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On the Algebraic Independence of Certain Numbers Connected with the Exponential and the Elliptic Functions

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Introduction

It has been conjectured in transcendental number theory that π and $\log 2$ are algebraically independent. It has also been conjectured that at least one of the numbers $\sum_{n=0}^{\infty} 2^{-n^2}$ and $\sum_{n=0}^{\infty}(-1)^n 2^{-n^2}$ is transcendental. Though no one has ever proved these conjectures, the authors have proved the following

PROPOSITION. *At least two of the numers*

$$\pi\,,\quad \log 2\,,\quad \sum_{n=0}^{\infty} 2^{-n^2}\,,\quad \sum_{n=0}^{\infty}(-1)^n 2^{-n^2}$$

are algebraically independent over $\boldsymbol{Q}$. (This is a special case of Example 2.1, §1.)

Let x be a transcendental number, and let κ be a real number $\geqq 2$. We shall say that x is of *transcendence type* $\leqq\kappa$ if there exists a constant $c>0$ depending only on x and κ such that

$$\log|P(x)|\geqq -c(\deg P+\log H(P))^{\kappa}$$

for all non-trivial polynomials P in $\boldsymbol{Z}[X]$. Here, $\deg P$ denotes the degree of P, and $H(P)$ denotes the height of P, i.e. the maximum of the absolute values of the coefficients of P.

The idea of transcendence type was introduced by Lang in his book [4]. For example, it follows from Fel'dman's result [2, Theorem 4] that

(1) π is of transcendence type $\leqq 2+\varepsilon$, for every $\varepsilon>0$.

This is a well-known result in transcendental number theory.

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The above proposition can be deduced from the following general theorem.

THEOREM. *Let $\mathscr{P}(z)$ be a Weierstrass $\mathscr{P}$-function with invariants g_2, g_3, and let ω_1, ω_2 be a pair of fundamental periods of $\mathscr{P}(z)$. Assume that x is a transcendental number of transcendence type$\leqq\kappa$, and that $2\leqq\kappa<2+(1/3)$. Let a be any non-zero complex number. Then at least two of the numbers*

$$x,\ a,\ g_2,\ g_3,\ \omega_1,\ \omega_2,\ e^{a\omega_1},\ e^{a\omega_2}$$

are algebraically independent over $\boldsymbol{Q}$.

The purpose of the present paper is to prove the above theorem.

We wish to express our sincere gratitude to our teacher Prof. T. Mitsui who has shown great interest in the results of the present paper and has given friendly encouragement to the authors.

§1. **Corollaries.** In the first place we introduce a notation for brevity.

NOTATION. For a finite set $S\subset \boldsymbol{C}$, ∂S denotes the maximal number of algebraically independent elements in S.

We state some of the interesting consequences of the theorem.

From (1), we can take $x=\pi$ in our theorem. Moreover, let us take $a=i\pi/\omega_1$, then we obtain the following results:

COROLLARY 1.

$$\partial\{g_2,\ g_3,\ \omega_1,\ \omega_2,\ \pi,\ e^{i\pi\tau}\}\geqq 2\ ,$$

where $\tau=\omega_2/\omega_1$.

We remark here that

$$g_2=g_2(\omega_1,\ \omega_2)=60\sum_{(m,n)\in \boldsymbol{Z}^2-\{o\}}(m\omega_1+n\omega_2)^{-4}$$

and

$$g_3=g_3(\omega_1,\ \omega_2)=140\sum_{(m,n)\in \boldsymbol{Z}^2-\{o\}}(m\omega_1+n\omega_2)^{-6}\ .$$

EXAMPLE 1.1. If r is a rational number, then

$$\partial\{g_2(1,\ i\pi^r),\ g_3(1,\ i\pi^r),\ \pi,\ e^{\pi^{r+1}}\}\geqq 2\ .$$

Let $\theta_0(v,\ \tau)$, $\theta_2(v,\ \tau)$, $\theta_3(v,\ \tau)$ be the theta functions defined by

$$\theta_0(v,\tau)=\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}e^{2n\pi i v},$$

$$\theta_2(v,\tau)=\sum_{n=-\infty}^{\infty}q^{(n-1/2)^2}e^{2n\pi i v},$$

$$\theta_3(v,\tau)=\sum_{n=-\infty}^{\infty}q^{n^2}e^{2n\pi i v},$$

where $q=e^{i\pi\tau}$ and $\mathrm{Im}(\tau)>0$. Put

$$\theta_0=\theta_0(0,\tau),\qquad \theta_2=\theta_2(0,\tau),\qquad \theta_3=\theta_3(0,\tau),$$

then the following classical formulas hold:

$$\theta_0^4+\theta_2^4=\theta_3^4,$$

$$g_2(1,\tau)=\frac{2\pi^4}{3}(\theta_0^8+\theta_2^8+\theta_3^8),$$

$$g_3(1,\tau)=\frac{4\pi^6}{27}(\theta_0^4-\theta_2^4)(2\theta_3^8+\theta_0^4\theta_2^4).$$

Therefore, from Corollary 1, we obtain the following

COROLLARY 2.

$$\partial\{\pi,\tau,e^{i\pi\tau},\theta_3(0,\tau),\theta_0(0,\tau)\}\geqq 2.$$

Let us take $\tau=-(\log\alpha)/\pi i$ in this corollary, then we have

EXAMPLE 2.1. If α is an algebraic number with $|\alpha|>1$, then

$$\partial\left\{\pi,\log\alpha,\sum_{n=0}^{\infty}\alpha^{-n^2},\sum_{n=0}^{\infty}(-1)^n\alpha^{-n^2}\right\}\geqq 2.$$

Furthermore, we have

EXAMPLE 2.2.

$$\partial\left\{\pi,\log\pi,\sum_{n=0}^{\infty}\pi^{-n^2},\sum_{n=0}^{\infty}(-1)^n\pi^{-n^2}\right\}\geqq 2\qquad(\tau=-(\log\pi)/\pi i).$$

From this example, we see that *at least one of the numbers*

$$\log\pi,\qquad \sum_{n=0}^{\infty}\pi^{-n^2},\qquad \sum_{n=0}^{\infty}(-1)^n\pi^{-n^2}$$

must be transcendental.

Next, we put

$$q_0=\prod_{n=1}^{\infty}(1-q^{2n}) , \qquad q_1=\prod_{n=1}^{\infty}(1+q^{2n}) ,$$

$$q_2=\prod_{n=1}^{\infty}(1+q^{2n-1}) , \quad q_3=\prod_{n=1}^{\infty}(1-q^{2n-1}) .$$

Then we have the following classical formulas:

$$\theta_3=q_0q_2^2 , \qquad \theta_0=q_0q_3^2 ,$$

$$q_1q_2q_3=1 , \quad 16qq_1^8=q_2^8-q_3^8 .$$

Therefore, from Corollary 2, we obtain

COROLLARY 3.

$$\partial\left\{\pi, \tau, q, \prod_{n=1}^{\infty}(1-q^{2n}), \prod_{n=1}^{\infty}(1+q^{2n})\right\}\geqq 2 .$$

From this corollary we have two examples similar to Examples 2.1 and 2.2.

It follows from Fel'dman's result [3, Theorem] that if g_2, g_3 are algebraic and $\mathscr{P}(z)$ has complex multiplication, then any non-zero period ω of $\mathscr{P}(z)$ is of transcendence type$\leqq 2+\varepsilon$, for every $\varepsilon>0$. Hence, we can take $x=\omega_1$ in our theorem, and obtain the following result:

COROLLARY 4. *If g_2, g_3 are algebraic and $\mathscr{P}(z)$ has complex multiplication, then*

$$\partial\{a, \omega_1, e^{a\omega_1}, e^{a\omega_2}\}\geqq 2 ,$$

for any non-zero complex number a.

EXAMPLE 4.1. If r is a rational number, then

$$\partial\{\omega_1, e^{\omega_1^r}, e^{\tau\omega_1^r}\}\geqq 2 ,$$

where $\tau=\omega_2/\omega_1$. $(a=\omega_1^{r-1}.)$

EXAMPLE 4.2. If α is an algebraic number $\neq 0$, and $\log\alpha\neq 0$, then

$$\partial\{\omega_1, \log\alpha, \alpha^{\tau}\}\geqq 2 . \qquad (a=(\log\alpha)/\omega_1.)$$

EXAMPLE 4.3.

$$\partial\{\omega_1, \log\omega_1, \omega_1^{\tau}\}\geqq 2 . \qquad (a=(\log\omega_1)/\omega_1.)$$

§2. **Preliminary lemmas.** Let x be a transcendental number, and let K be a finite extension of the field $\mathbf{Q}(x)$. Then we can write

$$K=\mathbf{Q}(x, \theta) ,$$

where θ is algebraic over $\mathbf{Q}(x)$, and is integral over the ring $\mathbf{Z}[x]$. Let d be the degree of θ over $\mathbf{Q}(x)$, then an element Γ of the ring $\mathbf{Z}[x, \theta]$ can be expressed uniquely in the form

$$\Gamma=\sum_{\nu=0}^{d-1} P_\nu(x)\theta^\nu ,$$

where $P_0(x), \cdots, P_{d-1}(x) \in \mathbf{Z}[x]$.

With respect to x and θ, we define the *degree* of Γ by

$$\deg \Gamma=\max_{0\leqq\nu\leqq d-1} \{\deg P_\nu\} ,$$

and the *height* of Γ by

$$H(\Gamma)=\max_{0\leqq\nu\leqq d-1} \{H(P_\nu)\} .$$

Further, we define the *size* of Γ by

$$s(\Gamma)=\max\{1+\deg \Gamma, \log H(\Gamma)\} .$$

(See Waldschmidt [5].)

Let $\Gamma_1, \cdots, \Gamma_n$ be n elements of $\mathbf{Z}[x, \theta]$, then the following inequalities hold:

$$s\left(\sum_{j=1}^{n} \Gamma_j\right)\leqq\max_{1\leqq j\leqq n} \{s(\Gamma_j)\}+\log n ,$$

$$s\left(\prod_{j=1}^{n} \Gamma_j\right)\leqq c \cdot \sum_{j=1}^{n} s(\Gamma_j) ,$$

where $c>0$ is a constant depending only on x and θ. These inequalities are easily verified.

LEMMA 1. *Let Γ be a non-zero element of $\mathbf{Z}[x, \theta]$, and let*

$$P=P(x)=N_{K/\mathbf{Q}(x)}(\Gamma) \in \mathbf{Z}[x]$$

be the norm of $\Gamma \in K$ over $\mathbf{Q}(x)$. Then we have

$$s(P)\leqq c \cdot s(\Gamma)$$

and

$$\log |P(x)| \leqq \log |\Gamma| + c \cdot s(\Gamma) ,$$

where $c>0$ *is a constant depending only on* x *and* θ.

PROOF. See Lemma 4.2.20 of [5].

LEMMA 2 (Siegel's lemma). *Let*

$$\sum_{j=1}^{n} A_{ij} X_j = 0 \quad (i=1, \cdots, m)$$

be a system of linear equations with coefficients A_{ij} *in* $Z[x, \theta]$, *and* $n \geqq 2m$. *Let* σ *be a number* $\geqq 1$ *such that* $s(A_{ij}) \leqq \sigma$, *for all* i, j. *Then there exists a non-trivial solution* $X_j \in Z[x, \theta]$, $j=1, \cdots, n$, *with*

$$s(X_j) < c(\sigma + \log n) , \quad \textit{for each } j ,$$

where $c>0$ *is a constant depending only on* x *and* θ.

PROOF. See Lemma 4.3.1 of [5].

LEMMA 3. *Let* $t \geqq 0$, $\lambda \geqq 0$ *be integers. There exists a polynomial* $R_{t,\lambda} \in Z[X, X', Y]$, *of degree at most* $\lambda + 2t$ *in* X, t *in* X' *and* t *in* Y, *such that*

$$\frac{d^t}{dz^t}(\wp(z)^\lambda) = R_{t,\lambda}\left(\wp(z), \wp'(z), \frac{1}{2}g_2\right) ,$$

$$L(R_{t,\lambda}) \leqq 9^t(\lambda + t)^t ,$$

where $L(R_{t,\lambda})$ *denotes the length of* $R_{t,\lambda}$, i.e. *the sum of the absolute values of the coefficients of* $R_{t,\lambda}$.

PROOF. The proof is easy by induction on t, starting from $\wp''(z) = 6\wp(z)^2 - (1/2)g_2$.

LEMMA 4. *If* a *is a non-zero complex number, then the following three functions*

$$z, \ e^{az}, \ \wp(z)$$

are algebraically independent over C.

PROOF. Assume that the lemma is false, then there exists a non-zero polynomial $P(X_1, X_2, X_3)$ in $C[X_1, X_2, X_3]$ such that $P(z, e^{az}, \wp(z))$ vanishes identically.

The polynomial P can be expressed in the form

$$P=\sum_{j=0}^{m} A_j(X_1, X_2)X_3^j ,$$

where $A_0(X_1, X_2), \cdots, A_m(X_1, X_2) \in \boldsymbol{C}[X_1, X_2]$. Without loss of generality, we can assume that $A_0(X_1, X_2) \neq 0$.

Since the two functions z, e^{az} are algebraically independent over $\boldsymbol{C}$ (see Lemma 1.4.1 of [**5**]), the function $f(z)$ defined by

$$f(z)=A_0(z, e^{az})$$

does not vanish identically. Hence we see that $m \geqq 1$, and so we have that

$$f(z)=-\mathscr{P}(z) \sum_{j=1}^{m} A_j(z, e^{az})\mathscr{P}(z)^{j-1} .$$

From this equality, we find that the number $n(f, r)$ of zeros of $f(z)$ in the disc $|z| \leqq r$ satisfies

$$n(f, r) \geqq c_1 r^2 , \quad \text{for all sufficiently large } r ,$$

where $c_1 > 0$ is independent of r.

On the other hand, using (1.5.5) of [**5**], we have

$$n(f, r) \leqq c_2 r , \quad \text{for all sufficiently large } r ,$$

where $c_2 > 0$ is independent of r. But this upper estimate for $n(f, r)$ contradicts the lower estimate for $n(f, r)$, and the contradiction proves the lemma.

§**3. Proof of the Theorem.** Let K_0 be the field

$$K_0=\boldsymbol{Q}(x, a, g_2, g_3, \omega_1, \omega_2, e^{a\omega_1}, e^{a\omega_2})$$

and let K be the field

$$K=K_0(e_1, e^{a\omega_1/2}, e^{a\omega_2/2}) ,$$

where $e_1=\mathscr{P}(\omega_1/2)$. We prove our theorem by contradiction. Assume that the transcendence degree of the field K_0 over $\boldsymbol{Q}$ is 1. Then the transcendence degree of the field K over $\boldsymbol{Q}$ is also 1, since K is algebraic over K_0. Therefore we can write

$$K=\boldsymbol{Q}(x, \theta) ,$$

where θ is algebraic over $\boldsymbol{Q}(x)$, and is integral over $\boldsymbol{Z}[x]$.

Let N be a sufficiently large integer, and define

$$L_0=N^3[\log N]\ ,\qquad L_1=[N^{3-3\mu}]\ ,\qquad L_2=[N^{1+\mu}]\ ,$$
$$T=N^3\ ,\qquad H=[N^{2-\mu}]\ ,$$

where $\mu>0$ is a sufficiently small number.

Hereafter, $c_1, c_2, \cdots$ denote positive constants which are independent of N.

LEMMA 5. *There exists a non-zero polynomial*

$$P_0=\sum_{\lambda_0=0}^{L_0}\sum_{\lambda_1=0}^{L_1}\sum_{\lambda_2=0}^{L_2}\varphi(\lambda_0, \lambda_1, \lambda_2)X_0^{\lambda_0}X_1^{\lambda_1}X_2^{\lambda_2}\in \boldsymbol{Z}[x, \theta][X_0, X_1, X_2]$$

such that the function $F(z)$ *defined by*

$$F(z)=P_0(z, \mathscr{P}(z), e^{az})$$

satisfies the following two conditions:

$$\frac{d^t}{dz^t}F\left(\frac{\omega_1}{2}+h_1\omega_1+h_2\omega_2\right)=0\quad \text{for}\quad 0\leqq t<T\ ,\quad 0\leqq h_1, h_2<H\ ,\quad (h_1, h_2)\in \boldsymbol{Z}\times\boldsymbol{Z}\ ;\tag{2}$$

$$s(\varphi(\lambda_0, \lambda_1, \lambda_2))\leqq c_1N^3(\log N)^2\quad \text{for all}\quad \lambda_0, \lambda_1, \lambda_2\ .\tag{3}$$

PROOF. We can regard (2) as the linear homogeneous system

$$\sum_{\lambda_0=0}^{L_0}\sum_{\lambda_1=0}^{L_1}\sum_{\lambda_2=0}^{L_2}\varphi(\lambda)G_{t,(\lambda)}\left(\frac{\omega_1}{2}+h_1\omega_1+h_2\omega_2\right)=0\qquad (0\leqq t<T;\ 0\leqq h_1, h_2<H)\tag{4}$$

of TH^2 equations in $(L_0+1)(L_1+1)(L_2+1)$ unknowns $\varphi(\lambda)=\varphi(\lambda_0, \lambda_1, \lambda_2)\in \boldsymbol{Z}[x, \theta]$, where

$$G_{t,(\lambda)}(z)=\frac{d^t}{dz^t}(z^{\lambda_0}\mathscr{P}(z)^{\lambda_1}e^{\lambda_2 az})\ .$$

Since any element of K is a quotient of two elements of $\boldsymbol{Z}[x, \theta]$, we can write

$$\omega_1=2\Gamma_1/\Gamma_0\ ,\qquad \omega_2=\Gamma_2/\Gamma_0\ ,\qquad e_1=\Gamma_3/\Gamma_0\ ,\qquad g_2=2\Gamma_4/\Gamma_0\ ,$$
$$e^{a\omega_1/2}=\Gamma_5/\Gamma_0\ ,\qquad e^{a\omega_2/2}=\Gamma_6/\Gamma_0\ ,\qquad a=\Gamma_7/\Gamma_0\ ,$$

where $\Gamma_0, \cdots, \Gamma_7\in \boldsymbol{Z}[x, \theta]$.

By Leibnitz's rule, we have

$$G_{t,(\lambda)}(z)=\sum_{\substack{\sigma_0+\sigma_1+\sigma_2=t\\ \sigma_0\leqq\lambda_0}}\frac{t!}{\sigma_0!\sigma_1!\sigma_2!}\cdot\frac{\lambda_0!}{(\lambda_0-\sigma_0)!}\cdot z^{\lambda_0-\sigma_0}$$
$$\times R_{\sigma_1,\lambda_1}\left(\mathscr{P}(z),\ \mathscr{P}'(z),\ \frac{1}{2}g_2\right)\cdot(\lambda_2 a)^{\sigma_2}\cdot e^{\lambda_2 az}\ ,$$

where R_{σ_1,λ_1} are the polynomials of Lemma 3. Hence we obtain

$$G_{t,(\lambda)}\left(\frac{\omega_1}{2}+h_1\omega_1+h_2\omega_2\right)$$
$$(5)\qquad =\sum_{(\sigma)}\frac{t!}{\sigma_0!\sigma_1!\sigma_2!}\cdot\frac{\lambda_0!}{(\lambda_0-\sigma_0)!}\cdot R_{\sigma_1,\lambda_1}(\Gamma_3/\Gamma_0,\ 0,\ \Gamma_4/\Gamma_0)$$
$$\times((1+2h_1)\Gamma_1+h_2\Gamma_2)^{\lambda_0-\sigma_0}(\lambda_2\Gamma_7)^{\sigma_2}\Gamma_5^{\lambda_2+2\lambda_2h_1}\Gamma_6^{2\lambda_2h_2}$$
$$\times\Gamma_0^{-(\lambda_0-\sigma_0+\sigma_2+\lambda_2+2\lambda_2h_1+2\lambda_2h_2)}\ .$$

Let us multiply each equation of (4) by $\Gamma_0^{2L_0}$, then, by (5) and Lemma 3, we see that the coefficients

$$\Gamma_0^{2L_0}G_{t,(\lambda)}\left(\frac{\omega_1}{2}+h_1\omega_1+h_2\omega_2\right)$$

of the new system lie in $Z[x,\theta]$ and have

$$\text{sizes}\leqq c_2L_0\log N\leqq c_2N^3(\log N)^2\ .$$

Therefore, by Lemma 2, the system (4) has non-trivial solution $\varphi(\lambda)\in Z[x,\theta]$ satisfying the condition (3), so that the required result follows.

From Lemma 4, we see that the function $F(z)$ is not identically zero. Let N_1 be the maximal integer such that

$$(6)\qquad \frac{d^t}{dz^t}F\left(\frac{\omega_1}{2}+h_1\omega_1+h_2\omega_2\right)=0\quad\text{for}\quad 0\leqq t<N_1^3\ ,$$
$$0\leqq h_1,\ h_2<[N_1^{2-\mu}]\ .$$

Then there exist integers p, l_1, l_2 such that

$$0\leqq p<(N_1+1)^3\ ,\qquad 0\leqq l_1,\ l_2<[(N_1+1)^{2-\mu}]\ ,$$
$$(7)\qquad \frac{d^p}{dz^p}F\left(\frac{\omega_1}{2}+l_1\omega_1+l_2\omega_2\right)\neq 0\ ,$$
$$\frac{d^t}{dz^t}F\left(\frac{\omega_1}{2}+l_1\omega_1+l_2\omega_2\right)=0\ ,\quad\text{for}\quad 0\leqq t<p\ .$$

We define T_1, H_1, w, ξ by

$$T_1=N_1^3\ ,$$

$$H_1=[N_1^{2-\mu}]\ ,$$

$$w=\frac{\omega_1}{2}+l_1\omega_1+l_2\omega_2\ ,$$

$$\xi=\frac{d^p}{dz^p}F(w)\ .$$

An upper estimate for ξ. Let $\sigma(z)$ be the Weierstrass sigma function associated with $\mathscr{P}(z)$, then both $\sigma(z)$ and $\sigma(z)^2\mathscr{P}(z)$ are entire functions. Hence the function $F_0(z)$ defined by

$$F_0(z)=\sigma(z)^{2L_1}F(z)$$

is also entire. From (7), we see that

$$\frac{d^p}{dz^p}F_0(w)=\sigma(w)^{2L_1}\xi\ . \tag{8}$$

We put

$$r=N_1^{2-(\mu/2)}\ ,\qquad R=N_1^2\ .$$

Let us denote by $|F_0|_\rho$ the maximum of $|F_0(z)|$ on $|z|=\rho$. By Cauchy's estimate and the maximum principle, we have

$$\left|\frac{d^p}{dz^p}F_0(w)\right|\leqq\frac{p!}{(r-|w|)^p}|F_0|_r\leqq p^p|F_0|_r\ . \tag{9}$$

From (6), we see that the number of zeros of $F_0(z)$ in the disc $|z|\leqq r$, counted with multiplicities, is at least $T_1H_1^2$. Hence, by Schwarz lemma (see Lemma 1.3.1 of [7]), we have

$$\begin{aligned}\log|F_0|_r&\leqq\log|F_0|_R-T_1H_1^2\log(R/2r)\\&\leqq\log|F_0|_R-\frac{1}{3}\mu N_1^{7-2\mu}\log N_1\ .\end{aligned} \tag{10}$$

We note here that (3) implies

$$|\varphi(\lambda_0,\lambda_1,\lambda_2)|\leqq\exp\{c_3N^3(\log N)^2\}\quad\text{for all}\quad\lambda_0,\ \lambda_1,\ \lambda_2\ ;$$

and that the function $\sigma(z)$ is entire of order 2. Hence we find that

$$|F_0|_R\leqq\exp(c_4L_1R^{2+\varepsilon})\ , \tag{11}$$

where $\varepsilon>0$ is a sufficiently small number.

From (9), (10) and (11), we obtain

$$\log\left|\frac{d^p}{dz^p}F_0(w)\right|\leqq-\frac{1}{4}\mu N_1^{7-2\mu}\log N_1 . \tag{12}$$

Let $\zeta(z)$ be the Weierstrass zeta function associated with $\mathscr{P}(z)$, then

$$\sigma(w)=(-1)^{l_1+l_2+l_1l_2}\sigma(\omega_1/2)$$
$$\times\exp\{(l_1\zeta(\omega_1/2)+l_2\zeta(\omega_2/2))(\omega_1+l_1\omega_1+l_2\omega_2)\} .$$

By this equality, we have

$$|\sigma(w)|\geqq e^{-R^2} . \tag{13}$$

Thus we obtain from (8), (12) and (13) that

$$\log|\xi|\leqq-\frac{1}{5}\mu N_1^{7-2\mu}\log N_1 . \tag{14}$$

A lower estimate for ξ. From the definition of ξ, we see that

$$\Gamma_0^{2L_{01}}\xi\in \boldsymbol{Z}[x, \theta] ,$$

where $L_{01}=N_1^3[\log N_1]$. Hence, putting $\Gamma=\Gamma_0^{2L_{01}}\xi$, we see that the norm

$$P=P(x)=N_{K/\boldsymbol{Q}(x)}(\Gamma)$$

of $\Gamma\in K$ over $\boldsymbol{Q}(x)$ is a non-zero element of $\boldsymbol{Z}[x]$. If we estimate the size of Γ, then we have

$$s(\Gamma)\leqq c_5L_{01}\log N_1\leqq c_6N_1^3(\log N_1)^2 .$$

By this and Lemma 1, we have that

$$s(P)\leqq c_6\cdot s(\Gamma)\leqq c_7N_1^3(\log N_1)^2 , \tag{15}$$

and that

$$\begin{aligned}\log|P(x)|&\leqq\log|\Gamma|+c_8\cdot s(\Gamma)\\ &\leqq c_9L_{01}+\log|\xi|+c_8\cdot s(\Gamma)\\ &\leqq\log|\xi|+c_{10}N_1^3(\log N_1)^2 .\end{aligned} \tag{16}$$

Since x is of transcendence type$\leqq\kappa$, we have by (15) that

$$\log|P(x)|\geqq-c_{11}s(P)^\kappa\geqq-c_{12}N_1^{3\kappa}(\log N_1)^{2\kappa} . \tag{17}$$

Thus we obtain from (16) and (17) that

$$-c_{13}N_1^{3\kappa}(\log N_1)^{2\kappa} \leqq \log|\xi| \ . \tag{18}$$

CONCLUSION. We obtain from (14) and (18) that

$$-c_{13}N_1^{3\kappa}(\log N_1)^{2\kappa} \leqq -\frac{1}{5}\mu N_1^{7-2\mu} \log N_1 \ .$$

But this inequality contradicts the hypothesis that $\kappa < 2+(1/3)$, and the contradiction proves the theorem.

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