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# An Intermediate Value Theorem in Neighbourhood Spaces

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## Introduction

In [6], the intermediate value theorem for fuzzy spaces was proved. These spaces were considered as an alternative of topological spaces. Hazy spaces were devised in [2] as an extension for fuzzy spaces. Neighbourhood spaces were considered in [3] as a generalization of hazy spaces. In this note an intermediate value theorem for neighbourhood maps on any connected neighbourhood space with its values on the standard neighbourhood space is given.

## § 1. Preliminaries.

We fix our terminology as in the following, for the detail of which one can see [3, 4].

DEFINITION. A *hazy space* is a pair $(X, \tau)$, where $\tau$, the haze, is a reflexive and symmetric relation from the non-empty set $X$ to its set of all subsets, Sub $X$. That is, $\tau \subseteqq X \times \mathrm{Sub}\, X$, with for all $x, y \in X$

(i) $x \in \tau(x) = \bigcup_{(x,A)\in\tau} A$

(ii) $x \in \tau(y)$ iff $y \in \tau(x)$.

$\tau(x)$ is called the *neighbourhood* (abbr. to *nbd*) of $x$.

The set $\boldsymbol{Z}$ of all integers with standard haze,

$$\upsilon = \{(k, \{k, k-1\}), (k, \{k, k+1\}); k \in \boldsymbol{Z}\}$$

plays a role in hazy spaces corresponding to that of the Euclidean 1-dimensional space $\boldsymbol{R}$ in topological spaces. The observation that the nbds $\tau(x)$ play a much more prominent role than the subsets $A$ was the motivation to develop the neighbourhood spaces in [3]. Structures of this kind were also introduced in [1] as neighbourhood systems.

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DEFINITION. A *neighbourhood space* (abbr. to *nbd-space*) is a pair $(X, \tau)$, where $\tau$ is a reflexive and symmetric map from the non-empty set $X$ to the set of all subsets of $X$, Sub $X$. That is, for all $x, y \in X$

(i) $x \in \tau(x)$

(ii) $x \in \tau(y)$ iff $y \in \tau(x)$.

The *standard nbd-structure* on $\boldsymbol{Z}$ is denoted by $\upsilon$ and is given by

$$\upsilon: \boldsymbol{Z} \longrightarrow \text{Sub } \boldsymbol{Z}: k \longmapsto \{k-1, k, k+1\} .$$

If $(X, \tau)$ is an nbd-space and $A$ is a subset of $X$, then the map $\tau \cap A: A \to \text{sub } A$ defined by $(\tau \cap A)(x) = \tau(x) \cap A$ for any $x \in A$ induces an nbd-structure on $A$ and $(A, \tau \cap A)$ is called an *nbd-subspace* of $(X, \tau)$.

We shall denote by $[i, j]$, called an *interval*, the set $\{i, i+1, \cdots, j\}$ considered as an nbd-subspace of $(\boldsymbol{Z}, \upsilon)$, where $i, j \in \boldsymbol{Z}$ and $i \leqq j$.

DEFINITION. A *neighbourhood map* (abbr. to *nbd-map*) $f$ between nbd-spaces $(X, \tau)$ and $(Y, \sigma)$ is any set-valued map $f: X \to \text{sub } Y$ such that for all $x \in X$, $f(\tau(x)) \subseteq \sigma(y)$ for some $y \in f(x)$, where $f(\tau(x)) = \bigcup_{x' \in \tau(x)} f(x')$, and is denoted by $f: (X, \tau) \to (Y, \sigma)$.

The composition $g \circ f$ of two nbd-maps $f: (X, \tau) \to (Y, \sigma)$ and $g: (Y, \sigma) \to (W, \delta)$ is the map $g \circ f: x \mapsto \bigcup_{y \in (x)} g(y)$, such composition is indeed an nbd-map. We denote the identity nbd-map on an nbd-space $(X, \tau)$ by $I_X$, induced by the usual identity map on $X$.

DEFINITION. Let $(X, \tau)$ and $(Y, \sigma)$ be nbd-spaces. An nbd-map $f: (X, \tau) \to (Y, \sigma)$ is called an *isomorphism* if there is an nbd-map $g: (Y, \sigma) \to (X, \tau)$ with $g \circ f = I_X$, $f \circ g = I_Y$. In this case $(X, \tau)$ and $(Y, \sigma)$ are said to be *isomorphic*.

DEFINITION. An *l-path* or a *path of length l* in an nbd-space $(X, \tau)$ is an nbd-map $p: [0, l] \to (X, \tau)$. Moreover we say that $p$ is an *l-path from about x to about y if* $x \in p(0)$, $y \in p(l)$ and is denoted by $\sim x \underset{(l)}{\xrightarrow{p}} \sim y$.

DEFINITION. An nbd-space $(X, \tau)$ is *path-connected* if for all $x, y \in X$, there is a path $\sim x \underset{(l)}{\xrightarrow{p}} \sim y$.

Let $(X, \tau)$ be the product nbd-space of nbd-spaces $(X_i, \tau_i)$ for $i=1, 2$; i.e., $X = X_1 \times X_2$ and for $x = (x_1, x_2) \in X$, $\tau(x) = \tau_1(x_1) \times \tau_2(x_2)$. Then we have,

THEOREM 1.1 (§2.5 of [4]). *$(X, \tau)$ is the largest nbd-space such that the set-theoretic projection map $\pi_i$ is an nbd-map for $i=1, 2$. Moreover for* *any nbd-space* $(Y, \sigma)$ *the set-theoretic map* $f: X \to \mathrm{Sub}\, Y$ *is an nbd-map iff* $\pi_i \circ f$ *nbd-map for* $i=1, 2$.

DEFINITION. A non-empty subset $A$ in an nbd-space $(X, \tau)$ is said to be *open* if $(\tau \cap A)(x) = \tau(x)$, for all $x \in A$. An nbd-space $(X, \tau)$ is *connected* if it is not the union of disjoint non-empty open subsets.

THEOREM 1.2 (§1.8 of [5]). *An nbd-space is path-connected iff it is connected.*

Since for any $k \in \boldsymbol{Z}$, nbds $\upsilon(k)$ are the same for the hazy space $(\boldsymbol{Z}, \upsilon)$ and nbd-space $(\boldsymbol{Z}, \upsilon)$, by §5.4 of [4] we have the following:

THEOREM 1.3. $(\boldsymbol{Z}, \upsilon)$ *is a path-connected nbd-space and its only path-connected subsets are intervals.*

## §2. Intermediate Value Theorem (I.V.T.).

THEOREM 2.1 (Intermediate Value Theorem). *Suppose* $i, j \in \boldsymbol{Z}$ *and* $f: [i, j] \to (\boldsymbol{Z}, \upsilon)$ *be an nbd-map. If* $\min f([i, j]) \times \max f([i, j]) < 0$, *then there is at least one* $k \in [i, j]$ *such that* $0 \in f(k)$ *and moreover* $f(k) \subseteqq \upsilon(0)$.

In order to prove I.V.T. we need the following lemma. Note that connectedness and path-connectedness are equal conditions by Theorem 1.2, therefore for simplicity we only use the term connectedness in the sequel.

LEMMA 2.1. *The nbd-map image of a connected nbd-space is connected.*

PROOF. Let $f: (X, \tau) \to (Y, \sigma)$ be a nbd-map. If $y_1, y_2 \in f(X)$, then $y_i \in f(x_i)$, for some $x_i \in X$, $i=1, 2$. Suppose $(X, \tau)$ is connected, then there is a path $\sim x_1 \xrightarrow[(l)]{p} \sim x_2$ such that $x_1 \in p(0)$ and $x_2 \in p(l)$. Hence $y_1 \in f(x_1) \subseteqq f \circ p(0)$ and $y_2 \in f(x_2) \subseteqq f \circ p(l)$. Since the composition $f \circ p$ of two nbd-maps $f$ ane $p$ is also an nbd-map, therefore we have the path $\sim y_1 \xrightarrow[(l)]{f \circ p} \sim y_2$ in $f(X)$, thereby $(f(X), f(X) \cap \sigma)$ is connected.

PROOF OF THEOREM 2.1. Let $m = \min f([i, j])$ and $M = \max f([i, j])$. Since $[i, j]$ is an interval, by Theorem 1.3 it is connected. Therefore by Lemma 2.1, $f([i, j])$ is a connected nbd-subspace of $(\boldsymbol{Z}, \upsilon)$. Hence, by Theorem 1.3, it is the interval $[m, M]$. Since $m < 0 < M$, $0 \in f([i, j])$. Thereby there is $k \in [i, j]$ such that $0 \in f(k)$.

To prove the existence of $k$ such that $0 \in f(k) \subseteqq \upsilon(0)$, we suppose the contrary, that is, we assume that $f(k) \not\subseteq \upsilon(0)$ for all $k \in [i, j]$ such that $0 \in f(k)$. Then $f(k) \subseteq \upsilon(1) = \{0, 1, 2\}$ or $f(k) \subseteq \upsilon(-1) = \{-2, -1, 0\}$. The cases $f(k) = \{0\}$, $f(k) = \{0, 1\}$ or $f(k) = \{-1, 0\}$ can not occur, since otherwise we get a contradiction to our assumption. The cases $f(k) = \{0, 2\}$ or $f(k) = \{-2, 0\}$ can not also occur, because $\{k\}$ is connected and by Lemma 2.1 $f(k)$ must be connected; i.e., an interval. Therefore we have only one of the cases (a): $f(k) = \{0, 1, 2\}$ or (b) $f(k) = \{-2, -1, 0\}$.

(a) In this case by checking the definition of an nbd-map at $k$, if $f(\upsilon(k)) \subseteq \upsilon(0)$ or $f(\upsilon(k)) \subseteq \upsilon(2)$ we get a contradiction to our assumption or to the fact that $0 \notin \upsilon(2)$, respectively. Hence $f(\upsilon(k)) \subseteq \upsilon(1)$ and therefore $f(k \pm 1) \subseteq \{0, 1, 2\}$. Now by induction on $n$ we show that for all $n \in \boldsymbol{Z}^+$ such that $k \pm n \in [i, j]$, then $f(k \pm n) \subset \boldsymbol{Z}^+$, where $\boldsymbol{Z}^+$ is the set of all non-negative integers. If $n = 1$, then already we proved $f(k \pm 1) \subseteq \{0, 1, 2\}$. Suppose $f(k \pm (n-1)) \subset \boldsymbol{Z}^+$, if $f(\upsilon(k \pm (n-1))) \subseteq \upsilon(0)$, and $0 \in f(k \pm (n-1))$, in this case $0 \in f(k \pm (n-1)) \subseteq \upsilon(0)$ is a contradiction to our assmuption. Hence $f(\upsilon(k \pm (n-1))) \subseteq \upsilon(c)$, for some $c$, $1 \leqq c \in f(k \pm (n-1))$. Therefore $f(k \pm n) \subset \boldsymbol{Z}^+$. So, for all $d \in [i, j]$, $f(d) \subset \boldsymbol{Z}^+$. Thereby we have $m \geqq 0$, which is a contradiction to the fact $m < 0$.

(b) In this case one can use the same argument as in case (a) to get for all $d \in [i, j]$, $f(d) \subset \{c \in \boldsymbol{Z}, c \leqq 0\}$. Thereby we have $M \leqq 0$, which is a contradiction to the fact that $M > 0$.

The following corollary is a generalization of I.V.T.

COROLLARY 2.1. *Suppose $i$, $j \in \boldsymbol{Z}$ and $f$: $[i, j] \to (\boldsymbol{Z}, \upsilon)$ be an nbd-map. If $m = \min f([i, j])$ and $M = \max f([i, j])$, then for all $d$, $m < d < M$, there is at least one $k \in [i, j]$ such that $d \in f(k) \subseteq \upsilon(d)$.*

PROOF. The proof follows from the fact that for any $a \in \boldsymbol{Z}$, the translation $t_a$: $(\boldsymbol{Z}, \upsilon) \to (\boldsymbol{Z}, \upsilon)$ defined by $t_a(m) = \{m + a\}$ is an isomorphism.

It is possible to extend Corollary 2.1 to the following:

COROLLARY 2.2. *Let $(X, \tau)$ be a connected nbd-space, and $f$: $(X, \tau) \to (\boldsymbol{Z}, \upsilon)$ an nbd-map. If $M = \sup f(X)$ and $m = \inf f(X)$, then for all $d$, $m < d < M$, there is at least one $x \in X$ such that $d \in f(x) \subseteq \upsilon(d)$.*

PROOF. Since $M = \sup f(X)$ and $m = \inf f(X)$, therefore $M \in f(x_M)$, $m \in f(x_m)$ for some $x_M$ and $x_m$ in $X$. Connectedness of $(X, \tau)$ implies that there is a path $\sim x_m \underset{(l)}{\overset{p}{\to}} \sim x_M$ such that $x_m \in p(0)$ and $x_M \in p(l)$. Since $f \circ p$: $[0, l] \to (\boldsymbol{Z}, \upsilon)$ is an nbd-map such that $d \in [m, M] \subseteq f \circ p([0, l])$. Therefore by Corollary 2.1 there is a $k' \in [0, l]$ such that $d \in f \circ p(k') \subseteq \upsilon(d)$.

Hence for some $x \in p(k')$, $d \in f(x) \subseteqq \upsilon(d)$.

Let $I_k = [i_k, j_k]$ be an interval of $(\boldsymbol{Z}, \upsilon)$, and $I^n = \times_{1 \leqq k \leqq n} I_k$ the product nbd-subspace of $(\boldsymbol{Z}^n, \upsilon^n)$. We apply Corollary 2.2 to extend the I.V.T. to the following:

COROLLARY 2.3. *Let $f: I^n \to (\boldsymbol{Z}, \upsilon)$ be an nbd-map. If $M = \sup f(I^n)$ and $m = \inf f(I^n)$, then for all $d$, $m < d < M$, there is at least one $k \in I^n$ such that $d \in f(k) \subseteqq \upsilon(d)$.*

In order to prove Corollary 2.3 we first prove the following lemma.

LEMMA 2.2. *Let $(X, \tau)$ be the product nbd-space of nbd-spaces $(X_i, \tau_i)$ for $i = 1, 2$. $(X, \tau)$ is connected iff $(X_i, \tau_i)$ is connected for $i = 1, 2$.*

PROOF. Let $(X, \tau)$ be connected. For $i = 1, 2$, if $x_i, y_i \in X_i$, then there are $x, y \in X$ such that $\{x_i\} = \pi_i(x)$ and $\{y_i\} = \pi_i(y)$, where $\pi_i$ is the set-theoretic projection map. Connectedness of $(X, \tau)$ implies that there is a path $\sim x \underset{(l)}{\xrightarrow{p}} \sim y$ such that $x \in p(0)$ and $y \in p(l)$. Hence $\{x_i\} = \pi_i(x) \subseteqq \pi_i \circ p(0)$ and $\{y_i\} = \pi_i(y) \subseteqq \pi_i \circ p(l)$. By Theorem 1.1, $\pi_i \circ p$ is an nbd-map, therefore it is the path $\sim x_i \underset{(l)}{\xrightarrow{\pi_i \circ p}} \sim y_i$, hence $(X_i, \tau_i)$ is connected. Conversely let $(X_i, \tau_i)$ be connected for $i = 1, 2$. Then for any $x, y \in Y$, $\{x_i\} \pi_i(x)$ and $\{y_i\} = \pi_i(y)$ are in $X_i$. Connectedness of $(X_i, \tau_i)$ for all $i = 1, 2$, implies the existence of the paths $\sim x_i \underset{(l_i)}{\xrightarrow{q_i}} \sim y_i$ such that $x_i \in q_i(0)$ and $y_i \in q_i(l_i)$. Without loss of generality we may assume that $l_1 \leqq l_2$. Define $r: [0, l_2] \to (X, \tau)$ as,

$$r(k) = \begin{cases} q_1(k) \times q_2(k) & \text{if} \quad O \leqq k \leqq l_1 \\ q_1(l_1) \times q_2(k) & \text{if} \quad l_1 < k \leqq l_2 \, . \end{cases}$$

Since

$$\pi_1 \circ r(k) = \begin{cases} q_1(k) & \text{if} \quad O \leqq k \leqq l_1 \\ q_1(l_1) & \text{if} \quad l_1 < k \leqq l_2 \, , \end{cases}$$

and

$$\pi_2 \circ r(k) = q_2(k) \quad \text{if} \quad O \leqq k \leqq l_2 \, ,$$

therefore $r$ is an nbd-map by Theorem 1.1, also $x \in r(0)$ and $y \in r(l_2)$. Hence $\sim x \underset{(l_2)}{\xrightarrow{r}} \sim y$, thereby $(X, \tau)$ is connected.

PROOF OF COROLLARY 2.3. It is enough to show $I^n$ is connected. This follows from Lemma 2.2 and Theorem 1.3.

REMARK. Corollary 2.2 may be interpreated as the existence of a point $x \in X$ such that not only $d \in f(x)$ but $f(x)$ is also totally indistinguishable of $\upsilon(d)$ in the sense of [5].

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