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Lipschitz Classes of Periodic Stochastic Processes and Fourier Series

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Introduction

Let $X(t, \omega)$ be a complex valued stochastic process on a complete probability space (Ω, F, P), $t \in R^1$, $\omega \in \Omega$. Suppose throughout that $X(t, \omega)$ is measurable $L \times F$ on $R^1 \times \Omega$, L being the class of Lebesgue measurable sets on R^1. Assume also that $X(t, \omega)$ is an L^r-process, namely $X(t, \omega) \in L^r(\Omega)$ for each $t \in R^1$, $1 \leqq r < \infty$ and that $X(t, \omega)$ is 2π-periodic in the sense that

$$(0.1) \qquad E|X(t+2\pi, \omega) - X(t, \omega)| = 0 ,$$

for each $t \in R^1$. For an L^2-process $X(t, \omega)$, (0.1) is equivalent to $E|X(t+2\pi, \omega) - X(t, \omega)|^2 = 0$ which, as we easily see, is equivalent also to the condition that the covariance function $\rho(u, v)$ of $X(t, \omega)$ is 2π-periodic with respect to each of u and v.

Write

$$(0.2) \qquad \|X(t, \cdot)\|_r = \|X(t, \omega)\|_r = [E|X(t, \omega)|^r]^{1/r} ,$$
$$\|X(\cdot, \cdot)\|_{s,r} = \|X(t, \omega)\|_{s,r} = \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \left\|X(t, \cdot)\right\|_r^s dt\right]^{1/s} .$$

The class of $X(t, \omega)$ for which $\|X(\cdot, \cdot)\|_{s,r} < \infty$ for some $1 \leqq r < \infty$, $1 \leqq s \leqq \infty$ is denoted by $L^{s,r} = L^{s,r}(T \times \Omega)$, $T = [-\pi, \pi]$.

Write, for a positive integer p, the p-th difference of $X(t, \omega)$ with increment h of t, by

$$(0.3) \qquad \Delta_h^{(p)} X(t, \omega) = \sum_{k=0}^{p} (-1)^{p-k} \binom{p}{k} X(t+kh, \omega)$$

and define, for $\delta > 0$,

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(0.4) $$M_{s,r}^{*(p)}(\delta)=M_{s,r}^{*(p)}(\delta, X)=\sup_{|h|\leq\delta}\|\Delta_h^{(p)}X(\cdot, \cdot)\|_{s,r} ,$$

(0.5) $$M_r^{(p)}(\delta)=M_r^{(p)}(\delta, X)=M_{\infty,r}^{*(p)}(\delta)=\sup_{|h|\leq\delta}\sup_{|t|\leq\pi}\|\Delta_h^{(p)}X(t, \cdot)\|_r .$$

We agree to call (0.5) and (0.4) the *mean modulus of continuity of order p* and the *mean integrated modulus of continuity of order p* respectively.

It is obvious that $M_{s,r}^{(p)}(\delta)$ and $M_r^{(p)}(\delta)$ are nondecreasing functions of $\delta>0$. The following inequalities are obvious too.

(0.6) $$M_{s,r}^{*(p)}(\delta)\leq 2^p M_{s,r}^{*(1)}(\delta) , \qquad M_r^{(p)}(\delta)\leq 2^p M_r^{(1)}(\delta) .$$

For any $\lambda>0$, we see

(0.7) $$M_{s,r}^{*(p)}(\lambda\delta)\leq(1+\lambda)^p M_{s,r}^{*(p)}(\delta) , \qquad M_r^{(p)}(\lambda\delta)\leq(1+\lambda)^p M_r^{(p)}(\delta) .$$

This is proved based on the identity

(0.8) $$\Delta_{nh}^{(p)}X(t, \omega)=\sum_{k_1=0}^{n-1}\cdots\sum_{k_r=0}^{n-1}\Delta_h^{(p)}X(t+k_1h+\cdots+k_rh, \omega), n\geq 1 ,$$

(see [4] Problem 1.5. 3. p. 76).

Let $\psi(t)$ be a nondecreasing continuous function on [0, 1] with $\psi(0)=0$. The class of $X(t, \omega)\in L^{s,r}(T\times\Omega)$ for which $M_{s,r}^{*(p)}(\delta)\leq\psi(\delta)$, $0\leq\delta\leq 1$, is called *Lipschitz class* $\Lambda_{s,r}^{*(p)}(\psi)=\Lambda_{s,r}^{*(p)}(\psi(\delta))$. $\Lambda_{\infty,r}^{*(p)}(\psi)$ is defined by $\Lambda_r^{(p)}(\psi)$.

In §1, we consider the approximation of $X(t, \omega)\in L^{s,r}(T\times\Omega)$ by trigonometric polynomials of the form

(0.9) $$P_n(t, \omega)=\sum_{k=-n}^{n}a_k(\omega)e^{ikt} ,$$

where $a_k(\omega)\in L^r(\Omega)$, $k=0, \pm1, \pm2, \cdots, \pm n$, and investigate some relationship between the magnitude of approximation error and the continuity modulus of $X(t, \omega)$.

In §3, some basic theorems concerning the membership of a function to Lipschitz class Λ_α of order α, $0<\alpha<1$ are generalized to the case of stochastic processes. In §4, we give the results on the relationship between Lipschitz classes defined above and the magnitude of Fourier coefficients. This sort of problems for the ordinary Fourier series is classical (see, f. ex. [6]).

In §6, we give theorems on the almost sure absolute convergence of Fourier series. In §7 and §8, we consider the mean derivatives of stochastic processes and give the conditions in terms of such derivatives for the almost sure absolute convergence of Fourier series. In §9, we give the results on sample continuity and sample differentiability of stochastic processes.

Results in this paper are generalizations of what have been previously shown for the case $s=r$ and are announced in [10], some corrections, improvements and additions being made.

§1. Trigonometric approximation and continuity modulus.

We begin with some remarks stating as lemmas. Write throughout this paper

$$\theta=\min(s, r) , \tag{1.1}$$

so that $1\leqq\theta<\infty$.

LEMMA 1.1. *For* $X(t, \omega) \in L^{s,r}(T\times\Omega)$, $1\leqq s\leqq\infty$, $1\leqq r<\infty$,

$$X(t, \omega) \in L^{\theta}(T) , \quad a.s. \tag{1.2}$$

PROOF. Suppose first $1\leqq s\leqq r$ so that $\theta=s$. By the Minkowski inequality, we have

$$\left\{E\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}|X(t, \omega)|^{s}dt\right]^{r/s}\right\}^{1/r}\leqq\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}[E|X(t, \omega)|^{r}]^{s/r}dt\right\}^{1/s}<\infty$$

which implies

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|X(t, \omega)|^{s}dt<\infty , \quad \text{a.s.}$$

Suppose next $1\leqq r\leqq s$. The Hölder inequality gives us

$$E\frac{1}{2\pi}\int_{-\pi}^{\pi}|X(t, \omega)|^{r}dt\leqq\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\|X(t, \omega)\|_{r}^{s/r}dt\right]^{r/s}<\infty$$

which gives us $X(t, \omega) \in L^{r}(T)$, a.s.

We may now define the Fourier series

$$X(t, \omega)\sim\sum_{n=-\infty}^{\infty}c_{n}(\omega)e^{int} , \tag{1.3}$$

where

$$c_{n}(\omega)=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(t, \omega)e^{-int}dt , \quad n=0, \pm 1, \cdots \tag{1.4}$$

a.s. The following is shown by the standard method [13].

LEMMA 1.2. *$L^{s,r}(T\times\Omega)$ is a Banach space with natural addition and scaler multiplication endowed with norm $\|X(\cdot,\cdot)\|_{s,r}$.*

We now consider the approximation of $X(t,\omega)\in L^{s,r}(T\times\Omega)$ by trigonometric polynomials of the form (0.9). Write

$$a(\omega)=\{a_k(\omega),\ |k|\leqq n\}\ , \tag{1.5}$$

$$\|a(\cdot)\|_r=\left[\sum_{k=-n}^{n}\|a_k(\cdot)\|_r^r\right]^{1/r}\ . \tag{1.6}$$

Define

$$e_n^{s,r}(X)=\inf_{a(\omega)}\|X(t,\omega)-\sum_{k=-n}^{n}a_k(\omega)e^{ikt}\|_{s,r}\ , \tag{1.7}$$

where the inf is taken over all $a(\omega)$ with $\|a(\cdot)\|_r<\infty$.

We note here that in [10] we erroneously announced that inf is actually attained by a trigonometric polynomial.

Now let $\sigma_n(t,\omega)$ be the $(C,1)$ mean of the Fourier series of a 2π-periodic stochastic process of $L^{s,r}(T\times\Omega)$ and form the De la Vallée Poussin mean

$$\tau_n(t,\omega)=2\sigma_{2n-1}(t,\omega)-\sigma_{n-1}(t,\omega)\ ,\qquad n\geqq 1\ . \tag{1.8}$$

We then have ([15] p. 115)

$$\tau_n(t,\omega)=\frac{2}{n\pi}\int_{-\infty}^{\infty}X(t+u,\omega)h(nu)/u^2du\ , \tag{1.9}$$

where

$$h(t)=\frac{1}{2}(\cos t-\cos 2t)\ ,\quad \frac{2}{\pi}\int_0^{\infty}h(u)/u^2du=1\ . \tag{1.10}$$

The following theorem is an analogue of Theorem 13.5 of [15].

THEOREM 1.1. *For $X(t,\omega)\in L^{s,r}(T\times\Omega)$, $1\leqq r<\infty$, $1\leqq s\leqq\infty$,*

$$\|\tau_n(\cdot,\cdot)-X(\cdot,\cdot)\|_{s,r}\leqq 4e_n^{s,r}(X)\ . \tag{1.11}$$

The proof is carried out in just a similar way. In fact, in arguing as in the proof of Theorem 13.5 of [15], we take a trigonometric polynomial $P_n(t,\omega)$ of the form (0.9) such that $\|X(\cdot,\cdot)-P_n(\cdot,\cdot)\|_{s,r}\leqq e^{s,r}(X)+1/N$, N being arbitrary positive number. We then obtain

$$\|\tau_n(\cdot,\ \cdot)-X(\cdot,\ \cdot)\|_{s,r}\leqq 4[e_n^{s,r}(X)+1/N]$$

from which (1.11) follows.

Let p be a positive integer and n be a multiple of $p!$ and define

$$\xi_n(t,\ \omega)=(-1)^{p-1}\sum_{\nu=1}^{p}(-1)^{p-\nu}\binom{p}{\nu}\tau_{n/\nu}(t,\ \omega) \tag{1.12}$$

which is a trigonometric polynomial of order $2n-1$.

LEMMA 1.3. *For* $X(t,\ \omega)\in L^{s,r}(T\times\Omega)$, $1\leqq r<\infty$, $1\leqq s\leqq\infty$,

$$\|X(\cdot,\ \cdot)-\xi_n(\cdot,\ \cdot)\|_{s,r}\leqq 2^{p+2}e_{n/p}^{s,r}(X)\ . \tag{1.13}$$

PROOF. We have

$$X(t,\ \omega)-\xi_n(t,\ \omega)=(-1)^p\sum_{\nu=1}^{p}(-1)^{p-\nu}\binom{p}{\nu}[\tau_{n/\nu}(t,\ \omega)-X(t,\ \omega)]\ .$$

Using Theorem 1.1, we have

$$\begin{aligned}\|X(\cdot,\ \cdot)-\xi_n(\cdot,\ \cdot)\|_{s,r}&\leqq\sum_{\nu=1}^{p}\binom{p}{\nu}\|X(\cdot,\ \cdot)-\tau_{n/\nu}(\cdot,\ \cdot)\|_{s,r}\\&\leqq 4\sum_{\nu=1}^{p}\binom{p}{\nu}e_{n/\nu}^{s,r}(X)\leqq 4\sum_{\nu=1}^{p}\binom{p}{\nu}e_{n/p}^{s,r}(X)\\&\leqq 4(2^p-1)e_{n/p}^{s,r}(X)\ .\end{aligned}$$

In what follows, n is not necessarily a multiple of $p!$.

THEOREM 1.2. *Let* $X(t,\ \omega)\in L^{s,r}(T\times\Omega)$, $1\leqq r<\infty$, $1\leqq s\leqq\infty$ *and* p *be a positive integer. Then there is a positive integer* n_0 *depending only on* p *such that for* $n>n_0$

$$e_n^{s,r}(X)\leqq 4M_{s,r}^{*(p)}\left(\frac{8\pi}{n}\right)\ . \tag{1.14}$$

PROOF. Let $n_0=2p!$. Choose a positive integer q such that $2qp\leqq n\leqq 2(q+1)p$ for any $n>n_0$. We have, writing $m=qp!$,

$$\begin{aligned}I&=\frac{2}{\pi}\int_{-\infty}^{\infty}\Delta_{u/m}^{(p)}X(t,\ \omega)h(u)/u^2du\\&=\sum_{\nu=0}^{p}(-1)^{p-\nu}\binom{p}{\nu}\frac{2}{\pi}\int_{-\infty}^{\infty}X(t+u/m,\ \omega)h(u)/u^2du\end{aligned}$$

$$=\sum_{\nu=0}^{p}(-1)^{p-\nu}\binom{p}{\nu}\tau_{m/\nu}(t,\omega),$$

where $\tau_{m/\nu}(t,\omega)$ is interpreted to be $X(t,\omega)$ when $\nu=0$. This is legitimate by (1.10). The last expression is

$$\sum_{\nu=1}^{p}(-1)^{p-\nu}\binom{p}{\nu}[\tau_{m/\nu}(t,\omega)-X(t,\omega)].$$

Hence

(1.15) $$I=(-1)^{p}[X(t,\omega)-\xi_m(t,\omega)].$$

On the other hand

(1.16) $$I=\frac{2}{\pi}\int_0^\infty D_t(u/m,\omega)h(u)/u^2du,$$

where

$$D_t(v,\omega)=\Delta_v^{(p)}X(t,\omega)-\Delta_{-v}^{(p)}X(t,\omega).$$

Hence

$$I=\frac{2}{\pi m}\int_0^\infty D_t(u,\omega)h(u)/u^2du,$$
$$=\frac{2}{\pi m}\sum_{k=0}^{\infty}\int_{2k\pi}^{2(k+1)\pi}D_t(u,\omega)h(mu)/u^2du$$

which is, by periodicity of $D_t(u,\omega)$ and $h(u)$, equal to

$$\frac{2}{\pi m}\int_0^{2\pi}D_t(u,\omega)h(mu)\sum_{k=0}^{\infty}(u+2k\pi)^{-2}du.$$

The last one is, in absolute value, not greater than

$$\frac{4}{\pi m}\int_0^{2\pi}|D_t(u,\omega)|\,|h(mu)|/u^2du.$$

Therefore from (1.15), we have

$$\|X(\cdot,\cdot)-\xi_m(\cdot,\cdot)\|_{s,r}\leqq\frac{4}{\pi m}\int_0^{2\pi}\|D_t(u,\omega)\|_{s,r}|h(mu)|/u^2du$$
$$\leqq\frac{8}{\pi m}\sup_{0\leqq u\leqq 2\pi}\|\Delta_{u/m}^{(p)}X(\cdot,\cdot)\|_{s,r}\int_0^{2\pi}|h(mu)|/u^2du$$
$$\leqq 4M_{s,r}^{*(p)}(2\pi/m).$$

Since $\|X(\cdot,\cdot)-\xi_m(\cdot,\cdot)\|_{s,r}\geqq e_{2m-1}^{s,r}(X)$, we have, for $n>n_0$,

$$e_n^{s,r}(X)\leqq e_{2m}^{s,r}\leqq e_{2m-1}^{s,r}(X)\leqq 4M_{s,r}^{*(p)}(2\pi/m)\leqq 4M_{s,r}^{*(p)}(2\pi/qp!)$$

which is, in view of $qp!>n/4$, not greater than $4M_{s,r}^{*(p)}(8\pi/n)$.

We shall now give an analogue of the Bernstein inequality for trigonometric polynomaials, the proof of which is carried out by a known argument (see f. ex. [4] 99–100).

THEOREM 1.3. *For*

$$P_n(t,\omega)=\sum_{k=-n}^{n}a_k(\omega)e^{ikt}\,, \tag{1.17}$$

the following inequality holds:

$$\left\|\frac{d^j}{dt^j}P_n(\cdot,\cdot)\right\|_{s,r}\leqq(2n)^j\|P_n(\cdot,\cdot)\|_{s,r} \tag{1.18}$$

for any nonnegative integer j, $1\leqq r<\infty$, $1\leqq s\leqq\infty$.

§ 2. Stochastic process of bounded variation.

A 2π-periodic L^r-process $X(t,\omega)$, $1\leqq r<\infty$, for which

$$\sup_D\sum_{j=1}^{n}\|X(t_j,\cdot)-X(t_{j-1},\cdot)\|_r<\infty\,, \tag{2.1}$$

sup being taken over all divisions D: $-\pi\leqq t_0<t_1<\cdots\leqq t_n\leqq\pi$, is called of bounded variation in $L^r(\Omega)$. The class of such processes is denoted by BV^r. The quantity of (2.1) is denoted by $V_r=V_r(X)$. This notion was used in [7]. For a later use, we prove

LEMMA 2.1. *Let $X(t,\omega)\in L^{s,r}(T\times\Omega)$, $1\leqq r<\infty$, $1\leqq s\leqq\infty$, be of bounded variation in $L^r(\Omega)$. Then, for every positive integer p, we have*

$$M_{s,r}^{*(p)}(\delta)\leqq C[M_r^{(p)}(\delta)]^{1-1/s}\delta^{1/s}\,,\quad \delta>0\,, \tag{2.2}$$

where $C=C_{p,s}V_r^{1/s}$, $C_{p,s}$ being a constant depending only on p and s.

PROOF. We know [8] that, for $1\leqq r<\infty$,

$$\int_{-\pi}^{\pi}\|\Delta_h^{(p)}X(t,\cdot)\|_r dt\leqq 2^p|h|\,V_r\,. \tag{2.3}$$

Now

$$M^{*(p)}_{s,r}(\delta)=\sup_{|h|\leq\delta}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\|\Delta_h^{(p)}X(t,\cdot)\|_r^{s-1}\|\Delta_h^{(p)}X(t,\cdot)\|_r dt\right]^{1/s}$$

$$\leq\sup_{|h|\leq\delta}\left[\sup_t\|\Delta_h^{(p)}X(t,\cdot)\|_r^{1-1/s}\right]\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\|\Delta_h^{(p)}X(t,\cdot)\|_r dt\right]^{1/s}.$$

This is true also for $s=\infty$. Using (2.3) we have

$$M^{*(p)}_{s,r}(\delta)\leq[M^{(p)}_r(\delta)]^{1-1/s}2^{p/s}(2\pi)^{-1/s}V_r^{1/s}\delta^{1/s}$$

which shows (2.2) with $C_{p,s}=2^{p/s}(2\pi)^{-1/s}$.

§ 3. Lipschitz classes and trigonometric approximation.

In the approximation theory the relationship between the membership of a function to a Lipschitz class Λ_α and the order of approximation of the function by trigonometric polynomials provides a fundamental problem. We shall consider the similar problem for the Lipschitz class $\Lambda^{*(p)}_{s,r}(\delta^\alpha)$ and the approximations of a periodic process in $L^{s,r}$. Actually we give analogues of basic Jackson and Bernstein theorems (see [4]). We also show an analogue of a basic result of Alexits and Kralik on the strong approximation [1], [2]. More detailed results in approximation and generalizations on the strong approximation [12], [13] and [16] will be able to be extended to the case of stochastic processes, but we do not attempt to do in this paper and we just show the very basic results.

THEOREM 3.1. *Let $X(t,\omega)\in L^{s,r}(T\times\Omega)$, $1\leq r<\infty$, $1\leq s\leq\infty$.*
(i) *If*

$$X(t,\omega)\in\Lambda^{*(p)}_{s,r}(\delta^\alpha)\,,\tag{3.1}$$

for some positive integer p and for some $\alpha>0$, then

$$e_n^{s,r}(X)=O(n^{-\alpha})\,,\quad n\to\infty\,.\tag{3.2}$$

(ii) *If* (3.2) *holds for some $\alpha>0$, then for any positive integer p,* (3.1) *holds, when $p>\alpha$, and when $1\leq p<\alpha$*

$$X(t,\omega)\in\Lambda^{*(p)}_{s,r}(\delta^p)\,,\tag{3.3}$$

holds. When $p=\alpha$, (3.2) *implies*

$$X(t,\omega)\in\Lambda^{*(p)}_{s,r}(\delta^p|\log\delta|)\,.\tag{3.4}$$

This is an analogue of Theorem 2.3.3 or Theorem 2.3.5 of [4].

THEOREM 3.2. *Let $X(t,\omega)\in L^{s,r}(T\times\Omega)$, $1<s, r<\infty$ and let $0<\alpha<1$.*

The following four statements (3.5)$\sim$(3.8) *are equivalent to each other.*

$$X(t, \omega) \in \Lambda^{*(p)}_{s,r}(\delta^{\alpha}) \text{ , for some positive integer } p \text{ ,} \tag{3.5}$$

$$e^{s,r}_{n}(X) = O(n^{-\alpha}) \text{ , as } n \to \infty \text{ ,} \tag{3.6}$$

$$\|\sigma_n(\cdot, \cdot) - X(\cdot, \cdot)\|_{s,r} = O(n^{-\alpha}) \text{ , as } n \to \infty \text{ ,} \tag{3.7}$$

and

$$\frac{1}{n}\sum_{k=1}^{n} \|S_k(\cdot, \cdot) - X(\cdot, \cdot)\|_{s,r} = O(n^{-\alpha}), \text{ as } n \to \infty \text{ ,} \tag{3.8}$$

where $S_n(t, \omega) = S_n(t, \omega; X)$ *and* $\sigma_n(t, \omega) = \sigma_n(t, \omega; X)$ *are respectively the partial sum and the* $(C, 1)$ *mean of the Fourier series of* $X(t, \omega)$.

REMARK. If $0 < \alpha < 1$ and (3.5) holds for some positive integer p, then it does for any positive integer p.

This is obvious from Theorem 3.1 (i) and the first part of (ii). The proof of Theorem 3.2 as well as of the following theorem will be given in the following section. The equivalence of (3.5), (3.6) and (3.7) is true also for $s = r = 1$.

THEOREM 3.3. *Suppose* $X(t, \omega) \in L^{s,r}(T \times \Omega)$, $1 \leqq s,\ r < \infty$, *is* $L^{s,r}$-*continuous. For* $0 < \alpha < 1$, *each of four statements in Theorem* 3.2 *is also equivalent to*

$$\|\sigma_{2n}(\cdot, \cdot) - \sigma_n(\cdot, \cdot)\|_{s,r} = O(n^{-\alpha}) \text{ , as } n \to \infty \text{ .} \tag{3.9}$$

Note that $X(t, \omega)$ is always $L^{r,r}$-continuous, since

$$\|X(\cdot + h, \cdot) - X(\cdot, \cdot)\|^{r}_{r,r} = E\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(t+h, \omega) - X(t, \omega)|^{r} dt \text{ ,}$$

in which the inner integral is bounded and $X(t, \omega)$ is $L^{r}(T)$-continuous for a.s. and the dominated convergence theorem is applied.

We shall give the proof of Theorem 3.1.

PROOF OF THROREM 3.1. (i) Suppose $X(t, \omega) \in \Lambda^{*(p)}_{s,r}(\delta^{\alpha})$ for some positive integer p and some $\alpha > 0$. By Theorem 1.2, there is an n_0 such that for $n > n_0$,

$$e^{s,r}_{n}(X) \leqq 4M^{*(p)}_{s,r}(8\pi/n) = O(n^{-\alpha}) \text{ .}$$

(ii) is shown by the same way as in the proof of Theorem 2.3.3 or 2.3.5 of [4]. Suppose $e^{s,r}_{n}(X) = 0(n^{-\alpha})$, $0 < \alpha < p$. Let $\tau_n(t, \omega)$ be the De la Vallee

Poussin mean of $X(t, \omega)$ defined by (1.8). Write

$$U_2(t, \omega)=\tau_{2^2}(t, \omega)\ , \quad U_n(t, \omega)=\tau_{2^n}(t, \omega)-\tau_{2^{n-1}}(t, \omega)\ .$$

Then for $n\geqq 2$ using Theorem 1.1, we have

$$\begin{aligned}(3.10)\qquad \|U_n(\cdot,\ \cdot)\|_{s,r}&\leqq\|\tau_{2^n}(\cdot,\ \cdot)-X(\cdot,\ \cdot)\|_{s,r}+\|\tau_{2^{n-1}}(\cdot,\ \cdot)-X(\cdot,\ \cdot)\|_{s,r}\\ &=O(e^{s,r}_{2^n-1}(X))=O(2^{-n\alpha})\ .\end{aligned}$$

Thus $\sum_{k=2}^n U_k(t, \omega)=\tau_{2^n}(t, \omega)$ converges in $L^{s,r}(T\times\Omega)$ to $X(t, \omega)$. Then for any positive integer m,

$$\begin{aligned}(3.11)\qquad \|\Delta_h^{(p)}X(\cdot,\ \cdot)\|_{s,r}&=\lim_{n\to\infty}\left\|\sum_{k=2}^n \Delta_h^{(p)}U_k(\cdot,\ \cdot)\right\|_{s,r}\\ &\leqq\left\|\sum_{k=2}^m \Delta_h^{(p)}U_k(\cdot,\ \cdot)\right\|_{s,r}+\sum_{k=m+1}^{\infty}\|\Delta_h^{(p)}U_k(\cdot,\ \cdot)\|_{s,r}\\ &\leqq\left\|\sum_{k=2}^m \Delta_h^{(p)}U_k(\cdot,\ \cdot)\right\|_{s,r}+\sum_{k=m+1}^{\infty}\sum_{\nu=0}^{p}\binom{p}{\nu}\|U_k(\cdot+h,\ \cdot)\|_{s,r}\\ &=\left\|\sum_{k=2}^m \Delta_h^{(p)}U_k(\cdot,\ \cdot)\right\|_{s,r}+2^p\sum_{k=m+1}^{\infty}\|U_k(\cdot,\ \cdot)\|_{s,r}\ .\end{aligned}$$

Since for each h

$$\Delta_h^{(p)}U_k(t, \omega)=\int_0^h du_p\int_0^h du_{p-1}\cdots\int_0^h u_k^{(p)}(t+u_1+\cdots+u_p,\ \omega)du_1\ ,$$

where (p) in the integrand denotes the p-th differentiation, we have

$$\begin{aligned}&\left\|\sum_{k=2}^m \Delta_h^{(p)}U_k(\cdot,\ \cdot)\right\|_{s,r}\\ &\qquad\leqq\sum_{k=2}^m\int_0^{|h|}du_p\int_0^{|h|}du_{p-1}\cdots\int_0^{|h|}du_1\|u_k^{(p)}(\cdot+u_1+\cdots+u_p,\ \cdot)\|_{s,r}\end{aligned}$$

which is, because of Theorem 1.3 and (3.10)

$$\begin{aligned}(3.12)\qquad &\leqq\sum_{k=2}^m 2^{kp}|h|^p\|U_k(\cdot,\ \cdot)\|_{s,r}\\ &\leqq O\Big(|h|^p\sum_{k=2}^m 2^{kp-k\alpha}\Big)=O(|h|^p 2^{(p-\alpha)m})\ ,\end{aligned}$$

when $0<\alpha<p$. Therefore from (3.11), for $0<\alpha<p$,

$$\begin{aligned}\|\Delta_h^{(p)}X(\cdot,\ \cdot)\|_{s,r}&=O(|h|^p 2^{(p-\alpha)m})+O\Big(\sum_{k=m+1}^{\infty}2^{-k\alpha}\Big)\\ &=O(|h|^p 2^{(p-\alpha)m})+O(2^{-m\alpha})\ .\end{aligned}$$

Choose m so that $2^{m-1}\leqq\delta^{-1}\leqq 2^m$. We then see that

$$\sup_{|h|\leqq\delta}\|\Delta_h^{(p)}X(\cdot,\ \cdot)\|_{s,r}=O(\delta^\alpha)\ .$$

Suppose next $e_n^{s,r}(X)=O(n^{-\alpha})$ and $\alpha=p$. Then from (3.12)

$$\left\|\sum_{k=2}^{m}\Delta_h^{(p)}U_k(\cdot,\ \cdot)\right\|_{s,r}=O(|h|^p m) \tag{3.13}$$

and hence with the same choice of m as before,

$$\sup_{|h|\leqq\delta}\|\Delta_h^{(p)}X(\cdot,\ \cdot)\|_{s,r}=O(\delta^p m)+O(2^{-mp})=O(\delta^p|\log\delta|)\ .$$

If $\alpha>p$, then the right hand side of (3.12) is $O(|h|^p)$, from which we have (3.3). The proof of Theorem 3.1 is thus complete.

§ 4. Proofs of Theorem 3.2 and Theorem 3.3.

The proof of equivalence of (3.5) and (3.6) in Theorem 3.2 is included in the proof of Theorem 3.1 (i). Therefore we have only to prove that (3.6) is equivalent to (3.7) and also to (3.8). Obviously (3.8) implies (3.7) and (3.7) implies (3.6). Hence in order to show Theorem 3.2, it is sufficient to prove that (3.6) implies (3.8).

The following proof of this fact is an adaptation of that of Lindler's generalization [11] of a theorem of Alexits and Kralik. It will be convenient to begin with the following lemma.

LEMMA 4.1. *If $X(t,\omega)\in L^{s,r}(T\times\Omega)$, $1<r,\ s<\infty$, then for any $2\leqq q<\infty$, $1<r,\ s\leqq q$,*

$$\sum_{k=m}^{n}\|S_k(\cdot,\ \cdot)\|_{s,r}^q\leqq Cn\|X(\cdot,\ \cdot)\|_{s,r}^q\ ,\quad n>m\ , \tag{4.1}$$

where C is a constant independent of m and n.

PROOF.

$$\begin{aligned}\sum_{k=m}^{n}\|S_k(\cdot,\ \cdot)\|_{s,r}^q&\leqq C\sum_{k=m}^{n}\left\|\frac{1}{\pi}\int_{|u|\leqq 1/n}X(\cdot+u,\ \cdot)D_k(u)du\right\|_{s,r}^q\\&\quad+C\sum_{k=m}^{n}\left\|\frac{1}{\pi}\int_{1/n<|u|\leqq\pi}X(\cdot+u,\ \cdot)D_k(u)du\right\|_{s,r}^q\\&=C(I_1+I_2)\ ,\end{aligned}$$

say, where $D_k(u)=\sin(k+1/2)u/(2\sin u/2)$ is the Dirichlet kernel.

$$I_1 \leqq \sum_{k=m}^{n}\left[\frac{1}{\pi}\int_{|u|\leq 1/n}\|X(\cdot+u,\ \cdot)\|_{s,r}|D_k(u)|du\right]^q \tag{4.2}$$
$$= \sum_{k=m}^{n}\|X(\cdot,\ \cdot)\|_{s,r}^q\left[\frac{1}{\pi}\int_{|u|\leq 1/n}|D_k(u)|du\right]^q$$
$$\leqq Cn\|X(\cdot,\ \cdot)\|_{s,r}^q .$$

Since $D_k(u)=1/2 \cot u/2 \sin(ku)+1/2 \cos(ku)$,

$$I_2 \leqq C\left[\sum_{k=m}^{n}\left\|\frac{1}{2\pi}\int_{1/n<|u|\leq\pi}X(\cdot+u,\ \cdot)\cot(u/2)\sin(ku)du\right\|_{s,r}^q\right.$$
$$\left.+\sum_{k=m}^{n}\left\|\frac{1}{2\pi}\int_{1/n<|u|\leq\pi}X(\cdot+u,\ \cdot)\cos(ku)du\right\|_{s,r}^q\right]=C[I_{21}+I_{22}]\ ,$$

say.

Repeated applications of the Minkowski inequality give us

$$I_{21}^{s/q} \leqq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\sum_{k=m}^{n}\left[E\left|\frac{1}{\pi}\int_{1/n<|u|\leq\pi}X(t+u,\ \omega)\frac{1}{2}\cot\frac{u}{2}\sin(ku)du\right|^r\right]^{q/r}\right\}^{s/q}dt$$
$$\leqq \frac{1}{2\pi}\int_{-\pi}^{\pi}E\left[\sum_{k=m}^{n}\left|\frac{1}{\pi}\int_{1/n<|u|\leq\pi}X(t+u,\ \omega)\frac{1}{2}\cot\frac{u}{2}\sin(ku)du\right|^q\right]^{r/q}dt\Big\}^{s/r}$$

which is, by the Hausdorff-Young inequality,

$$\leqq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{E\left[\frac{1}{\pi}\int_{1/n<|u|\leq\pi}\left|X(t+u,\ \omega)\frac{1}{2}\cot\frac{u}{2}\right]^{q'}du\right|^{r/q'}\right\}^{s/r}dt\ ,$$

where $q^{-1}+q'^{-1}=1$. This is, again by the Minkowski inequality,

$$\leqq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\frac{1}{\pi}\int_{1/n<|u|\leq\pi}\left[E\left|X(t+u,\ \omega)\frac{1}{2}\cot\frac{u}{2}\right|^r\right]^{q'/r}du\right\}^{s/q'}dt$$
$$\leqq C\int_{-\pi}^{\pi}\left[\frac{1}{\pi}\int_{1/n<|u|\leq\pi}\|X(t+u,\ \cdot)\|_r^{q'}|u|^{-q'}du\right]^{s/q'}dt\ .$$

Again by an application of the Minkowski inequality, we have

$$I_{21} \leqq C\left\{\int_{1/n<|u|\leq\pi}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\|X(t+u,\ \cdot)\|_r^s|u|^{-s}dt\right]^{q'/s}du\right\}^{q/q'} \tag{4.3}$$
$$=C\left(\int_{-1/n<|u|\leq\pi}|u|^{-q'}du\right)^{q/q'}\|X(\cdot,\ \cdot)\|_{s,r}^q$$
$$\leqq Cn\|X(\cdot,\ \cdot)\|_{s,r}^q .$$

I_{22} is handled in a simpler way to get

$$I_{22} \leqq C\|X(\cdot,\ \cdot)\|_{s,r}^q .$$

(4.2) and (4.3) with this complete the proof of the lemma.

PROOF OF THROREM 3.2. We shall prove that (3.6) implies (3.8). For $n>1$, choose m so that $2^{m-1}<n\leqq 2^m$.

Let $\tau(t, \omega; X)$ be the De la Vallee Poussin mean of $X(t, \omega)$ in (1.8). Note that τ_n is a trigonometric polynomial of order at most $2n-1$.

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}\|S_k(\cdot, \cdot; X)-X(\cdot, \cdot)\|_{s,r} &\leqq \frac{1}{n}\sum_{j=1}^{m}\sum_{k=2^{j-1}+1}^{2^j}\|S_k(\cdot, \cdot; X)-X(\cdot, \cdot)\|_{s,r}\\ &=\frac{1}{n}\sum_{j=1}^{m}\sum_{k=2^{j-1}}^{2^j}\|\tau_{2^{j-2}}(\cdot, \cdot; X)-X(\cdot, \cdot)-S_k(\cdot, \cdot; (\tau_{2^{j-2}}-X)\|_{s,r}\\ &\leqq\frac{1}{n}\sum_{j=1}^{m}\sum_{k=2^{j-1}+1}^{2^j}\|\tau_{2^{j-2}}(\cdot, \cdot; X)-X(\cdot, \cdot)\|_{s,r}\\ &\quad+\frac{1}{n}\sum_{j=1}^{m}\sum_{k=2^{j-1}}^{2^j}\|S_k(\cdot, \cdot; \tau_{2^{j-2}}-X)\|_{s,r}\\ &=J_1+J_2\ ,\end{aligned}$$

say. Using Theorem 1.1 and (3.6) we have

$$\begin{aligned}J_1&\leqq\frac{1}{n}\sum_{j=1}^{m}\sum_{k=2^{j-1}}^{2^j}e_{2^{j-2}}^{s,r}(X)=\frac{1}{n}\sum_{j=1}^{m}2^{j-1}O(2^{-(j-2)\alpha})\\ &=O(n^{-1}2^{m(1-\alpha)})=O(n^{-\alpha})\ .\end{aligned}\tag{4.4}$$

Take q as in Lemma 4.1 so that $q>s$, $r>q'$, $q\geqq 2$.

$$\begin{aligned}&\sum_{k=2^{j-1}+1}^{2^j}\|S_k(\cdot, \cdot; \tau_{2^{j-2}}-X)\|_{s,r}\\ &\qquad=O(2^{j/q'})\Big[\sum_{k=2^{j-1}+1}^{2^j}\|S_k(\cdot, \cdot; \tau_{2^{j-2}}-X)\|_{s,r}^q\Big]^{1/q}\end{aligned}$$

which is from Lemma 4.1, Theorem 1.1 and (3.6)

$$=O(2^{j/q'})O(2^{j/q}e_{2^{j-2}}^{s,r}(X))=O(2^{j(1-\alpha)})\ .$$

From this we have, as in (4.4),

$$J_2=O(n^{-\alpha})\ .\tag{4.5}$$

PROOF OF THROREM 3.3. Obviously (3.7) implies (3.9). Hence it is sufficient to prove that (3.9) implies (3.6). We note that because of the $L^{s,r}$-continuity of $X(t, \omega)$,

$$\sigma_n(t, \omega)\to X(t, \omega)\quad\text{in}\quad L^{s,r}(T\times\Omega)\ .\tag{4.6}$$

This is easily seen from the property of Fejér integral. Write

$$V_2(t, \omega)=\sigma_{2^2}(t, \omega),\ V_n(t, \omega)=\sigma_{2^n}(t, \omega)-\sigma_{2^{n-1}}(t, \omega)\ , \quad n>2\ .$$

From (4.6)

$$\sum_{k=2}^{n} V_k(t, \omega)=\sigma_{2^n}(t, \omega)$$

converges in $L^{s,r}(T\times\Omega)$ to $X(t, \omega)$. The proof of the first part of (ii) of Theorem 3.1 with $V_n(t, \omega)$ in place of $U_n(t, \omega)$ applies to have Theorem 3.3.

§ 5. Lipschitz classes and the magnitude of Fourier coefficients.

We study the relationship between the membership of $X(t, \omega)$ to the Lipschitz class $\Lambda_{s,r}^{*(p)}(\phi)$ and the magnitude of $\sum_{|k|>n}\|C_k(.)\|_\gamma^\beta$ for some β and γ. $C_n(\omega)$ is the Fourier coefficient of $X(t, \omega)$. We first indicate the following analogue of the Hausdorff-Young inequality.

THROREM 5.1. (i) *If* $X(t, \omega)\in L^{s,r}(T\times\Omega)$, $1\leqq s\leqq r\leqq s'$, $s^{-1}+s'^{-1}=1$, *then*

$$\left(\sum_{n=-\infty}^{\infty}\|C_n(\cdot)\|_r^{s'}\right)^{1/s'}\leqq\|X(\cdot, \cdot)\|_{s,r}\ . \tag{5.1}$$

(ii) *If* $\sum_{n=-\infty}^{\infty}\|C_n(\omega)\|_r^s<\infty$ *for* $1\leqq s\leqq r\leqq s'$, *then* $X(t, \omega)\in L^{s',r}(T\times\Omega)$ *and*

$$\|X(\cdot, \cdot)\|_{s',r}\leqq\left(\sum_{n=-\infty}\|C_n(\cdot)\|_r^s\right)^{1/s}\ . \tag{5.2}$$

By repeated use of the Minkowski inequality and the ordinary Hausdorff-Young inequality, we have

$$\begin{aligned}\left\{\sum_{n=-\infty}^{\infty}[E|C_n(\omega)|^r]^{s'/r}\right\}^{1/s'}&\leqq\left\{E\left[\sum_{n=-\infty}^{\infty}|C_n(\omega)|^{s'}\right]^{r/s'}\right\}^{1/r}\\ &\leqq\left\{E\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}|X(t, \omega)|^s dt\right]^{r/s}\right\}^{1/r}\\ &\leqq\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}[E|X(t, \omega)|^r dt]^{s/r}\right\}^{1/s}\\ &=\|X(\cdot, \cdot)\|_{s,r}\ .\end{aligned}$$

This shows (5.1). The second statement is similarly shown.

Let $\psi(t)$ be a continuous nondecreasing function on [0, 1] with $\psi(0)=0$ as in 1. Write as before $\theta=\min(s, r)$, $\theta^{-1}+\theta'^{-1}=1$.

THROREM 5.2. *Let* $1\leqq r, s\leqq 2$. *If* $X(t, \omega)\in \Lambda^{*(p)}_{s,r}(\psi)$ *for some positive integer* p, *then*

$$\sum_{|k|\geqq n}\|C_k(\cdot)\|_r^{\theta'}\leqq C_p\psi^{\theta'}(1/n)\ , \tag{5.3}$$

where C_p *is a constant depending only on* p.

This was shown in [6] when $r=s, 1<r$. (The proof in [6] is seen to be applied to the case $r=1$.)

PROOF. We shall prove

$$\sum_{|k|\geqq n}\|C_k(w)\|_r^{\theta'}\leqq C_pM^{*(p)}_{s,r}(1/n)\ . \tag{5.4}$$

Suppose first $1\leqq s\leqq r\leqq 2$. Since the Fourier coefficient of $\Delta_hX(t, \omega)$ is $C_k(w)(1-e^{ikh})^p$, we have, by Theorem 5.1, (5.1)

$$\|\Delta_h^{(p)}X(\cdot, \cdot)\|_{s,r}\geqq\Big[\sum_{k=-\infty}^{\infty}\|C_k(\cdot)\|_r^{s'}|2\sin(kh/2)|^{ps'}\Big]^{1/s'}\ . \tag{5.5}$$

Hence

$$M^{*(p)}_{s,r}(1/n)\geqq n\int_0^{1/n}\Big[\sum_{k=-\infty}^{\infty}\|C_k(\cdot)\|_r^{s'}|2\sin(kh/2)|^{ps'}\Big]^{1/s'}dh$$

which is, because of the Minkowski inequality,

$$\begin{aligned}&\geqq 2^pn\Big\{\sum_{|k|\geqq n}\Big[\int_0^{1/n}\|C_k(\cdot)\|_r|\sin(kh/2)|^pdh\Big]^{s'}\Big\}^{1/s'}\\ &=2^p\Big[\sum_{|k|\geqq n}\Big(n\int_0^{1/n}|\sin(kh/2)|^pdh\Big)^{s'}\|C_k(\cdot)\|_r^{s'}\Big]^{1/s'}\\ &\geqq C\Big[\sum_{|k|\geqq n}\|C_k(\cdot)\|_r^{s'}\Big]^{1/s'}\ ,\end{aligned}$$

for

$$n\int_0^{1/n}|\sin(kh/2)|^pdh=\frac{n}{k}\int_0^{k/n}\Big|\sin\frac{u}{2}\Big|^pdu\geqq C>0\ ,$$

where C is a constant depending only on p. (5.4) is thus proved.

The above proof was carried out for $1<s\leqq r\leqq 2$, but we easily see that it is adapted also for $s=1, s'=\infty$.

Suppose, second, $1\leqq r\leqq s$. By successive use of the Hölder inequality, the oridinary Hausdorff-Young inequality and the Minkowski inequality, we see that

$$\|\Delta_k^{(p)}X(\cdot,\ \cdot)\|_{s,r}\geqq\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}E|\Delta_k^{(p)}X(t,\ \omega)|^r dt\right]^{1/r}$$

$$=E\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}|\Delta_k^{(p)}X(t,\ \omega)|^r dt\right]^{1/r}$$

$$\geqq\left\{E\left[\sum_{k=-\infty}^{\infty}|C_k(\omega)|^{r'}|1-e^{ikh}|^{r'p}\right]^{r/r'}\right\}^{1/r}$$

$$\geqq\left\{\sum_{k=-\infty}^{\infty}[E|C_k(\omega)|^r]^{r'/r}\left|2\sin\frac{kh}{2}\right|^{rp}\right\}^{1/r'} .$$

This corresponds to (5.5), from which we have, as before, (5.4). We also see that the proof also goes on for $r=1$, $r'=\infty$, and the proof of Theorem 5.2 is complete.

The converse of Theorem 5.2 is true in the following form.

THROREM 5.3. *Let* $1\leqq r,\ s\leqq 2$. *If*

$$\int_t^1\psi^\theta(h)h^{-\theta p-1}dh\leqq C\psi^\theta(t)t^{-\theta p}\ ,\quad 0<t<1 \tag{5.6}$$

holds for some positive integer p and some constant C independent of t, then

$$\left[\sum_{|k|\geqq n}\|C_k(\cdot)\|_{r'}^{\theta}\right]^{1/\theta}\leqq C_1\psi\left(\frac{1}{n}\right) \tag{5.7}$$

implies $X(t,\ \omega)\in\Lambda_{\theta',r'}^{*,(p)}(\psi)$, C_1 *being a constant independent of n, and* $\theta^{-1}+\theta'^{-1}=1$.

PROOF. Suppose, first, $1\leqq s\leqq r\leqq 2$. The convergence of the series in (5.7) assures, by Theorem 5.1 (ii), that $X(t,\ \omega)\in L^{s'r'}(T\times\Omega)$. Here we note that $r'\leqq s'$. (5.2) with $\Delta_k^{(p)}X(t,\ \omega)$ in place of $X(t,\ \omega)$ gives us

$$\|\Delta_k^{(p)}X(\cdot,\ \cdot)\|_{s',r'}\leqq\left\{\sum_{k=-\infty}^{\infty}[E|C_k(\omega)|^{r'}|2\sin(kh/2)|^{pr'}]^{s/r'}\right\}^{1/s}$$

$$=\left\{\sum_{k=1}^{\infty}A_k|2\sin(kh/2)|^{ps}\right\}^{1/s} ,$$

where

$$A_k=\|C_k(\cdot)\|_{r'}^{s}+\|C_{-k}(\cdot)\|_{r'}^{s} .$$

Now we have

$$\|\Delta_h^{(p)}X(\cdot,\ \cdot)\|_{s',r'} \leqq \left[\sum_{k=1}^{[1/|h|]} A_k|2\sin(kh/2)|^{ps} + \sum_{k=[1+|h|]+1} 2^{ps}A_k\right]^{1/s}$$
$$\leqq |h|^p\left[\sum_{k=1}^{[1/|h|]} A_k k^{ps}\right]^{1/s} + 2^p\left[\sum_{k=[1+|h|]+1} A_k\right]^{1/s}$$
$$= K_1 + K_2\ ,$$

say. From (5.7)

$$K_2 = 2^p C_1 \psi(1/[1/|h|]+1) \leqq C\psi(|h|)\ , \tag{5.9}$$

where C is a constant which may differ on each occurrence in what follows.

Writing $R_m = \sum_{k\geqq m} A_k$, we have

$$K_1^s = |h|^{ps}\sum_{k=1}^{[1/|h|]} k^{ps}(R_k - R_{k+1})$$
$$= |h|^{ps}\sum_{k=1}^{[1/|h|]} (k^{ps}-(k-1)^{ps})R_k - |h|^{ps}[1/|h|]^{ps}R_{[1/|h|]}\ .$$

Since $R_k \leqq C^s\psi^s(1/k)$, we have

$$K_1^s \leqq C|h|^{ps}\sum_{k=1}^{[1/|h|]} k^{ps-1}\psi^s(1/k)$$
$$\leqq C|h|^{ps}\left[\psi^s(1) + \int_{|h|}^1 t^{-1-ps}\psi^s(t)dt\right]$$
$$\leqq C|h|^{ps}\int_{|h|}^1 t^{-1-ps}\psi^s(t)dt \leqq C\psi^s(|h|)\ .$$

Inserting this and (5.9) into (5.8), we have

$$\|\Delta_h^{(p)}X(\cdot,\ \cdot)\|_{s',r'} \leqq C\psi(|h|)\ ,$$

which shows the theorem.

Second, suppose $1 \leqq r \leqq s \leqq 2$. A similar arguement gives us

$$\|\Delta_h^{(p)}X(\cdot,\ \cdot)\|_{r',r'} \leqq \left(\sum_{k=1}^{\infty} B_k|2\sin(kh/2)|^{ps}\right)^{1/r}\ ,$$

where $B_k = \|C_k(\cdot)\|_{r'}^{r} + \|C_{-k}(\cdot)\|_{r'}^{r}$. From this we can, in the same way as above, get $X(t,\omega) \in \Lambda_{r',r'}^{*(p)}$. Thus the proof of the theorem is complete.

§ 6. Almost sure absolute convergence of Fourier series.

Let $\phi(t)$ be a nondecreasing continuous function on [0, 1] such that either $\phi(0)=0$ and $\phi(t)/t$ is nonincreasing on (0, 1], or $\phi(t)$ is identically 1 on [0.1].

THEOREM 6.1. *Let $X(t, \omega) \in L^{s,r}(T\times\Omega)$, $1\leqq r, s\leqq 2$. If*

$$\sum_{n=1}^{\infty} n^{k-1/\theta'}[\phi(1/n)]^{-1}M_{s,r}^{*(p)}(1/n)<\infty , \tag{6.1}$$

for some nonnegative integer k and some positive integer p, then

$$\sum_{n=-\infty}^{\infty} |n|^k[\phi(1/|n|)]^{-1}|C_n(\omega)|<\infty , \tag{6.2}$$

almost surely.

This is a generalization of Theorem 3.1 in [7] in which $r=s$.

PROOF.

$$\begin{aligned}
S&=\sum_{n=3}^{\infty} n^k[\phi(n^{-1})]^{-1}E|C_n(\omega)| \\
&=\sum_{n=1}^{\infty}\sum_{j=2^n+1}^{2^{n+1}} j^k[\phi(j^{-1})]^{-1}E|C_j(\omega)| \\
&\leqq\sum_{n=1}^{\infty} 2^{(n+1)k}[\phi(2^{-n-1})]^{-1}\sum_{j=2^n+1}^{2^{n+1}}\|C_j(\cdot)\|_1 \\
&\leqq\sum_{n=1}^{\infty} 2^{(n+1)k}[\phi(2^{-n-1})]^{-1}\sum_{j=2^n+1}^{2^{n+1}}\|C_j(\cdot)\|_r \\
&\leqq\sum_{n=1}^{\infty} 2^{(n+1)k}[\phi(2^{-n-1})]^{-1}\Big[\sum_{j=2^n+1}^{2^{n+1}}\|C_j(\cdot)\|_r^{\theta'}\Big]^{1/\theta'}2^{n/\theta} \\
&\leqq 2^{2k+2}\sum_{n=1}^{\infty}\sum_{m=2^n+1}^{2^n} m^{k-1/\theta'}[\phi(1/4m)]^{-1}\Big[\sum_{j=m}^{\infty}\|C_j(\cdot)\|_r^{\theta'}\Big]^{1/\theta'}
\end{aligned}$$

which is, from (5.4) and $\phi(t\lambda)\geqq\lambda\phi(t)$ for $0<\lambda<1$,

$$\leqq C\sum_{m=1}^{\infty} m^{k-1/\theta'}[\phi(1/m)]^{-1}M_{s,r}^{*(p)}(1/m)<\infty .$$

The same is true for $S'=\sum_{n=-\infty}^{-3}|n|^k[\phi(1/|n|)]^{-1}E|C_n(\omega)|$. From these we have that (6.2) holds almost surely.

Theorem 3.1 suggests that the same conclusion in Theorem 6.1 holds with the condition (6.1) with $e_n^{s,r}(X)$, $z_n^{s,r}=\|\sigma_n(\cdot,\cdot)-X(\cdot,\cdot)\|_{s,r}$ or $t_n^{s,r}(X)=n^{-1}\sum_{k=1}^{n}\|S_k(\cdot,\cdot;X)-X(\cdot,\cdot)\|_{s,r}$, in place of $M_{s,r}^{*(p)}(1/n)$. We now show that actually it is.

Since

$$e_n^{s,r}(X)\leqq z_n^{s,r}(X)\leqq t_n^{s,r}(X) , \tag{6.3}$$

it is sufficient for our purpose to prove the following theorem.

THEOREM 6.2. *Let $X(t, \omega) \in L^{s,r}(T\times \Omega)$, $1\leqq r, s\leqq 2$. If*

$$\sum_{n=1}^{\infty} n^{k-1/\theta'}[\phi(1/n)]^{-1}e_n^{s,r}<\infty \,, \tag{6.4}$$

for some nonnegative integer k, then (6.2) *holds almost surely.*

PROOF. Define, as in the proof of Theorem 3.1 $U_2(t, \omega)=\tau_{2^2}(t, \omega)$, $U_n(t, \omega)=\tau_{2^n}(t, \omega)-\tau_{2^{n-1}}(t, \omega)$, $n\geqq 2$, where $\tau_n(t, \omega)$ is the De la Vallee Poussin mean. As is easily seen from (6.4) and the fact that $e_n^{s,r}\to 0$ monotonously as $n\to\infty$, we see that

$$\sum_{\nu=2}^{n} U_\nu(t, \omega)=\tau_{2^n}(t, \omega)$$

converges to $X(t, \omega)$ in $L^{s,r}(T\times\Omega)$ as $n\to\infty$.

Now let m be a positive integer.

$$\begin{aligned}
\|\Delta_h^{(p)}X(\cdot, \cdot)\|_{s,r}&=\lim_{N\to\infty}\Big\|\sum_{\nu=2}^{N}\Delta_h^{(p)}U_\nu(\cdot, \cdot)\Big\|_{s,r}\\
&\leqq\Big\|\sum_{\nu=2}^{m}\Delta_h^{(p)}U_\nu(\cdot, \cdot)\Big\|_{s,r}+\lim_{N\to\infty}\Big\|\sum_{\nu=m+1}^{N}\Delta_h^{(p)}U_\nu(\cdot, \cdot)\Big\|_{s,r}\\
&\leqq\Big\|\sum_{\nu=2}^{m}\Delta_h^{(p)}U_\nu(\cdot, \cdot)\Big\|_{s,r}+\sum_{j=0}^{p}\binom{p}{j}\lim_{N\to\infty}\Big\|\sum_{\nu=m+1}^{N}U_\nu(\cdot+jh, \cdot)\Big\|_{s,r}\\
&=L_1(h)+L_2(h),
\end{aligned}$$

say.

$$\begin{aligned}
L_2(h)&=\sum_{j=1}^{p}\binom{p}{j}\lim_{N\to\infty}\|\tau_{2^N}(\cdot+jh), \cdot)-\tau_{2^m}(\cdot+jh, \cdot)\|_{s,r}\\
&=\sum_{j=1}^{p}\binom{p}{j}\lim_{N\to\infty}\|\tau_{2^N}(\cdot, \cdot)-\tau_{2^m}(\cdot, \cdot)\|_{s,r}\\
&\leqq 2^p\Big[\lim_{N\to\infty}\|\tau_{2^N}(\cdot, \cdot)-X(\cdot, \cdot)\|_{s,r}+\|\tau_{2^m}(\cdot, \cdot)-X(\cdot, \cdot)\|_{s,r}\Big].
\end{aligned}$$

Since from Theorem 1.1 $\|\tau_{2^N}(\cdot, \cdot)-X(\cdot, \cdot)\|_{s,r}\leqq 4e_{2^N}^{s,r}$ which converges to zero as $N\to\infty$ from (6.4), we have

$$L_2(h)\leqq 2^{p+2}e_{2^m}^{s,r}\,,$$

and hence choosing m so that $2^{m-1}\leqq n\leqq 2^m$, we obtain for a given n

$$\sup_{|h|\leqq 1/n} L_2(h)\leqq 2^{p+2}e_{2^m}^{s,r}\leqq 2^{p+2}e_n^{s,r}\,. \tag{6.5}$$

On the other hand

$$L_1(h) \leqq \sum_{\nu=2}^{m} \|\Delta_h^{(p)} U_\nu(\cdot, \cdot)\|_{s,r}$$

which is, from (3.12),

$$\leqq \sum_{\nu=2}^{m} 2^{\nu p}|h|^p \| U_\nu(\cdot, \cdot)\|_{s,r}$$

$$\leqq \sum_{\nu=2}^{m} 2^{\nu p}|h|^p [\|\tau_{2^\nu}(\cdot, \cdot) - X(\cdot, \cdot)\|_{s,r} + \|\tau_{2^{\nu-1}}(\cdot, \cdot) - X(\cdot, \cdot)\|_{s,r}]$$

$$\leqq 2|h|^p \sum_{\nu=2}^{m} e_{2^{\nu-1}}^{s,r} 2^{\nu p+2}$$

and hence

$$\sup_{|h| \leqq 1/n} L_1(h) \leqq Cn^{-p} \sum_{\nu=1}^{m-1} 2^{\nu p} e_{2^\nu}^{s,r} . \tag{6.6}$$

Therefore we have, from (6.5) and (6.6)

$$M_{s,r}^{*(p)}(1/n) \leqq C\left(n^{-p} \sum_{\nu=1}^{m-1} 2^{\nu p} e_{2^\nu}^{s,r} + e_n^{s,r}\right) ,$$

and

$$\sum_{n=1}^{\infty} n^{k-1/\theta'}[\phi(1/n)]^{-1} M_{s,r}^{*(p)}(1/n)$$

$$\leqq C \sum_{n=1}^{\infty} n^{k-p-1/\theta'}[\phi(1/n)]^{-1} \sum_{\nu=1}^{m-1} 2^{\nu p} e_{2^\nu}^{s,r} + C \sum_{n=1}^{\infty} n^{k-1/\theta'}[\phi(1/n)]^{-1} e_n^{s,r}$$

$$= CL_{1,1} + CL_{1,2} , \tag{6.7}$$

say.

By (6.4), $L_{1,2}$ is finite.

$$L_{1,1} = \sum_{n=1}^{\infty} \sum_{\mu=2^{n-1}+1}^{2^n} \mu^{k-1/\theta'-p}[\phi(1/\mu)]^{-1} \sum_{\nu=1}^{m'-1} 2^{\nu p} e_{2^\nu}^{s,r} ,$$

where $2^{m'-1} < \mu \leqq 2^{m'}$,

$$\leqq \sum_{n=1} 2^{n(k-1/\theta'-p+1)}[\phi(1/2^n)]^{-1} \sum_{\nu=1}^{n} 2^{\nu p} e_{2^\nu}^{s,r}$$

$$\leqq \sum_{\nu=1}^{\infty} 2^{\nu p} e_{2^\nu}^{s,r} \sum_{n=\nu}^{\infty} 2^{n(k-1/\theta'-p+1)}[\phi(1/2^n)]^{-1} .$$

Since $[\phi(1/2^n)]^{-1} \leqq 2^{n-\nu}[\phi(1/2^\nu)]^{-1}$, the last one is

$$\leqq \sum_{\nu=1}^{\infty} e_{2^\nu}^{s,r} 2^{\nu p-\nu}[\phi(1/2^\nu)]^{-1} \sum_{n=\nu}^{\infty} 2^{n(k-1/\theta'-p+1)+n} .$$

Taking $p\geqq k+3$ we have

$$L_{1,1}\leqq\sum_{\nu=1}^{\infty}2^{\nu(k-1/\theta'+1)}[\phi(1/2^{\nu})]^{-1}e_{2^{\nu}}^{s,r}$$
$$\leqq C\sum_{\nu=1}^{\infty}\sum_{n=2^{\nu-1}}^{2^{\nu}}n^{k-1/\theta'}[\phi(1/n)]^{-1}e_{n}^{s,r}$$
$$\leqq C\sum_{n=1}^{\infty}n^{k-1/\theta'}[\phi(1/n)]^{-1}e_{n}^{s,r}$$

Therefore from (6.6), we have

$$\sum_{n=1}n^{k-1/\theta'}[\phi(1/n)]^{-1}M_{s,r}^{*(p)}(1/n)<\infty ,$$

for some positive integer p. Then we obtain Theorem 6.2 from Theorem 6.1.

We remark that, because of Theorem 1.2 and the proof of Theorem 6.2, Theorems 6.1 and 6.2 are substantially equivalent.

THEOREM 6.3. *Let* $1\leqq r\leqq 2, 1<s\leqq 2$. *Suppose* $X(t, \omega)\in L^{s,r}(T\times\Omega)$ *is of bounded variation in* $L^{r}(\Omega)$. *If*

$$\sum_{n=1}^{\infty}n^{k-1/\theta'-1/s}[\phi(1/n)]^{-1}[M_{r}^{(p)}(1/n)]^{1/s'}<\infty , \tag{6.8}$$

for some nonnegative integer k *and some positive integer* p, *then* (6.2) *holds almost surely.*

PROOF. Using Lemma 2.1 and (6.8), we have

$$\sum_{n=1}^{\infty}n^{k-1/\theta'}[\phi(1/n)]^{-1}M_{s,r}^{*(p)}(1/n)$$
$$\leqq C\sum_{n=1}^{\infty}n^{k-1/\theta'}[\phi(1/n)]^{-1}[M_{r}^{(p)}(1/n)]^{1/s'}n^{-1/s}<\infty .$$

Hence Theorem 6.3 follows from Theorem 6.1.

We remark that if $s=1$, then (6.8) never holds.

§ 7. Mean derivative and the absolute convergence Fourier series.

Let $X(t, \omega)$ be of $L^{s,r}(T\times\Omega), 1\leqq r<\infty, 1\leqq s\leqq\infty$. If there exists an $X_{M}'(t, \omega)\in L^{s,r}(T\times\Omega)$ such that

$$\|(1/h)[X(\cdot+h, \cdot)-X(\cdot, \cdot)]-X_{M}'(\cdot, \cdot)\|_{s,r}\to 0 , \tag{7.1}$$

as $h\to 0$, we say that $X(t, \omega)$ has the mean derivative $X_{M}'(t, \omega)$ in $L^{s,r}(T\times\Omega)$.

If $X'_M(t, \omega)$ has the mean derivative $X''_M(t, \omega)$ in $L^{s,r}(T\times\Omega)$, then $X(t, \omega)$ is said to have the second mean derivative $X''_M(t, \omega)$ in $L^{s,r}(T\times\Omega)$. In a similar way, we successively define the k-th mean derivative $X_M^{(k)}(t, \omega)$. For $s=r$, these were defined in [8]. The following lemmas are generalizations of the correspondings with $r=s$ and the proofs are carried out without any sustantial change.

LEMMA 7.1. *If $X(t, \omega)\in L^{s,r}(T\times\Omega)$, $1\leqq r<\infty$, $1\leqq s\leqq\infty$ and $X'_M(t, \omega)$ exists in $L^{s,r}(T\times\Omega)$, then as $h\to 0$,*

$$\left\|(1/h)\int_t^{t+h} X(u, \cdot)du - X(t, \cdot)\right\|_{s,r} = O(h) . \tag{7.2}$$

LEMMA 7.2. *If $X(t, \omega)\in L^{s,r}(T\times\Omega)$, $1\leqq r<\infty$, $1\leqq s\leqq\infty$, has the mean derivative $X'_M(t, \omega)$ in $L^{s,r}(T\times\Omega)$, then*

$$X(t+h, \omega)-X(t, \omega)=\int_t^{t+h} X'_M(u, \omega)du \tag{7.3}$$

almost everywhere in $T\times\Omega$ for each h.

Now we shall prove

THEOREM 7.1. *If $X(t, \omega)\in L^{s,r}(T\times\Omega)$, $1\leqq r<\infty$, $1\leqq s\leqq\infty$ has the p-th mean derivative $X_M^{(p)}(t, \omega)$ in $L^{s,r}(T\times\Omega)$, p being a positive integer, then*

$$M_{s,r}^{*(p)}(\delta)\leqq 2^{p/s}\|X_M^{(p)}(\cdot, \cdot)\|_{s,r}\delta^p , \quad \delta>0 . \tag{7.4}$$

The proof goes through just as in that of Theorem 1 of [8]. In fact, writing

$$Y_k(u)=\int_u^{u+h} dt_{p-k+1}\cdots\int_{t_{p-1}}^{t_{p-1}+h} \|X_M^{(p)}(t_p, \cdot)\|_r^s dt_p ,$$

we have, for $h>0$ and $1\leqq s<\infty$, by Lemma 7.2,

$$\begin{aligned}\|\Delta_h^{(p)}X(t, \cdot)\|_r^s &= \left[\int_t^{t+h} dt_1\cdots\int_{t_{p-1}}^{t_{p-1}+h}\|X_M^{(p)}(t_p, \cdot)\|_r dt_p\right]^s \\ &\leqq h^{p(s-1)}\int_t^{t+h} Y_{p-1}(t_1)dt_1 ,\end{aligned} \tag{7.5}$$

from which we have

$$\begin{aligned}\|\Delta_h^{(p)}X(\cdot, \cdot)\|_{s,r} &\leqq h^{p(1-1/s)}(2h)^{1/s}\frac{1}{2\pi}\int_{-\pi}^{\pi} Y_{p-1}(t_1)dt_1 \\ &\leqq h^{p(1-1/s)}(2h)^{p/s}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} Y_0(t_p)dt_p\right]^{1/s}\end{aligned} \tag{7.6}$$

$$\equiv 2^{p/s}h^p \| X_M^{(p)}(\cdot, \cdot)\|_{s,r}$$

which proves (7.4). The similar is true for $h<0$ and Theorem 7.1 is proved for $1 \leqq s < \infty$.

If $s=\infty$, then (7.4) is easily obtained from (7.5).

We now give a theorem which assures the almost sure validity of (6.2) for a stochastic process which has the mean derivative.

THEOREM 7.2. *Suppose* $X(t, \omega) \in L^{s,r}(T\times\Omega)$, $1\leqq r<\infty$, $1\leqq s\leqq\infty$, *has the* $k+1$*-st mean derivative* $X_M^{(k+1)}(t, \omega)$ *in* $L^{s,r}(T\times\Omega)$ *for some nonnegative integer* k. *If*

$$\sum_{n=1}^{\infty} n^{-2+1/\theta}[\phi(1/n)]^{-1} < \infty \ , \tag{7.7}$$

then (6.2) *holds almost surely.*

This theorem immediately follows from Theorem 6.1 and 7.1. The following corollary is also immediate with $\phi(t)=t^\alpha$, $0\leqq\alpha<1$.

COROLLARY 7.1. *If* $X(t, \omega) \in L^{s,r}(T\times\Omega)$ *has the mean derivative* $X_M^{(k+1)}(t, \omega)$ *in* $L^{s,r}(T\times\Omega)$ *for some nonnegative integer* k *and*

$$0<\alpha<1-\theta^{-1} \ , \tag{7.8}$$

then

$$\sum_{n=-\infty}^{\infty} |n|^{k+\alpha}|C_n(\omega)| < \infty \tag{7.9}$$

almost surely.

Theorem 7.2 and Corollary 7.1 with $r=s$ were shown in [8]. If $s<r$, the conditions (7.7) and (7.8) are stronger than the corresponding ones with $r=s$. However the class of $X(t, \omega)$ is broader.

§ 8 Sample properties.

Once the theorems on the absolute convergence of Fourier series of stochastic processes as in the foregoing section are obtained, we can derive, from them, the results which give the conditions for the sample continuity or sample differentiability by the argument same as in [7] Theorem 6.1.

We throughout this section assume that a *2π-periodic L^r-process $X(t, \omega)$ is stochastically continuous*, that is, for every t

$$(8.1) \qquad P(|X(t+h, \omega)-X(t, \omega)|>\varepsilon)\to 0 ,$$

as $h\to 0$, for every $\varepsilon>0$. In this case (8.1) holds uniformly and the $(C, 1)$ mean $\sigma_n(t, \omega)$ of the Fourier series of $X(t, \omega)$ converges uniformly in probability to $X(t, \omega)$. [7]

Let $\phi(t)$ be a function considered in 7. Λ_ϕ denotes the ordinary Lipschitz class $\{f(t); \sup_{t,|h|<\delta}|f(t+h)-f(t)|=O(\phi(\delta))\}$ of 2π-periodic functions and when $\phi(t)\equiv 1$, denotes the class of continuous functions.

We have the following theorems from Theorems 6.1, 6.2, 2.3, 7.2 and Corollary 7.1. We do not think that we have to repeat their proofs.

THEOREM 8.1. *If $X(t, \omega)\in L^{s,r}(T\times\Omega)$, $1\leqq r$, $s\leqq 2$ satisfies the condition* (6.1) *in Theorem* 6.1 *for some nonnegative integer k and some positive integer p, then there is a modification $X_0(t, \omega)$ of $X(t, \omega)$ with the property that $X_0(t, \omega)$ has almost surely the k-th derivative which belongs to Λ_ϕ.*

THEOREM 8.2 *If $X(t, \omega)\in L^{s,r}(T\times\Omega)$, $1\leqq r$, $s\leqq 2$ satisfies the condition* (6.4) *in Theorem* 6.2 *for some nonnegative integer k, then the conclusion of Theorem* 8.1 *is valid.*

THEOREM 8.3. *If $X(t, \omega)\in L^{s,r}(T\times\Omega)$, $1\leqq r\leqq 2$, $1<s\leqq 2$, is of bounded variation in $L^r(\Omega)$ and satisfies the condition* (6.8) *in Theorem* 6.3 *for some nonnegative integer k and some positive integer p, then the conclusion of Theorem* 8.1 *is valid.*

THEOREM 8.4. *If $X(t, \omega)\in L^{s,r}(T\times\Omega)$, $1\leqq r<\infty$, $1\leqq s\leqq\infty$, has the mean derivative $X_M^{(k+1)}(t, \omega)$ in $L^{s,r}(T\times\Omega)$ for some nonnegative integer k and the condition* (7.7) *in Theorem* 7.2 *is satisfied, then the conclusion of Theorem* 8.1 *is valid.*

THEOREM 8.5. *Let $X(t, \omega)\in L^{s,r}(T\times\Omega)$, $1\leqq r<\infty$, $1\leqq s\leqq\infty$, have the mean derivative $X_M^{(k+1)}(t, \omega)$ in $L^{s,r}(T\times\Omega)$ for some nonnegative integer k. If a number α satisfies* (7.8) *in Corollary* 7.1, *then there is a modification $X_0(t, \omega)$ of $X(t, \omega)$ with the property that $X_0(t, \omega)$ has almost surely the k-th derivative which belongs to Λ_α.*

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