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Superficial Saturation

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Introduction

Let A be a Cohen-Macaulay semi-local ring of dimension d, I an ideal of definition of A and $P(I, t)=\sum_{n>0}^{\infty}\lambda(I^n/I^{n+1})t^n$ the associated Poincaré series where $\lambda(\ )$ denotes the length of A-module. Then $P(I, t)$ is of the form $e_0(1-t)^{-d}-e_1(1-t)^{1-d}+\cdots+(-1)^{d-1}e_{d-1}(1-t)^{-1}+(-1)^d e_d^{(0)}+(-1)^d e_d^{(1)}t+\cdots+(-1)^d e_d^{(r)}t^r$. The coefficients e_k $(0\leqq k\leqq d)$ are the so called normalized Hilbert-Samuel coefficients of I with $e_d=e_d^{(0)}+e_d^{(1)}+\cdots+e_d^{(r)}$. Since $\sum_{i=0}^{k}\binom{d+i-1}{i}=\binom{k+d}{d}$, the Hilbert-Samuel function $\lambda(A/I^{n+1})$ of I equals $e_0\binom{n+d}{d}-e_1\binom{n+d-1}{d-1}+\cdots+(-1)^{d-1}e_{d-1}\binom{n+1}{1}+(-1)^d e_d$ for each $n>r$. We say that $e_0(1-t)^{-d}+\cdots+(-1)^{d-1}e_{d-1}(1-t)^{-1}$ and $(-1)^d(e_d^{(0)}+e_d^{(1)}t+\cdots+e_d^{(r)}t^r)$ are respectively the principal part and the polynomial part of the Poincaré series. In this paper we assume that A/P is infinite for each maximal ideal P, which guarantees the existence of superficial elements. A superficial element x of I is said to be stable if $I^n: x=I^{n-1}$ for all $n>1$. We say that a sequence of d elements $x_1, \cdots, x_d$ of I is an I-superficial (resp. a stable I-superficial) sequence, if x_k mod. $(x_1, \cdots, x_{k-1})$ is a (resp. stable) superficial element of $I/(x_1, \cdots, x_{k-1})$ for each k $(1\leqq k\leqq d)$. For an I-superficial sequence $x_1, \cdots, x_d$, there exists $m>0$ such that $(x_1, \cdots, x_d)I^m=I^{m+1}$. We evaluate m in section 1.

Now in case $d=1$, I is said to be stable if it satisfies one of the following equivalent conditions.

(i) $\lambda(A/I^n)$ is a polynomial in n for all $n>0$.

(ii) $xI=I^2$ for some x in I.

(iii) $P(I, t)$ is of the form $e_0(1-t)^{-1}-e_1$ (see [6]).

In the case of dimension $d>1$, the theory of stable ideals can be extended in two directions. One is about the ideals such that $(x_1, \cdots, x_d)I=I^2$ for some $x_1, \cdots, x_d$ in I. The other is about the ideals satisfying the above

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condition (i). We are mainly concerned with the latter case. So we define the stability of I as such. First we define the superficial saturation of a decreasing sequence of ideals belonging to I. Then the sequence of ideals thus obtained has a stable superficial element and has the same associated Hilbert-Samuel polynomial. Therefore we get some information about the coefficients of the polynomial part of $P(I, t)$ by comparing them. In section 2, we show that this method is especially useful in dimension 2. As an application, we give another proof of K. Kubota's result ([5]) that, if the Hilbert-Samuel polynomial $P(n)$ equals $e_0\binom{n+d-1}{d}+\lambda(A/I)\binom{n+d-1}{d-1}$, then $\lambda(A/I^{n+1})=P(n)$ for all $n\geqq 0$.

REMARK 1. An ideal I in A is called open if $m^n\subset I$ for some $n>0$, where m is the Jacobson radical of A. In [14] Lemma 6, we assumed implicitly that the open ideal I is contained in m. Therefore we assume in this paper that I is an ideal of definition.

REMARK 2. The definition of Cohen-Macaulay ring is that of Nagata [8]. Therefore we assume that all maximal ideals of A have the same rank.

§1. Superficial saturation.

LEMMA 1. *Let $x_1, \cdots, x_r$ be elements of I. Then the following statements are equivalent.*

(i) *The sequence $x_1, \cdots, x_r$ is A-regular.*

(ii) $A/(x_1, \cdots, x_{i-1})\xrightarrow{x_i}A/(x_1, \cdots, x_{i-1})$ *is injective for each i ($1\leqq i\leqq r$).*

(iii) $(A/(x_1, \cdots, x_{i-1}))\otimes A_P\xrightarrow{x_i\otimes 1}(A/(x_1, \cdots, x_{i-1}))\otimes A_P$ *is injective for each i ($1\leqq i\leqq r$) and each maximal ideal P in A.*

(iv) $A_P/(x_1, \cdots, x_{i-1})\xrightarrow{x_i}A_P/(x_1, \cdots, x_{i-1})$ *is injective for each i ($1\leqq i\leqq r$) and each maximal ideal P in A.*

(v) $\mathrm{ht}_{A_P}(x_1, \cdots, x_r)=r$ *for each maximal ideal P in A.*

PROOF. (i) is equivalent to (ii) by the definition. (ii) is equivalent to (iii) by [1] Proposition 3.9. Since the functor $\otimes A_P$ is exact, (iii) is equivalent to (iv). (iv) is equivalent to (v) by [7] Theorem 31.

LEMMA 2. *Let I be an ideal of definition of A and $x_1, \cdots, x_d$ be elements of I such that $(x_1, \cdots, x_d)I^m=I^{m+1}$ for some $m>0$. Then the sequence $x_1, \cdots, x_d$ is A-regular.*

PROOF. Let P be a maximal ideal of A. Then we have $d=\mathrm{ht}_{A_P}(IA_P)=$

$\mathrm{ht}_{A_P}(I^{m+1}A_P) \leqq \mathrm{ht}_{A_P}(x_1, \cdots, x_d) \leqq \mathrm{ht}_{A_P}(IA_P) = d$. Hence $\mathrm{ht}_{A_P}(x_1, \cdots, x_d) = d$. By Lemma 1, $x_1, \cdots, x_d$ is an A-regular sequence.

LEMMA 3. *Let x_1 be a superficial element of I. Then there exists elements $x_2, \cdots, x_d$ of I such that $x_1, \cdots, x_d$ is an I-superficial sequence and $(x_1, \cdots, x_d)I^m = I^{m+1}$ for some $m > 0$.*

PROOF. By induction on d, the proof proceeds just as the one given in [14] Lemma 6.

LEMMA 4. *Let $x_1, \cdots, x_d$ be an A-regular sequence in I. If a is in $(x_1, \cdots, x_d)^r I^m : x_1$, then a is a homogeneous polynomial of degree $r-1$ in $I[x_1, \cdots, x_d]$.*

PROOF. Using the condition (*) of [7] (15.B) repeatedly, we know that a is a homogeneous polynomial of degree $r-1$ in $I[x_1, \cdots, x_d]$.

LEMMA 5. *If $(x_1, \cdots, x_d)I = I^2$ for some $x_1, \cdots, x_d$ in I, then $x_1, \cdots, x_d$ is a stable I-superficial sequence.*

PROOF. Since $x_1, \cdots, x_d$ is A-regular by Lemma 2, the lemma follows from Lemma 4.

LEMMA 6. *Let x_1 be a superficial element of I. Then there is an integer $s > 0$ such that $I^n : x_1 = I^{n-1}$ for each $n > s$.*

PROOF. Let $r > 0$ be an integer such that $(I^n : x_1) \cap I^r = I^{n-1}$ for each $n > r$ and $x_1, \cdots, x_d$ be an I-superficial sequence such that $(x_1, \cdots, x_d)I^m = I^{m+1}$ for some $m > 0$. Put $s = m + r$. Then, for each $n > s$ and each a in $I^n : x_1$, we have $ax_1 \in I^n = (x_1, \cdots, x_d)^r I^{n-r}$. By Lemma 4, a is in I^r. Thus a is in I^{n-1}.

PROPOSITION 7. *Let x be a superficial element and s the least integer $s > 0$ such that $I^n : x = I^{n-1}$ for each $n > s$. Then s is independent of the choice of the superficial element x.*

PROOF. Let x, y be superficial elements of I and $s(x)$ and $s(y)$ the least such integers respectively for x and y. Let $n > s(x)$. Then, for any z in $I^n : y$, we have $yzx^{s(y)} \in I^{n+s(y)}$. Hence $zx^{s(y)} \in I^{n+s(y)-1}$. Thus $z \in I^{n-1}$. Therefore $I^n : y = I^{n-1}$, which means $s(x) \geqq s(y)$. By the change of the role of x and y, we get $s(x) = s(y)$.

We denote this s by $s(I)$.

COROLLARY 8. *If I has a stable superficial element, then any superficial element of I is stable.*

Let $m(I)$ be the least integer $m \geqq 0$ such that $\lambda(A/I^n)$ is a polynomial in n for each $n>m$. If $m(I)>0$, then $m(I)$ is the degree of the polynomial part of the Poincaré series $P(I, t)$. We say that I is stable if $m(I)=0$, in other words, if the polynomial part of the Poincaré series $P(I, t)$ is a constant.

PROPOSITION 9. *Let x be a superficial element of I, $\bar{A}=A/(x)$ and $\bar{I}=I/(x)$. Then $s(I) \leqq \max\{m(I)+1, m(\bar{I})\}$. If $m(I)+1<m(\bar{I})$, then $s(I)=m(\bar{I})$.*

PROOF. Let $N_n = xA \cap I^n$. Then we have the exact sequence $0 \to xA/N_n \to A/I^n \to \bar{A}/\bar{I}^n \to 0$. Hence

$$(*) \qquad \lambda(A/I^n)=\lambda(xA/N_n)+\lambda(\bar{A}/\bar{I}^n) .$$

Let $m(N)$ be the least integer $m \geqq 0$ such that $\lambda(xA/N_n)$ is a polynomial in n for each $n>m$. (i) Assume that $m(I)<m(\bar{I})$. Then we know that $m(N)=m(\bar{I})$ by (∗). Since $\lambda(xA/N_n)=\lambda(xA+I^n/I^n)=\lambda(A/(I^n\colon x))=\lambda(A/I^{n-1})$ for each $n>s(I)$ and both $\lambda(xA/N_n)$ and $\lambda(A/I^{n-1})$ are polynomials in n for each $n>m(\bar{I})$, $\lambda(xA/N_n)=\lambda(A/I^{n-1})$ for each $n>m(\bar{I})$. Hence $I^n\colon x=I^{n-1}$ for each $n>m(\bar{I})$. Thus $s(I) \leqq m(N)=m(\bar{I})$. (ii) Assume that $m(\bar{I}) \leqq m(I)$. Then from (∗), $m(N) \leqq m(I)$. Since $\lambda(A/N_n)=\lambda(A/(I^n\colon x))=\lambda(A/I^{n-1})$ for each $n>m(I)+1$, we have $I^n\colon x=I^{n-1}$. Thus $s(I) \leqq m(I)+1$. Finally suppose $m(I)+1<m(\bar{I})$. Then $m(N)=m(\bar{I})$ by (∗). Assume that $s(I)<m(N)$. Since $m(N)-1 \geqq s(I)$, we have $I^n\colon x=I^{n-1}$ for each $n>m(N)-1$. Hence $\lambda(xA/N_n)=\lambda(A/I^{n-1})$ is a polynomial in n for each $n>m(N)-1$ because $n-1>m(N)-2 \geqq m(I)$, a contradiction.

PROPOSITION 10. *Let $x_1, \cdots, x_d$ be an I-superficial sequence, $I_i=I/(x_1, \cdots, x_i)$ $(1 \leqq i \leqq d-1)$ and $m=1+\max\{m(I), m(I_1), \cdots, m(I_{d-1})\}$. Then $(x_1, \cdots, x_d)I^m=I^{m+1}$.*

PROOF. If $d=1$, the proposition follows from [6] Theorem 1.9. Let $d>1$ and assume the proposition for $d-1$. Then $(x_1, \cdots, x_d)I^m \equiv I^{m+1} \bmod. x_1A$. Hence $x_1A+(x_2, \cdots, x_d)I^m \supset I^{m+1}$. Since $m \geqq s(I)$, we deduce that $(x_1, x_2, \cdots, x_d)I^m=I^{m+1}$.

PROPOSITION 11. *Under the same assumptions as in Proposition 10, the following statements are equivalent.*

(i) *$I, I_1, \cdots, I_{d-1}$ are stable.*

(ii) $P(I, t)=e_0(1-t)^{-d}-e_1(1-t)^{1-d}$.

(iii) $\lambda(A/I^{n+1})=e_0\binom{n+d}{d}-e_1\binom{n+d-1}{d-1}$ *for all* $n \geqq 0$.

(iv) $\lambda(A/I^{n+1})=e_0\binom{n+d}{d}-e_1\binom{n+d-1}{d-1}$ *for all large n and* $e_d^{(0)}=0$.

(v) $(x_1, \cdots, x_d)I=I^2$.

Proof. Assume (i). If $d=1$, (i) implies (ii) by [6] Theorem 1.9. Let $d>1$. By Proposition 9, there exists a stable I-superficial sequence $x_1, \cdots, x_d$. Since $\lambda(x_1A/N_n)=\lambda(A/I^{n-1})$, the equality (*) in the proof of Proposition 9 implies that $P(I, t)=P(I_1, t)/(1-t)$. Now (ii) follows by induction on d. Obviously (ii) is equivalent to (iii) and (iii) implies (iv). (v) follows from (iv) by [5] Corollary 6 and Proposition 8. Assume (v). By Lemma 5, $x_1, \cdots, x_d$ is a stable I-superficial sequence. Thus $P(I_{i-1}, t)=P(I_i, t)/(1-t)$ $(1\leqq i\leqq d-1)$. Since $P(I_{d-1}, t)=e_0(1-t)^{-1}-e_1$, $P(I_i, t)=e_0(1-t)^{i-d}-e_1(1-t)^{i+1-d}$. Therefore I_i is stable $(0\leqq i\leqq d-1)$.

Remark. The conditions (i) and (v) of Proposition 11 are independent of the choice of the I-superficial sequence $x_1, \cdots, x_d$, because the conditions (ii), (iii) and (iv) are so.

Definition. We say that a family of ideals of definition of A, $\{J^{(n)}\}_{n>0}$ is a decreasing sequence belonging to I if it satisfies the following conditions.

(i) $J^{(n)}\supset J^{(n+1)}$ for each $n>0$.

(ii) $J^{(n)}J^{(m)}\subset J^{(n+m)}$ for each $n, m>0$.

(iii) $I^n\subset J^{(n)}$ for each $n>0$ and $I^n=J^{(n)}$ for all large n.

Let $\{J^{(n)}\}_{n>0}$ be a decreasing sequence belonging to I. Then we say that $P(t)=\sum_{n\geqq 0}\lambda(J^{(n)}/J^{(n+1)})t^n$ $(J^{(0)}=A)$ and $H(n)=\lambda(A/J^{(n+1)})$ are respectively the Poincaré series and the Hilbert-Samuel function of $\{J^{(n)}\}_{n>0}$. By the condition (iii), I and $\{J^{(n)}\}_{n>0}$ have the same Hilbert-Samuel polynomial. Therefore the principal parts of their Poincaré series are the same.

Lemma 12. *If x is a superficial element of I, then x is a superficial element of IA_P for each maximal ideal P in A and $s(IA_P)\leqq s(I)$.*

Proof. Since $0\rightarrow I^{m-1}\rightarrow A\overset{x}{\rightarrow}A/I^m$ is exact for each $m>s(I)$ by Lemma 6, we have the exact sequence; $0\rightarrow I^{m-1}A_P\rightarrow A_P\overset{x}{\rightarrow}A_P/I^mA_P$ for each $m>s(I)$. Thus x is a superficial element of IA_P and $s(IA_P)\leqq s(I)$.

Let x be a superficial element of I. Then the least integer $s(J^{(*)})>0$ such that $J^{(n)}: x=J^{(n-1)}$ for each $n>s(J^{(*)})$ is independent of the choice of x. If $s(J^{(*)})=1$, we say that x is a stable superficial element of the decreasing sequence $\{J^{(n)}\}_{n>0}$. Put $I^{(n)}=\cup_{k>0}(J^{(n+k)}: x^k)$. As $(J^{(n+k)}: x^k)\subset(J^{(n+k+1)}: x^{k+1})$ for each $k>0$ and A is Noetherian, $I^{(n)}=I^{n+k}: x^k$ for all large k. Obviously $J^{(n)}\subset I^{(n)}$.

Lemma 13. *$I^{(n)}$ is independent of the choice of the superficial* *element* x.

PROOF. Let y be a superficial element of I and $m>s(I)$. Then $ax^m \in I^{n+m}$ if and only if $ax^m y^m \in I^{n+2m}$. Hence we have $I^{n+m}: x^m = I^{n+m}: y^m$.

LEMMA 14. *$I^{(n)}$ is contained in the Jacobson radical of A for each $n>0$.*

PROOF. Assume the contrary. Then there exists $a \in I^{(n)}$ for some $n>0$ which is not contained in some maximal ideal P in A. Since $ax^m \in I^{n+m}$ for some large $m>0$, $ax^m \in I^{n+m}A_P$. As a is not in P, a is a unit in A_P. Hence $x^m \in I^{n+m}A_P$. Let k be an integer such that $k \geqq s(I)$ and $k>n$. For any element b of $I^{k-n}A_P$, we have $bx^m \in I^{k+m}A_P$. By Lemma 12, $b \in I^k A_P$. Hence $I^{k-n}A_P \subset I^k A_P$. Therefore $I^{k-n}A_P = I^k A_P$. By Nakayama's lemma, $I^{k-n}A_P = 0$, a contradiction.

DEFINITION. We say that the decreasing sequence $\{I^{(n)}\}_{n>0}$ thus obtained is the superficial saturation of $\{J^{(n)}\}_{n>0}$. Remark that the superficial saturation is uniquely determined by I. We say that I is superficially saturated if $I^{(1)} = I$. A decreasing sequence $\{J^{(n)}\}_{n>0}$ is said to be superficially saturated if $J^{(n)} = I^{(n)}$ for each $n>0$.

PROPOSITION 15. *Let x be a superficial element of I, $\{J^{(n)}\}_{n>0}$ a decreasing sequence belonging to I and $\{I^{(n)}\}_{n>0}$ the superficial saturation of $\{J^{(n)}\}_{n>0}$. Then*

(i) *$\{I^{(n)}\}_{n>0}$ is a decreasing sequence belonging to I.*

(ii) *$\{I^{(n)}\}_{n>0}$ has a stable superficial element.*

(iii) *$\{I^{(n)}\}_{n>0}$ is superficially saturated.*

PROOF. (i) By Lemma 14, $I^{(n)}$ is an ideal of definition of A. Obviously $I^{(n)} \supset I^{(n+1)}$ for each $n>0$. Let $a \in I^{(n)}$ and $b \in I^{(m)}$. Then $ax^k \in I^{n+k}$ and $bx^k \in I^{m+k}$ for some k. Hence $abx^{2k} \in I^{n+m+2k}$. Thus $ab \in I^{(n+m)}$. Obviously $I^n \subset I^{(n)}$ for each $n>0$. Let m be an integer such that $I^{(n)} = I^{n+m}: x^m$. Then, if $n \geqq s(I)$, we have $I^{(n)} = I^n$. Thus $\{I^{(n)}\}_{n>0}$ is a decreasing sequence belonging to I. (ii) Let $k>1$ and $a \in I^{(k)}: x$. Then $ax \in I^{(k)}$. Hence $ax^{m+1} \in I^{k+m}$ for some m. This implies that $a \in I^{(k-1)}$. (iii) follows from (ii).

COROLLARY 16. *I has a stable superficial element if and only if $I^{(n)} = I^n$ for each $n>0$.*

LEMMA 17 (One dimensional case). *Assume that A is of dimension one. Let x be a regular element of A contained in I, e_0 the multiplicity of I and $\{J^{(n)}\}_{n>0}$ a decreasing sequence belonging to I. Then $e_0 = \lambda(A/xA) \geqq$*

$\lambda(J^{(n)}/J^{(n+1)})$ *for each* $n\geqq 0$. *The equality holds if and only if* $xJ^{(n)}=J^{(n+1)}$. *If* $xJ^{(n)}=J^{(n+1)}$, *then* $xJ^{(m)}=J^{(m+1)}$ *for each* $m\geqq n$.

PROOF. $e_0=\lambda(A/xA)$ by [6] Theorem 1.9. Since multiplication by x induces the isomorphism $A/J^{(n)}\to xA/xJ^{(n)}$, we have

$$\begin{aligned}\lambda(A/xA)&=\lambda(A/xA)+\lambda(xA/xJ^{(n)})-\lambda(A/J^{(n)})\\&=\lambda(A/xJ^{(n)})-\lambda(A/J^{(n)})\\&=\lambda(J^{(n)}/xJ^{(n)})\geqq\lambda(J^{(n)}/J^{(n+1)})\ .\end{aligned}$$

From this, it is clear that the equality holds if and only if $xJ^{(n)}=J^{(n+1)}$. Assume that $xJ^{(n)}=J^{(n+1)}$. It is obvious that $xJ^{(m)}\subset J^{(m+1)}$ for each $m\geqq 0$. Let $m\geqq n$ and $y\in J^{(m+1)}$. Then $yx^k\in J^{(m+1+k)}$ for some k. As $y\in J^{(n+1)}$, $y=xz$ for some $z\in J^{(n)}$. Thus $zx^{k+1}\in J^{(m+1+k)}$. This implies that $z\in J^{(m)}$.

COROLLARY 18. *Let A be of dimension one and* $\{J^{(n)}\}_{n>0}$ *a decreasing sequence belonging to I. Then all the normalized coefficients of the polynomial part of the Poincaré series of* $\{J^{(n)}\}_{n>0}$ *are non-negative.*

LEMMA 19. *Let x be a superficial element of I,* $\{I^{(n)}\}_{n>0}$ *the superficial saturation of* $\{I^n\}_{n>0}$ *and* $(-1)^d(a_0+a_1t+\cdots+a_rt^r)$ *the polynomial part of the Poincaré series of* $\{I^{(n)}\}_{n>0}$. *Then* $e_0(I^{(*)})=e_0(I^{(*)}/(x))$, $\cdots$, $e_{d-2}(I^{(*)})=e_{d-2}(I^{(*)}/(x))$ *and the polynomial part of* $\{I^{(n)}+(x)/(x)\}_{n>0}$ *is of the form* $(-1)^{d-1}(b_0+b_1t+\cdots+b_{r+1}t^{r+1})$, *where* $e_{d-1}=b_0+\cdots+b_{r+1}$, $a_0=b_1+b_2+\cdots+b_{r+1}$, $a_1=b_2+b_3+\cdots+b_{r+1}$, $\cdots$, $a_{r-1}=b_r+b_{r+1}$, $a_r=b_{r+1}$.

PROOF. Let $N_n=xA\cap I^{(n)}$. Then, from the exact sequence $0\to xA/N_n\to A/I^{(n)}\to\bar{A}/\bar{I}^{(n)}\to 0$, we have $(1-t)P(I^{(*)},t)=P(\bar{I}^{(*)},t)$, just as the proof of Proposition 9. Comparing the coefficients, we have the lemma.

PROPOSITION 20. *If there exists a stable I-superficial sequence, then all the normalized coefficients* e_i $(1\leqq i\leqq d)$ *of the Hilbert-Samuel polynomial of I and all the normalized coefficients* $e_d^{(i)}$ $(0\leqq i\leqq m(I))$ *of the polynomial part of the Poincaré series of I are non-negative.*

PROOF. This follows from Lemma 19 by using induction on the dimension of A.

LEMMA 21. *Let* $\{J^{(n)}\}_{n>0}$ *be a decreasing sequence belonging to I and let* $a_0+a_1t+\cdots+a_mt^m$, $b_0+b_1t+\cdots+b_lt^l$ *and* $a_0^*+a_1^*t+\cdots+a_n^*t^n$ *be respectively polynomial parts of the Poincaré series of I,* $\{J^{(n)}\}_{n>0}$ *and* $\{I^{(n)}\}_{n>0}$. *Then* $a_0+a_1+\cdots+a_k\geqq b_0+b_1+\cdots+b_k\geqq a_0^*+a_1^*+\cdots+a_k^*$ *for each* $k\geqq 0$.

PROOF. Since the Poincaré series of these decreasing sequence have the same principal part, the inequality $\lambda(A/I^{k+1}) \geqq \lambda(A/J^{(k+1)}) \geqq \lambda(A/I^{(k+1)})$ implies the lemma.

§2. Two dimensional case.

THEOREM 22. *Let A be of dimension 2 and $a_0^* + a_1^* t + \cdots + a_m^* t^m$ $(a_m^* \neq 0)$ the polynomial part of the Poincaré series of a superficially saturated decreasing sequence $\{I^{(n)}\}_{n>0}$ belonging to I. Then $a_0^* > a_1^* > \cdots > a_m^* > 0$.*

PROOF. Let x be a superficial element of I. Then $P(I^{(*)}, t) = P(I^{(*)}/(x), t)/(1-t)$. Now the theorem follows from Lemma 17 and Lemma 19.

COROLLARY 23. *Let A be of dimension $d \geqq 2$. Then $e_2(I) \geqq 0$.*

PROOF. Since the Hilbert-Samuel polynomials of $\{I^{(n)}\}_{n>0}$ and $\{I^n\}_{n>0}$ are the same, we may consider $\{I^{(n)}\}_{n>0}$ instead of $\{I^n\}_{n>0}$. By Lemma 19, it is sufficient to prove the lemma in the case of dimension 2. Now the lemma follows from Theorem 22.

THEOREM 24. *Let A be of dimension 2 and let I be stable and superficially saturated. Then $\{I^n\}_{n>0}$ is the only decreasing sequence belonging to I.*

PROOF. With the same notations as in Lemma 21, we have $a_0 = b_0 = a_0^*$ and $a_1 + \cdots + a_k \geqq b_1 + \cdots + b_k \geqq a_1^* + \cdots + a_k^*$ for each $k \geqq 1$. Since $a_k = 0$ and $a_k^* \geqq 0$ for each $k \geqq 1$, we have $b_k = 0$ for each $k \geqq 1$.

COROLLARY 25. *Let A be of dimension 2 and let I be stable and superficially saturated. Then there exists a stable superficial element of I.*

PROOF. By Theorem 24, $I^{(k)} = I^k$ for each $k \geqq 1$.

Let $m(J^{(*)})$ be the least non-negative integer such that $J^{(n)} : x = J^{(n-1)}$ for each $n > m(J^{(*)})$, where x is a superficial element of I and $\{J^{(n)}\}_{n>0}$ is a decreasing sequence belonging to I.

PROPOSITION 26. *Let $\{I^{(n)}\}_{n>0}$ be the superficial saturation of a decreasing sequence $\{J^{(n)}\}_{n>0}$ belonging to I, $m = m(J^{(*)})$ and $n = m(I^{(*)})$. Then:*

(i) *If A is of dimension 1, then $xJ^{(m+1)} = J^{(m+2)}$ for each superficial element x.*

(ii) *If A is of dimension 2, then $(x, y)I^{(n+2)} = I^{(n+3)}$ for each I-superficial sequence x, y. In particular, if I is stable and superficially saturated, then $(x, y)I^2 = I^3$.*

PROOF. (i) follows from Lemma 17. (ii) Let x, y be an I-superficial sequence. Then $m(I^{(*)}/(x))=m+1$. Hence $yI^{(m+2)}/(x)=I^{(m+3)}/(x)$ by (i). As x is a stable superficial element of $\{I^{(n)}\}_{n>0}$, we have $(x, y)I^{(m+2)}=I^{(m+3)}$.

PROPOSITION 27. *Let A be of dimension 2 and k a positive integer. Then the decreasing sequence $\{I^{(nk)}\}_{n>0}$ belonging to I^k is stable if and only if $k>m(I^{(*)})$.*

PROOF. Let $a_0+at_1+\cdots+a_mt^m$ ($a_m\neq 0$) be the polynomial part of the Poincaré series of $\{I^{(n)}\}_{n>0}$. Then the polynomial part of the Poincaré series of $\{I^{(nk)}\}_{n>0}$ is

$$(a_0+\cdots+a_{k-1})+(a_k+\cdots+a_{2k-1})t+\cdots .$$

Now the proposition follows from Theorem 22.

§3. An application.

As an application of superficial saturation, we prove the implication (iv)$\Rightarrow$(v) of Proposition 11.

THEOREM 28 (K. Kubota). *Let $\lambda(A/I)=e_0-e_1$ and $\lambda(A/I^{n+1})=e_0\binom{n+d}{d}-e_1\binom{n+d-1}{d-1}$ for all large n. Then $(x_1, \cdots, x_d)I=I^2$ for each I-superficial sequence $x_1, \cdots, x_d$.*

PROOF. Let x_1 be a superficial element of I, $\bar{I}=I/(x_1)$, $\{I^{(n)}\}_{n>0}$ the superficials aturation of $\{I^n\}_{n>0}$ and $\bar{I}^{(n)}=I^{(n)}+(x_1)/(x_1)$. If $d=1$, then the theorem follows from [6] Theorem 1.9. Assume that $d=2$. Since $e_d^{(0)}(I)=0$, we have $0=e_d^{(0)}(I^{(*)})=e_d^{(1)}(I^{(*)})=e_d^{(2)}(I^{(*)})=\cdots$ by Theorem 22. As $e_d^{(0)}(I^{(*)})=e_d^{(0)}(I)$ and the principal parts of the Poincaré series of $\{I^{(n)}\}_{n>0}$ and I are the same, we have $I^{(1)}=I$. By Lemma 19, $m(\bar{I}^{(*)})=0$. By Proposition 26 (i), $x_2\bar{I}^{(1)}=\bar{I}^{(2)}$ for some $x_2\in I$. Since $\bar{I}^{(2)}\supset\bar{I}^2$ and $x_2\bar{I}^{(1)}=x_2\bar{I}\subset\bar{I}^2$, $x_2\bar{I}=\bar{I}^2$, i.e. $x_2I\equiv I^2$ mod. (x_1). As I is superficially saturated, we have $(x_1, x_2)I=I^2$. Now assume that $d>2$ and we proceed by induction on d. By Lemma 19, the Poincaré series of $\{\bar{I}^{(n)}\}_{n>0}$ is of the form

$$e_0(1-t)^{1-d}-e_1(1-t)^{2-d}+\cdots+(-1)^{d-1}(b_0+b_1t+\cdots+b_rt^r) ,$$

where $b_0+b_1+\cdots+b_r=e_{d-1}=0$. Since $\{\bar{I}^{(n)}\}_{n>0}$ and $\bar{I}$ have the same Hilbert-Samuel polynomial, $e_{d-1}(\bar{I})=b_0+b_1+\cdots+b_r=0$. As $e_0-e_1=\lambda(A/I)=\lambda(\bar{A}/\bar{I})$, we can apply the induction assumption to $\bar{I}$ to know that $(\bar{x}_2, \cdots, \bar{x}_d)\bar{I}=\bar{I}^2$ for some $\bar{I}$-superficial sequence $\bar{x}_2, \cdots, \bar{x}_d$. By Lemma 5, $\bar{x}_2$ is a stable superficial element of $\bar{I}$. Therefore each decreasing sequence belonging to $\bar{I}$ coincides with $\{\bar{I}^n\}_{n>0}$ by Lemma 21. Thus $\bar{I}^{(n)}=\bar{I}^n$ for each $n>0$. As $e_0-e_1=\lambda(\bar{A}/\bar{I})=\lambda(\bar{A}/\bar{I}^{(1)})$, we have $b_0=0$. Therefore $e_d^{(0)}(I^{(*)})=b_1+b_2+\cdots+b_r=0$. Since $e_d^{(0)}(I)=0$, we know tha $I^{(1)}=I$, namely I is superficially saturated, because the principal parts of the Poincaré series of $\{I^{(n)}\}_{n>0}$ and I are the same. As $(x_2, \cdots, x_d)I\equiv I^2$ mod. (x_1) and I is superficially saturated, we have $(x_1, \cdots, x_d)I=I^2$.

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