=1}^{q} b_j w(n-|w|+q-j, x)+\sum_{j\geq p+q+1} b_j w(n-|w|+p+q-j, x)$$

$$+w(n-|w|+q, x)$$

$$=\sum_{j\geq q+1} b_j w(n-|w|+q-j, x)+\chi(n-|w|+q, x)$$

$$+\sum_{j\geq p+q+1} b_j w(n-|w|+p+q-j, x)$$

$$=\sum_{j\geq 1} (b_{j+q}+b_{j+p+q})w(n-|w|-j, x)+\chi(n-|w|+q, x) .$$

Therefore

$$b_j(w, m)=\begin{cases} b_{j+p} & \text{if } q=0 , \\ b_{j+q}+b_{j+p+q} & \text{if } q>0 . \end{cases} \tag{2.13}$$

This completes the proof.

THEOREM 2.6. *Let γ_1 be the greatest Fredholm eigenvalue, that is, $|\gamma_1|$ is greater than absolute value of any other Fredholm eigenvalues. Then the topological entropy $h(F)$ of the mapping F equals* $\log|\gamma_1|$.

PROOF. Clear from the fact that $z=\gamma_1$ is a singular point of $w(z; x)$.

THEOREM 2.7. *Suppose that the mapping F is piecewise linear with its slope λ. Then we get $\lambda=\gamma_1$.*

PROOF. We devide the proof into three steps.

i) Note that

$$F(x)=\begin{cases} \lambda(x-a_1^x\lambda^{-1}) & \text{if } \operatorname{sgn} a_1^x=1 , \\ -\lambda(x-a_1^x\lambda^{-1}) & \text{if } \operatorname{sgn} a_1^x=-1 . \end{cases} \tag{2.14}$$

Thus we get

$$x=(a_1^x+e^x)\lambda^{-1}+\operatorname{sgn} a_1^x F(x)\lambda^{-1} , \tag{2.15}$$

where

$$e^x=\begin{cases} 0 & \text{if } \operatorname{sgn} a_1^x=1 , \\ 1 & \text{if } \operatorname{sgn} a_1^x=-1 . \end{cases}$$

Therefore, repeating this procedure, we get that λ is one of the Fredholm eigenvalues. This shows $\lambda \leqq |\gamma_1|$.

ii) Assume that the mapping F is of periodic type. Then, since the number of subintervals of the form $F^{(|w|-1)}((w))$ ($w \in W(F)$) is finite, there exists a constant $B>0$ such that for any $w \in W(F)$, the length of the interval $F^{(|w|-1)}((w))$ is greater than B. This means that the length of (w) is greater than $\lambda^{-|w|+1}B$. Hence

$$w(n, \phi) \leqq \lambda^{n-1} B^{-1} . \tag{2.16}$$

This shows

$$\begin{aligned} h(F) &= \lim_{n\to\infty} \frac{1}{n} \log w(n, \phi) \\ &\leqq \log \lambda . \end{aligned} \tag{2.17}$$

Therefore, by Theorem 2.6, we get $\lambda \geqq |\gamma_1|$. Hence, by i), for a mapping which is of periodic type we get $\lambda = \gamma_1$.

iii) For any F which is aperiodic, there exists a sequence of mappings F_n which is M_n-periodic for some M_n such that

$$\lim_{n\to\infty} \lambda_n = \lambda$$

and there exists N_n such that

$$\lim_{n\to\infty} N_n = \infty$$

and

$$b_{i,n} = b_i \qquad 1 \leqq i \leqq N_n ,$$

where λ is the slope of F and λ_n is the slope of F_n, and

$$\Phi(z, F) = 1 - \sum_i b_i z^i ,$$

$$\Phi(z, F_n) = 1 - \sum_{i=1}^{M_n} b_{i,n} z^i - z^{M_n}$$

are the Fredholm determinant of F and F_n, respectively. Thus, by Rouché's theorem the Fredholm eigenvalues of F_n which is greater than 1 in modulus converge to that of F. This completes the proof.

Concerning the convergence of a Fredholm eigenvalue, we get:

THEOREM 2.8. *Let Φ be a Fredholm determinant and $\{\Phi_n\}$ be a sequence of Fredhlom determinants which satisfy*

$$0 \leqq b_{i,n} \leqq k$$

and

$$b_i = b_{i,n} \quad \textit{for} \quad 1 \leqq i \leqq n \ ,$$

where

$$\Phi(z) = 1 - \sum_i b_i z^i$$
$$\Phi_n(z) = 1 - \sum_i b_{i,n} z^i \ .$$

Let γ ($|\gamma|>1$) be a Fredholm eigenvalue of Φ with multiplicity p. Then for any $\varepsilon>0$ ($|\gamma|-\varepsilon>1$) and sufficiently large n, there exist Fredholm eigenvalues $\gamma_{1,n}, \cdots, \gamma_{p,n}$ of Φ_n such that

$$|\gamma - \gamma_{i,n}| < (|\gamma| - \varepsilon)^{-n/p} \qquad 1 \leqq i \leqq p \ . \tag{2.18}$$

PROOF. From the assumption, there exists a constant $C \neq 0$ such that

$$\Phi(z) = C(z - \gamma^{-1})^p + \text{small order} \tag{2.19}$$

as $z \to \gamma^{-1}$. On the other hand, there exists a constant $D>0$ such that

$$|\Phi_n(z) - \Phi(z)| < D(|\gamma|^{-1} + \delta)^n \tag{2.20}$$

for $|z - \gamma^{-1}| = \delta$. Take $\delta = (|\gamma| - \varepsilon)^{-n/p}$. Then

$$\delta^p (|\gamma|^{-1} + \delta)^{-n} = (|\gamma| - \varepsilon)^{-n} (|\gamma|^{-1} + (|\gamma| - \varepsilon)^{-n/p})^{-n}$$

tends to zero as $n \to \infty$. This shows, for sufficiently large n,

$$|\Phi_n(z) - \Phi(z)| < |\Phi(z)| \tag{2.21}$$

for $|z - \gamma^{-1}| = \delta$. By Rouché's theorem, this proves the theorem.

§3. The decay of correlation.

In this section, we will prove Theorem 0.1.

THEOREM 3.1. *Assume that*

$$\rho(x) = -\lambda(\Phi'(1/\lambda))^{-1} \chi(1/\lambda : x) \geqq 0 \tag{3.1}$$

Then $\rho(x)$ is the density of the invariant probability measure for the mapping F.

PROOF. Noticing the fact that

$$\chi(n, x) \operatorname{sgn} a(1, n) = \begin{cases} 1 & \text{if} \quad x < a^1_{n+1} a^1_{n+2} \cdots \\ 0 & \text{if} \quad x > a^1_{n+1} a^1_{n+2} \cdots \end{cases} \tag{3.2}$$

we can prove this theorem in the same way as in [4], thus we omit the proof. By μ, we denote the invariant measure with its density ρ.

DEFINITION. i)
BV= the set of functions with bounded variation on the unit interval,
$V(f)$= the total variation of a function $f \in BV$,
L^∞= the set of bounded measurable functions on the unit interval,
$\|f\| = \text{ess sup}_{x\in[0,1]} |f(x)|$,
L^1= the set of integrable functions with respect to the Lebesgue measure on the unit interval.

ii) For $f, g \in L^1$, we define operators $P = P_F$ and $Q^n = Q_F^n$ $(n \geqq 1)$ by

$$\int Pf(x)g(x)d\mu = \int f(x)g(F(x))d\mu ,$$

$$Q^n f(x) = P^n f(x) - \int f d\mu .$$

REMARK. The operator P is called the Perron-Frobenius operaror. On the property of Q^n, see [13] (cf. also [3], [7]).

iii) For a monotone function f, we define

$$(f)_0 = \min_x f(x) ,$$

and for $n \geqq 1$, we inductively define functions

$$(f)_n = \sum_{w;|w|=n} B_w I_w + (f)_{n-1} ,$$

where

$$B_w = \min_{x\in(w)} \{f(x) - (f)_{n-1}(x)\}$$

and I_w is the indicator function of the interval (w).

iv) We define ν^ε in the following way:

a) For a monotone function f, we define

$$\nu_n^\varepsilon(f) = \sum_{m=0}^{n} \sum_{w;|w|=m} |B_w|(\eta+\varepsilon)^{-m} .$$

b) For a function $f \in BV$, we define

$$\nu_n^\varepsilon(f) = \inf\{\nu_n^\varepsilon(f_1) + \nu_n^\varepsilon(f_2)\} ,$$

where infimum is taken over all f_1 and f_2 which are monotone and $f_1 + f_2 = f$.

c) For a function $f \in BV$, we define

$$\nu^{\varepsilon}(f)=\overline{\lim_{n\to\infty}}\,\nu_n^{\varepsilon}(f)\ .$$

LEMMA 3.2. *For a function $f\in BV$,*

$$\text{(3.3)}\quad \text{i)}\qquad \nu^{\varepsilon}(f)<\infty\ ,$$

$$\text{(3.4)}\quad \text{ii)}\qquad \left|\int [f(x)-\{(f_1)_n(x)+(f_2)_n(x)\}]dx\right|\leqq V(f)\lambda^{-n}\ ,$$

if $f_1+f_2=f$ and both f_i $(i=1, 2)$ are monotone.

PROOF. Assume that the function f is monotone. Then it is trivial that

$$\text{(3.5)}\qquad \nu_n^{\varepsilon}(f)\leqq V(f)\{1+(k+1)(\eta+\varepsilon-1)^{-1}\}\ .$$

This shows i). For $j=1$ or 2, we get

$$\text{(3.6)}\qquad \left|\int (f_j(x)-(f_j)_n(x))dx\right|\leqq V(f_j)\lambda^{-n}\ .$$

This shows ii).

LEMMA 3.3. *Assume that the greatest Fredholm eigenvalue λ is simple. Then for any $\varepsilon>0$, there exists a constant K such that*

$$\text{(3.7)}\qquad |w(n, x)-\rho(x)\lambda^n|<K(\eta+\varepsilon)^n\ .$$

PROOF. By Theorem 2.4 and the definition of ρ,

$$w(z; x)-\rho(x)(1-\lambda z)^{-1}=\chi(z; x)/\Phi(z)-\rho(x)(1-\lambda z)^{-1}$$

is analytic in $|z|<\eta^{-1}$. Thus for $|z|<(\eta+\varepsilon)^{-1}$ there exists a constant K such that

$$\text{(3.8)}\qquad |w(z; x)-\rho(x)(1-\lambda z)^{-1}|<K\ .$$

This proves the lemma.

For a word $w\in W(F)$ and $g\in L^{\infty}$, by Lemma 2.5, we get

$$\begin{aligned}\text{(3.9)}\quad \left|\int Q^n I_w(x)g(x)d\mu\right| &= \left|\lambda^{-n}\sum_{|u|=n}\int I_w(ux)\rho(ux)g(x)dx-\mu((w))\int gd\mu\right|\\ &= \left|\lambda^{-n}\sum_{|v|=n-|w|}\int \rho(wvx)g(x)dx-\mu((w))\int gd\mu\right|\\ &= \left|\lambda(\Phi'(1/\lambda))^{-1}\lambda^{-n}\sum_{m\geq 0}\lambda^{-m}\sum_{|v|=n-|w|}\int \chi(m, wvx)g(x)dx+\mu((w))\int gd\mu\right|\\ &= \left|\lambda(\Phi'(1/\lambda))^{-1}\sum_{m\geq 0}\lambda^{-n-m}\operatorname{sgn} a(1, m)\int w(w, m, n, x)g(x)dx\right.\end{aligned}$$

$$+\mu((w))\int g d\mu\Big|$$

$$\leq\Big|\lambda(\Phi'(1/\lambda))^{-1}\sum_{m\geq 0}\lambda^{-n-m}\operatorname{sgn} a(1, m)\sum_{j=1}^{n-|w|}{}' b_j(w, m)\int w(n-|w|-j, x)$$

$$\times g(x)dx+\mu((w))\int g d\mu\Big|+2|\lambda(\Phi'(1/\lambda))^{-1}|\,\|g\|\lambda^{-n+1}/(\lambda-1)$$

$$\leq\Big\{\Big|\lambda(\Phi'(1/\lambda))^{-1}\sum_{m\geq 0}\sum_{j=1}^{n-|w|}{}'\operatorname{sgn} a(1, m)b_j(w, m)\lambda^{-m-|w|-j}$$

$$+\mu((w))\Big|+|\lambda(\Phi'(1/\lambda))^{-1}\sum_{m\geq 0}\sum_{j\geq n-|w|-1}|b_j(w, m)|\lambda^{-m-|w|-j}$$

$$\times\sum_{m\geq 0}\lambda^{-n-m}2k\sum_{j=0}^{n-|w|}{}'\int|w(n-|w|-j, x)-\rho(x)\lambda^{n-|w|-j}|dx$$

$$+2\lambda^{-n+1}/(\lambda-1)\Big\}\|g\|\,.$$

Similarly as for the calculation of $w(w, m, n, x)$, we get

$$-\lambda(\Phi'(1/\lambda))^{-1}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\operatorname{sgn} a(1, m)b_j(w, m)\lambda^{-m-|w|-j}=\mu((w))\,. \tag{3.10}$$

Thus we conclude that the first term in (3.9) equals zero. On the other hand, by Lemma 3.3, there exists a constant K' such that for $n\geq|w|$

(3.11) the remaining term of $(4.9)<K'((\eta+\varepsilon)/\lambda)^n(\eta+\varepsilon)^{-|w|}\|g\|$
$=K'((\eta+\varepsilon)/\lambda)^n\|g\|\nu_n^\varepsilon(I_w)$.

For a function $f\in BV$, consider monotone functions f_1, f_2 such that $f=f_1+f_2$, then,

$$\Big|\int Q^n f(x)g(x)d\mu\Big|\leq\Big|\int Q^n\{(f_1)_n+(f_2)_n\}(x)g(x)d\mu\Big|$$
$$+\Big|\int Q^n[f-\{(f_1)_n+(f_2)_n\}](x)g(x)d\mu\Big|\,. \tag{3.12}$$

By the linearity of Q^n,

(3.13) the first term of right hand side of (3.12)
$<K'((\eta+\varepsilon)/\lambda)^n(\nu_n^\varepsilon(f_1)+\nu_n^\varepsilon(f_2))\|g\|$.

On the other hand,

the second term of right hand side of (3.12)

$$\leq\Big|\int P^n[f-\{(f_1)_n+(f_2)_n\}](x)g(x)d\mu\Big|$$

$$+\left|\int [f(x)-\{(f_1)_n(x)+(f_2)_n(x)\}]d\mu\int gd\mu\right|$$

$$\leqq 2\|\rho\|\,\|g\|\left|\int[f(x)-\{(f_1)_n(x)+(f_2)_n(x)\}]dx\right|,$$

which by Lemma 3.2 ii),

$$\leqq 2\|\rho\|\,\|g\|\,V(f)\lambda^{-n}. \tag{3.14}$$

Hence,

$$\left|\int Q^n f(x)g(x)d\mu\right| < K'((\eta+\varepsilon)/\lambda)^n\|g\|\,\nu_n^*(f)+2\|\rho\|\,\|g\|\,V(f)\lambda^{-n}. \tag{3.15}$$

Therefore we conclude that

$$\overline{\lim_{n\to\infty}}((\eta+\varepsilon)/\lambda)^{-n}\left|\int Q^n f(x)g(x)d\mu\right| < K'\|g\|\nu^*(f),$$

and this proves the latter half of ii) of Theorem 0.1. On the other hand if the dynamical system $([0, 1], \mu, F)$ is mixing and μ is absolutely continuous, then for $f, g\in L^\infty$,

$$\begin{aligned}\int fd\mu\int \rho(x)d\mu &= \lim_{n\to\infty}\int f(F^{(n)}(x))\rho(x)d\mu \\ &= \lim_{n\to\infty}\int f(F^{(n)}(x))\frac{d\mu}{dx}(x)\rho(x)dx \\ &= \int f\rho(x)dx\int\frac{d\mu}{dx}(x)\rho(x)dx.\end{aligned} \tag{3.16}$$

Therefore

$$\int fd\mu=\int f(x)\rho(x)dx. \tag{3.17}$$

This shows $(d\mu/dx)(x)=\rho(x)$ a.e., this completes the proof.

REMARK. In our case, all the mappings considered are mixing except unimodal linear transformations with its slope$\leqq\sqrt{2}$ (cf. [1], [4], [5], [7]).

§4. Critical phenomena.

Among the mappings which we considered in this paper, there are two critical states. One is the case when $\lambda\downarrow 1$ for β-transformations and the other is the case when $\lambda\downarrow\sqrt{2}$ for unimodal linear transformations.

For a β-transformation with its slope λ which is sufficiently close to 1, there exists N such that the Fredholm determinant is of the form:

$$\Phi(z)=1-z-z^N G(z)\ , \tag{4.1}$$

where

$$G(z)=1+z^{N_1}(1+z^{N_2}(\cdots)\cdots) \tag{4.2}$$
$$N\leqq N_i\leqq\infty \quad (\text{we consider } z^\infty=0)\ .$$

In [8], they considered the special case $N_1=\infty$. For a unimodal linear transformation F with its slope λ which is sufficiently close to $\sqrt{2}$, there exists N such that

$$(1+z)\Phi(z)=1-2z^2-2z^N G(z)\ , \tag{4.3}$$

where $G(z)$ is analytic in the domain $|z|<1$.

THEOREM 4.1. i) *For a β-transformation F with the Fredholm determinant of the form* (4.1), *the greatest Fredholm eigenvalues exist around* 1 *and they are of the asymptotic form*:

$$e^{\alpha+i\beta}\ , \tag{4.4}$$

where

$$\alpha=\frac{1}{N}\left(\log N-\log\log N+\frac{\log\log N}{\log N}+\text{small order}\right) \tag{4.5}$$

and

$$\beta=\frac{1}{N}\left(n\pi+(-1)^{n-1}\frac{n}{\log N}\pi+\text{small order}\right) \qquad n=0,\ \pm1,\ \pm2,\ \cdots \tag{4.6}$$

The greatest Fredholm eigenvalue (=the slope λ) is the case when $n=0$ and the second Fredholm eigenvalues are the case $n=\pm1$. The decay rate of correlation η/λ is asymptotically of the form:

$$\eta/\lambda=\exp-\frac{1}{2N}(\pi/\log N)^2+\text{small order}\ , \tag{4.7}$$

as $\lambda\downarrow 1$. N is expressed by λ asymptotically of the form:

$$N=-\frac{\log\log\lambda}{\log\lambda}\left[1+\frac{1}{2\log\log\lambda}\left\{\frac{\log(-\log\log\lambda)}{\log\log\lambda}\right\}^2+\text{small order}\right]. \tag{4.8}$$

ii) For a unimodal linear transformation F with the Fredholm determinant of the form (4.3),

$$\lambda=\sqrt{2}\exp 2^{N/2}(g+\text{small order})\,, \tag{4.9}$$

and the second Fredholm eigenvalue whose argument equals π is asymptotically of the form:

$$-\sqrt{2}\exp 2^{N/2}(-g+\text{small order})\,, \tag{4.10}$$

as $\lambda\downarrow\sqrt{2}$ and other eigenvalues are much smaller that the above two solutions, where

$$g=G(2^{-1/2})\,.$$

The decay rate of the correlation η/λ is asymptotically of the form:

$$\eta/\lambda=2\lambda^{-2}+\text{small order}\,. \tag{4.11}$$

PROOF. We will consider β-transformations. Let for an integer n

$$z=\exp\frac{1}{N}\{\varepsilon+i(n+\tau)\pi\}\,.$$

Then the real and imaginary part of the equation $\Phi(z)=0$ become

$$1-e^{\varepsilon/N}\cos\frac{(n+\tau)\pi}{N}=e^{\varepsilon}\{\cos(n+\tau)\pi\operatorname{Re}G+\sin(n+\tau)\pi\operatorname{Im}G\}\,, \tag{4.12}$$

$$-e^{\varepsilon/N}\sin\frac{(n+\tau)\pi}{N}=e^{\varepsilon}\{\sin(n+\tau)\pi\operatorname{Re}G+\cos(n+\tau)\pi\operatorname{Im}G\}\,, \tag{4.13}$$

where $\operatorname{Re}G$ and $\operatorname{Im}G$ are the real and the imaginary part of G, respectively.

LEMMA 4.2. *Let $\varepsilon=\varepsilon(n, N)$ be a solution of* (4.12) *and* (4.13). *Then*

$$\lim_{N\to\infty}(\varepsilon+\log N)/\log\log N\geqq 1\,. \tag{4.14}$$

PROOF. Since the minimum solution is $1/\lambda$ (i.e. $n=\tau=0$), it is sufficient to consider only this case. Suppose that (4.14) does not hold, then there exists a subsequence, which we also write by $\{N\}$, and $\alpha>0$ such that for sufficiently large N,

$$\varepsilon<-\log N+(1-\alpha)\log\log N\,.$$

Therefore, since

(4.15) $$|\mathrm{Re}\, G-1|<\left(1-\left(\frac{\log N}{N}\right)^{1-\alpha}\right)^{-1} \longrightarrow 1\,,$$

the right hand term of (4.12) $N(\log N)^{\alpha-1}<1$

for sufficiently large N. On the other hand

(4.16) the left hand term of (4.12) $=1-e^{\varepsilon/N}=-\varepsilon/N+$small order $>N^{-1}\log N+$small order .

This contradicts (4.15).

Now we will show the existence of $\xi=(\xi_1, \xi_2)$ ($\varepsilon=-\log N+\log\log N+\xi_1$, $\xi_2=\tau$). Let for a given n

$$\varphi_1^{N,G}(\xi)=(N/\log N)\mathrm{Re}\, \Phi\left(\exp \frac{1}{N}\{\varepsilon+i(n+\xi_2)\}\right)\,,$$

$$\varphi_2^{N,G}(\xi)=(N/\log N)\mathrm{Im}\, \Phi\left(\exp \frac{1}{N}\{\varepsilon+i(n+\xi_2)\}\right)\,,$$

and

$$\varphi^{N,G}(\xi)=(\varphi_1^{N,G}(\xi),\ \varphi_2^{N,G}(\xi))\ .$$

Then

(4.17) $$\varphi^{N,G}(0)\longrightarrow 0$$

and

(4.18) $$J(\varphi^{N,G})(0)=(\partial\varphi_i^{N,G}/\partial\xi_j)(0)\longrightarrow \begin{pmatrix}-1 & 0\\ 0 & \beta\end{pmatrix}$$

uniformly in G as $N\to\infty$, where

$$\beta=\begin{cases}\pi & \text{if } n \text{ is even}\,,\\ -\pi & \text{if } n \text{ is odd}\,.\end{cases}$$

LEMMA 4.3 (inverse function theorem). *For sufficiently large N, there exist $r>0$ and neighbourhood of* 0 $U^{N,G}\subset Q(r)$ *and* $V^{N,G}\subset Q(r/2)$ *such that $\varphi^{N,G}$ is one to one and onto from $U^{N,G}$ to $V^{N,G}$, where*

$$Q(r)=\{\xi=(\xi_1, \xi_2)\colon |\xi_i|<r \quad (i=1, 2)\}\ .$$

PROOF. Let

$$\psi^{N,G}(\xi)=J^{-1}\varphi^{N,G}(\xi)-\xi\ ,$$

where

$$J=\begin{pmatrix}-1 & 0\\ 0 & \beta\end{pmatrix}.$$

Then by the continuity in ξ, for sufficiently large N, there exists $r>0$ such that the absolute value of each component of $J(\psi^{N,G})(\xi)$ is less than 1/4 for $\xi \in Q(r)$. Therefore for $\xi, \xi' \in Q(r)$,

$$|\psi^{N,G}(\xi)-\psi^{N,G}(\xi')|<\frac{1}{2}|\xi-\xi'|, \tag{4.19}$$

where

$$|\xi|=\max_{i=1,2}|\xi_i|.$$

On the other hand,

$$|\xi-\xi'|<|J^{-1}(\varphi^{N,G}(\xi)-\varphi^{N,G}(\xi'))|+|\psi^{N,G}(\xi)-\psi^{N,G}(\xi')|. \tag{4.20}$$

Hence

$$|J^{-1}(\varphi^{N,G}(\xi)-\varphi^{N,G}(\xi'))|>\frac{1}{2}|\xi-\xi'|. \tag{4.21}$$

This shows that $\varphi^{N,G}$ is one to one in the domain $Q(r)$. For any $\delta>0$ $(\delta/2<r)$, take N sufficiently large to satisfy

$$|\psi^{N,G}(0)|<\delta/2.$$

For any $\zeta \in Q((r-\delta)/2)$, let

$$\xi_0=0$$

and

$$\xi_k=\zeta-\psi^{N,G}(\xi_{k-1}) \qquad \text{for} \quad k\geqq 1.$$

Then

$$|\xi_k-\xi_{k-1}|=|\psi^{N,G}(\xi_{k-1})-\psi^{N,G}(\xi_{k-2})|<\frac{1}{2}|\xi_{k-1}-\xi_{k-2}|<2^{-k+1}|\xi_1|. \tag{4.22}$$

Then

$$\begin{aligned}|\xi_k|&\leqq|\xi_0|+\sum_{i=1}^{k}|\xi_i-\xi_{i-1}|<2|\xi_1|=2|\zeta-\psi^{N,G}(0)|\\ &\leqq 2(|\zeta|+|\psi^{N,G}(0)|)<r.\end{aligned} \tag{4.23}$$

This shows $\xi_k \in Q(r)$. Since $\{\xi_k\}$ is a Cauchy sequence, there exists $\xi \in Q(r)$

suce that

$$\xi=\zeta-\psi^{N,G}(\xi) ,$$

that is,

$$\varphi^{N,G}(\xi)=J\zeta .$$

Hence, putting

$$U^{N,G}=Q(r)\cap(J^{-1}\varphi^{N,G})^{-1}(Q((r-\delta)/2))$$

and

$$V^{N,G}=J(Q((r-\delta)/2)) ,$$

the assertion of Lemma 4.3 is proved.

By Lemma 4.3, there exists, for a given n, a unique solution $\xi(N)$ which satisfies

$$\varphi^{N,G}(\xi(N))=0$$

and

$$\lim_{N\to\infty}\xi(N)=0 .$$

This shows, for a given n, unique solutions $\varepsilon=\varepsilon(n, N)$ and $\tau=\tau(n, N)$ which satisfy

$$\Phi\left(\exp\frac{1}{N}\{\varepsilon+i(n+\tau)\pi\}\right)=0 .$$

Moreover, it is easy to show that the solutions of (4.12) and (4.13) satisfy

(4.24) $$\varepsilon=\varepsilon(n, N)=-\log N+\log\log N-\frac{\log\log N}{\log N}+\text{small order} ,$$

(4.25) $$\tau=\tau(n, N)=(-1)^{n-1}\frac{n}{\log N}+\text{small order} .$$

Now we will consider the decay rate of the correlation. Let

$$\sigma(n)=\varepsilon(n, N)-\varepsilon(0, N)$$

and

$$G(n)=G\left(\exp\left[\frac{1}{N}\{\varepsilon(n, N)+i(n+\tau(n, N))\}\right]\right) .$$

Since

(4.26) $$|1-z|^2=|z|^{2N}|G|^2 ,$$

we get

(4.27) $$1-2e^{\varepsilon(n,N)/N}\cos\frac{1}{N}(n\pi+\tau(n, N)\pi)+e^{2\varepsilon(n,N)/N} = e^{2\varepsilon(n,N)}|G(n)|^2 ,$$

(4.28) $$1-2e^{\varepsilon(0,N)/N}+e^{2\varepsilon(0,N)/N}=e^{2\varepsilon(0,N)}|G(0)|^2 .$$

Hence

(4.29) $$(n+\tau(n, N))^2\pi^2N^{-2}+\text{small order}=e^{2\varepsilon(0,N)}(e^{2\sigma(n)}-1) + e^{2\varepsilon(n,N)}(|G(n)|^2-1)-e^{2\varepsilon(0,N)}(|G(0)|^2-1) .$$

Therefore

(4.30) $$\sigma(n)=\frac{1}{2}\left(\frac{n\pi}{\log N}\right)^2+\text{small order} .$$

On the other hand, it is trivial that if $n\geqq 0(\log N)$, then there exists no solutions with order as one of (4.24) and if $n=o(\log N)$, (4.24) and (4.25) holds. Thus the second greatest Fredholm eigenvalue are the case $n=\pm 1$ and

(4.31) $$\eta/\lambda=e^{-\sigma(1)/N}=\exp\left\{-\frac{1}{2N}\left(\frac{\pi}{\log N}\right)^2+\text{small order}\right\} .$$

This completes i) of Theorem 4.1. The proof of ii) is almost the same, thus we omit it.

For a noncritical state, we get:

THEOREM 4.4. *Let F be a piecewise linear mapping and $\{F_N\}$ be a sequence of piecewise linear mappings such that*

$$b_i=b_i^N \qquad \text{for} \quad 1\leqq i\leqq N ,$$

where

$$\Phi(z, F)=1-\sum_i b_i z^i \qquad \text{(the Fredholm determinant of } F) ,$$

$$\Phi(z, F_N)=1-\sum_i b_i^N z^i \qquad \text{(the Fredholm determinant of } F_N) .$$

Then the decay rate of the correlation η/λ for F and η_N/λ_N for F_N satisfy, for any $\varepsilon>0$ and for sufficiently large N,

(4.32) $$|\eta/\lambda-\eta_N/\lambda_N|<(|\gamma|-\varepsilon)^{-N/p} ,$$

where λ (λ_N) is the slope of F (F_N), respectively,

$$\gamma=\begin{cases}\gamma_2 & (=\text{the second greatest Fredholm eigenvalue of } \Phi(z, F)) \\ & \quad \text{if } |\gamma_2|>1\,, \\ \lambda & \quad \text{if } |\gamma_2|<1\,,\end{cases}$$

and p is the multiplicity of the Fredholm eigenvalue γ.

REMARK. If $|\gamma_2|=1$, we cannot evaluate the order.

PROOF.

$$|\eta/\lambda-\eta_N/\lambda_N|\leqq|\eta/\lambda|\,|(\lambda-\lambda_N)/(\lambda-(\lambda-\lambda_N))| + |(\eta-\eta_N)/(\lambda-(\lambda-\lambda_N))|\,.$$

Thus by Theorem 2.8, the assertion of Theorem 4.4 easily follows.

REMARK. It seems that the rigorous studies on the power spectrum of mappings are important application, but we find them only in [8], [17] and [18].

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