TOKYO J. MATH.
VOL. 9, No. 2, 1986

Hilbert Transforms on One Parameter Groups of Operators

Shiro ISHIKAWA

Keio University

Introduction

In [1], M. Cotlar showed that M. Riesz's theorem could be extended to the case of a measure preserving flow as well as a real line or a circle. In this paper, more generally, we shall consider Hilbert transform on a one parameter group of operators on a complete locally convex space. For this, we define several terms and prepare some lemmas in what follows.

DEFINITION 1. Let $\boldsymbol{R}$ be a real field and let X be a complete locally convex space. Then $\{U_t; t\in \boldsymbol{R}\}$ is said to be a one parameter group of operators on X, if the following conditions are satisfied;

(i) U_t is a continuous linear operator on X for all $t\in\boldsymbol{R}$, and U_0 is an identity operator on X,

(ii) $U_tU_s=U_{t+s}$ for all $t, s\in\boldsymbol{R}$,

(iii) for any $t\in\boldsymbol{R}$ and any $x\in X$, $(U_{t+h}-U_t)x$ converges to 0 as $h\to 0$ in the topology of X (for short, in X).

DEFINITION 2. A continuous linear operator $H_{\varepsilon,N}(0<\varepsilon<N<\infty)$ on X is defined as follows;

$$H_{\varepsilon,N}x=\frac{1}{\pi}\int_{\varepsilon<|t|<N}\frac{U_tx}{t}dt \quad (x\in X)$$

(this integral can be well defined since a mapping $t\in R\to(U_tx)/t\in X$ is continuous on a compact set $\{t\in\boldsymbol{R}; \varepsilon\leqq|t|\leqq N\}$). Also, if $\lim_{\varepsilon\to 0+,N\to\infty}H_{\varepsilon,N}x$ exists in X, we denote it by Hx and call it a Hilbert transform of x. And the domain of H (i.e. $\{x\in X; Hx \text{ exists}\}$) is denoted by $D(H)$.

LEMMA 1. *Let X be a complete locally convex space and let $\{U_t; t\in\boldsymbol{R}\}$ be a one parameter group of operators on X. Let x be any element in X represented by*

Received May 7, 1985

$$x=\frac{1}{2\delta}\int_{-\delta}^{\delta} U_t v dt$$

where $\delta>0$ *and* $v\in X$. *Then* $\lim_{\varepsilon\to 0+} H_{\varepsilon,1}x$ *exists in* X.

PROOF. Note that

$$\begin{aligned}\lim_{h\to 0}\frac{U_h x-U_{-h}x}{2h}&=\lim_{h\to 0}\frac{1}{2h}\Big[U_h\Big(\frac{1}{2\delta}\int_{-\delta}^{\delta} U_t v dt\Big)-U_{-h}\Big(\frac{1}{2\delta}\int_{-\delta}^{+\delta} U_t v dt\Big)\Big]\\&=\lim_{h\to 0}\frac{1}{2h}\Big[\frac{1}{2\delta}\int_{h-\delta}^{h+\delta} U_t v dt-\frac{1}{2\delta}\int_{-h-\delta}^{-h+\delta} U_t v dt\Big]\\&=\lim_{h\to 0}\frac{1}{2\delta}\Big[\frac{1}{2h}\int_{\delta-h}^{\delta+h} U_t v dt-\frac{1}{2h}\int_{-\delta-h}^{-\delta+h} U_t v dt\Big]\\&=\frac{1}{2\delta}(U_\delta v-U_{-\delta}v)\,.\end{aligned}$$

Let q be any semi-norm from the system of semi-norms $\{q\}$ defining the topology of X. From above equality, there exists $\eta>0$ such that

$$q\Big(\frac{U_h x-U_{-h}x}{2h}-\frac{U_\delta v-U_{-\delta}v}{2\delta}\Big)\leqq 1\,,\quad \text{for all } 0<|h|<\eta\,.$$

We have, for any $0<|h|<\eta$,

$$\begin{aligned}q\Big(\frac{U_h x-U_{-h}x}{2h}\Big)&\leqq q\Big(\frac{U_h x-U_{-h}x}{2h}-\frac{U_\delta v-U_{-\delta}v}{2\delta}\Big)+q\Big(\frac{U_\delta v-U_{-\delta}v}{2\delta}\Big)\\&\leqq 1+q\Big(\frac{U_\delta v-U_{-\delta}v}{2\delta}\Big)\,.\end{aligned}$$

Hence we see that, for any ε, ε' such that $0<\varepsilon<\varepsilon'<\eta$,

$$\begin{aligned}q(H_{\varepsilon,1}x-H_{\varepsilon',1}x)&=q\Big(\frac{1}{\pi}\int_{\varepsilon<|t|<\varepsilon'}\frac{U_t x}{t}dt\Big)\\&=q\Big(\frac{2}{\pi}\int_{\varepsilon}^{\varepsilon'}\frac{U_t x-U_{-t}x}{2t}dt\Big)\\&\leqq\frac{2(\varepsilon'-\varepsilon)}{\pi}\Big(1+q\Big(\frac{U_\delta x-U_{-\delta}x}{2\delta}\Big)\Big)\end{aligned}$$

which implies that $\{H_{\varepsilon,1}x\}_{\varepsilon>0}$ is a Cauchy net as $\varepsilon\to 0+$. Therefore, from the completeness of X, $\lim_{\varepsilon\to 0+} H_{\varepsilon,1}x$ exists in X. This completes the proof.

LEMMA 2. *Let* X *be a complete locally convex space and let* $\{U_t\colon t\in \boldsymbol{R}\}$ *be a one parameter group of operators on* X. *Let* x *be any element of*

X represented by

$$x=z-\frac{1}{2T}\int_{-T}^{T}U_s z\,ds\,,$$

where $T>0$ *and* $z\in X$ *and* $\{U_t z\}$ *is supposed to be bounded in* X *uniformly for* $t\in \boldsymbol{R}$. *Then* $\lim_{N\to\infty} H_{1,N}x$ *exists in* X.

PROOF. Since X is complete, it is sufficient to prove that $\{H_{1,N}x\}_{N=1}^{\infty}$ is a Cauchy sequence as $N\to\infty$.

Let q be any semi-norm from the system of semi-norms $\{q\}$ defining the topology of X. Now we get that, for any N, N' such that $0<T<N<N'$,

$$\begin{aligned}
&q(H_{1,N'}x-H_{1,N}x)\\
&\quad=\frac{1}{\pi}q\Big(\int_{N<|t|<N'}\frac{U_t x}{t}dt\Big)\\
&\quad=\frac{1}{\pi}q\Big(\int_{N<|t|<N'}\frac{1}{t}U_t\Big(z-\frac{1}{2T}\int_{-T}^{T}U_s z\,ds\Big)dt\Big)\\
&\quad=\frac{1}{\pi}q\Big(\frac{1}{2T}\int_{-T}^{T}\Big(\int_{N}^{N'}\frac{(U_t-U_{t+s})z}{t}dt+\int_{-N'}^{-N}\frac{(U_t-U_{t+s})z}{t}dt\Big)ds\Big)\\
&\quad\leqq\frac{1}{\pi}q\Big(\frac{1}{2T}\int_{-T}^{T}\Big(\int_{N}^{N'}\frac{(U_t-U_{t+s})z}{t}dt\Big)ds\Big)\\
&\qquad+\frac{1}{\pi}q\Big(\frac{1}{2T}\int_{-T}^{T}\Big(\int_{-N'}^{-N}\frac{(U_t-U_{t+s})z}{t}dt\Big)ds\Big)\\
&\quad=I_1+I_2\,,\quad say\,.
\end{aligned}$$

Since we can, from the boundedness of $\{U_t z\colon t\in\boldsymbol{R}\}$, take $M>0$ such that $q(U_t z)<M$ for all $t\in\boldsymbol{R}$, we see that, for any N, N' such that $0<T\leqq N\leqq N+T\leqq N'$,

$$\begin{aligned}
I_1&=\frac{1}{\pi}q\Big(\frac{1}{2T}\int_{-T}^{T}\Big(\int_{N}^{N'}\frac{U_t z}{t}dt-\int_{N+s}^{N'+s}\frac{U_t z}{t-s}dt\Big)ds\Big)\\
&\leqq\frac{1}{2\pi T}q\Big(\int_{0}^{T}\Big(\int_{N}^{N'}\frac{U_t z}{t}dt-\int_{N+s}^{N'+s}\frac{U_t z}{t-s}dt\Big)ds\Big)\\
&\quad+\frac{1}{2\pi T}q\Big(\int_{-T}^{0}\Big(\int_{N}^{N'}\frac{U_t z}{t}dt-\int_{N+s}^{N'+s}\frac{U_t z}{t-s}dt\Big)ds\Big)\\
&=\frac{1}{2\pi T}q\Big(\int_{0}^{T}\Big(\int_{N}^{N+s}\frac{U_t z}{t}dt+\int_{N+s}^{N'}\Big(\frac{1}{t}-\frac{1}{t-s}\Big)U_t z\,dt-\int_{N'}^{N'+s}\frac{U_t z}{t-s}dt\Big)ds\Big)\\
&\quad+\int_{-T}^{0}\Big(-\int_{N+s}^{N}\frac{U_t z}{t-s}dt+\int_{N}^{N'+s}\Big(\frac{1}{t}-\frac{1}{t-s}\Big)U_t z\,dt+\int_{N'+s}^{N'}\frac{U_t z}{t}dt\Big)ds\Big)
\end{aligned}$$

$$\leqq \frac{M}{2\pi T}\int_0^T\left(\int_N^{N+s}\frac{1}{t}dt+\int_{N+s}^{N'}\left(\frac{1}{t-s}-\frac{1}{t}\right)dt+\int_{N'}^{N'+s}\frac{1}{t-s}dt\right)ds$$
$$+\frac{M}{2\pi T}\int_{-T}^0\left(\int_{N+s}^{N}\frac{1}{t-s}dt+\int_N^{N'+s}\left(\frac{1}{t}-\frac{1}{t-s}\right)dt+\int_{N'+s}^{N'}\frac{1}{t}dt\right)ds$$
$$\leqq \frac{M}{2\pi T}\int_0^T\left(\log\frac{N+s}{N}+\log\frac{(N'-s)(N+s)}{NN'}+\log\frac{N'+s}{N'}\right)ds$$
$$+\frac{M}{2\pi T}\int_{-T}^0\left(\log\frac{N}{N+s}+\log\frac{(N'+s)(N-s)}{NN'}+\log\frac{N'}{N'+s}\right)ds$$
$$(1)\qquad \to 0 \quad (\text{as } N, N'\to\infty)\,.$$

Also we see as in the above estimation of I_1 that, for any N, N' such that $0<T\leqq N\leqq N'\leqq N+T$ $(<N+2T)$,

$$I_1=\frac{1}{\pi}q\left(\frac{1}{2T}\int_{-T}^T\left(\int_N^{N'}\frac{(U_t-U_{t+s})z}{t}dt\right)ds\right)$$
$$\leqq \frac{1}{\pi}q\left(\frac{1}{2T}\int_{-T}^T\left(\int_N^{N+2T}\frac{(U_t-U_{t+s})z}{t}dt\right)ds\right)$$
$$+\frac{1}{\pi}q\left(\frac{1}{2T}\int_{-T}^T\left(\int_{N'}^{N+2T}\frac{(U_t-U_{t+s})z}{t}dt\right)ds\right)$$
$$\to 0 \quad (\text{as } N, N'\to\infty)\,.$$

From this and (1), we see that $I_1\to 0$ as $N, N'\to\infty$. In a similar way, we can also see that $I_2\to 0$ as $N, N'\to\infty$. Hence this implies that $\{H_{1,N}x\}_{N=1}^{\infty}$ is a Cauchy sequence in X. This completes the proof.

§1. Main theorems.

Now we can show the following theorems by Lemma 1 and Lemma 2.

THEOREM 1. *Let X be a complete locally convex space and let $\{U_t: t\in \boldsymbol{R}\}$ be a one parameter group of operators on X. Let x be any element in X represented by*

$$x=\frac{1}{2\delta}\int_{-\delta}^{\delta}U_s\left(z-\frac{1}{2T}\int_{-T}^T U_t z dt\right)ds+v$$

where δ, $T>0$, $z\in X$ and $v\in X$ such that $U_t v=v$ for all $t\in\boldsymbol{R}$ and $\{U_t z\}$ is supposed to be bounded in X uniformly for $t\in\boldsymbol{R}$.

Then $\lim_{\epsilon\to 0+,N\to\infty}H_{\epsilon,N}x$ exists in X (i.e. $x\in D(H)$).

PROOF. Since it is clear that $H_{\epsilon,N}v=0$, we see

$$H_{\varepsilon,N}x=H_{\varepsilon,N}\Big(\frac{1}{2\delta}\int_{-\delta}^{\delta}U_s\Big(z-\frac{1}{2T}\int_{-T}^{T}U_tzdt\Big)ds\Big)$$
$$=H_{\varepsilon,1}\Big(\frac{1}{2\delta}\int_{-\delta}^{\delta}U_s\Big(z-\frac{1}{2T}\int_{-T}^{T}U_tzdt\Big)ds\Big)$$
$$+H_{1,N}\Big(\frac{1}{2\delta}\int_{-\delta}^{\delta}U_szds-\frac{1}{2T}\int_{-T}^{T}U_t\Big(\frac{1}{2\delta}\int_{-\delta}^{\delta}U_szds\Big)dt\Big)$$

which implies, from Lemma 1 and 2, that $\lim_{\varepsilon\to 0+,N\to\infty}H_{\varepsilon,N}x$ exists in X, since $\Big\{U_t\Big(\frac{1}{2\delta}\int_{-\delta}^{\delta}U_szds\Big)\Big\}$ is clearly bounded in X uniformly for $t\in \boldsymbol{R}$.

THEOREM 2. *Let X be a complete locally convex space and let $\{U_t: t\in\boldsymbol{R}\}$ be a one parameter group of operators on X such that the set $\Big\{x\in X: \lim_{T\to\infty}(1/2T)\int_{-T}^{T}U_txdt$ (denoted by $\bar{x}$) exists in X and $\{U_tx\}$ is bounded in X uniformly for $t\in\boldsymbol{R}\Big\}$ is dense in X. Then the domain of H (denoted by $D(H)$) is a dense set in X.*

PROOF. Let u be any element in X and let V be any balanced convex neighborhood of 0 in X. Then, we can find δ, $T>0$ and an x in the dense set in the assumption such that

$$u-x\in\frac{V}{3},\quad x-\frac{1}{2\delta}\int_{-\delta}^{\delta}U_sxds\in\frac{V}{3}$$

and

$$\frac{1}{2\delta}\int_{-\delta}^{\delta}U_s\Big(\frac{1}{2T}\int_{-T}^{T}U_txdt\Big)ds-\bar{x}\in\frac{V}{3}\,.$$

Then, we see that

$$u-\Big[\frac{1}{2\delta}\int_{-\delta}^{\delta}U_s\Big(x-\frac{1}{2T}\int_{-T}^{T}U_txdt\Big)ds+\bar{x}\Big]$$
$$=[u-x]+\Big[x-\frac{1}{2\delta}\int_{-\delta}^{\delta}U_sxds\Big]+\Big[\frac{1}{2\delta}\int_{-\delta}^{\delta}U_s\Big(\frac{1}{2T}\int_{-T}^{T}U_txdt\Big)ds-\bar{x}\Big]$$
$$\in\frac{V}{3}+\frac{V}{3}+\frac{V}{3}=V\,. \tag{2}$$

Also, since we can easily see that $U_t\bar{x}=\bar{x}$ for all $t\in\boldsymbol{R}$, we get, by Theorem 1, that

$$\frac{1}{2\delta}\int_{-\delta}^{\delta}U_s\Big(x-\frac{1}{2T}\int_{-T}^{T}U_txdt\Big)ds+\bar{x}\in D(H)\,.$$

From this and (2), it follows that $D(H)$ is dense in X, since $u(\in X)$ and neighborhood V of 0 in X are arbitrary.

THEOREM 3. *Let X be a complete locally convex space and let $\{U_t: t \in \boldsymbol{R}\}$ be a one parameter group of operators on X such that the set $\left\{x \in X: \lim_{T\to\infty}(1/2T)\int_{-T}^{T} U_t x dt \text{ (denoted by } \bar{x}) \text{ exists in } X \text{ and } \{U_t x\} \text{ is bounded in } X \text{ uniformly for } t \in \boldsymbol{R}\right\}$ is dense in X. Assume that, for any neighbourhood V of 0 in X, there exists a neighborhood W of 0 in X such that*

$$H_{\varepsilon,N} z \in V \quad \text{for all } z \in W \text{ and } 0<\varepsilon<N<\infty .$$

Then, for any $x \in X$, Hx exists in X. Moreover, H is a continuous linear operator on X.

PROOF. Let x be any element in X. It is sufficient to prove that $\{H_{\varepsilon,N}x\}$ is a Cauchy net as $\varepsilon \to 0+$, $N \to \infty$. Let V be any balanced convex neighbourhood of 0 in X. Take a balanced convex neighbourhood W of 0 in X such that

$$H_{\varepsilon,N} z \in \frac{V}{3} \quad \text{for all } z \in W \text{ and } 0<\varepsilon<N<\infty .$$

From Theorem 2, there exist y in $D(H)$ and $0<\varepsilon_0<N_0<\infty$ such that

$$x-y \in W$$

and

$$H_{\varepsilon,N} y - H_{\varepsilon',N'} y \in \frac{V}{3}$$

for all ε, ε', N and N' such that $0<\varepsilon$, $\varepsilon'<\varepsilon_0$ and $N_0<N$, $N'<\infty$. Then we see that, for any ε, ε', N and N' such that $0<\varepsilon$, $\varepsilon'<\varepsilon_0$ and $N_0<N$, $N'<\infty$,

$$\begin{aligned} &H_{\varepsilon,N}x - H_{\varepsilon',N'}x \\ &\quad = (H_{\varepsilon,N}y - H_{\varepsilon',N'}y) + H_{\varepsilon,N}(x-y) - H_{\varepsilon',N'}(x-y) \\ &\quad \in \frac{V}{3} + \frac{V}{3} + \frac{V}{3} = V , \end{aligned}$$

which implies that $\{H_{\varepsilon,N}x\}$ is a Cauchy net as $\varepsilon \to 0+$ and $N \to \infty$. Then we get, from the completeness of X, that Hx exists in X.

Next we shall prove that H is a continuous linear operator on X. Since the linearity of H trivially follows, it is sufficient to prove the continuity of H at 0 in X. Let K be any balanced convex neibourhood of 0 in X. Then by the assumption there exists a balanced convex neibourhood G of 0 in X such that

$$(3) \qquad H_{\varepsilon,N}z \in \frac{K}{2} \quad \text{for all } z \in G \text{ and } \quad 0<\varepsilon<N<\infty .$$

Let u be any element in G. Since $\lim_{\varepsilon\to 0+, N\to\infty} H_{\varepsilon,N}u = Hu$ in X, there exist ε_1 and N_1 such that

$$Hu - H_{\varepsilon_1,N_1}u \in \frac{K}{2} .$$

Hence we see, from this and (3), that

$$\begin{aligned} Hu &= Hu - H_{\varepsilon_1,N_1}u + H_{\varepsilon_1,N_1}u \\ &\in \frac{K}{2} + \frac{K}{2} = K \end{aligned}$$

which implies the continuity of H at 0 in X. Therefore we have that H is a continuous linear operator on X. This completes the proof.

COROLLARY 1. *Let X be a Banach space and let $\{U_t \in \boldsymbol{R}\}$ be a one parameter group of operators on X such that*

(i) *the set* $\left\{x \in X: \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} U_t x dt \text{ exists in } X \text{ and } \{U_t x\} \text{ is bounded in } X \text{ uniformly for } t \in \boldsymbol{R}\right\}$ *is dense in X*

and

(ii) *there exists a $C>0$ such that*

$$\|H_{\varepsilon,N}x\| \leqq C\|x\| \quad \text{for all } x \in X \text{ and } 0<\varepsilon<N<\infty .$$

Then, for any $x \in X$, Hx exists in X. Moreover, H is a continuous linear operator on X.

PROOF. The proof immediately follows from Theorem 3.

COROLLARY 2. *Let X be a Hilbert space and let $\{U_t: t \in \boldsymbol{R}\}$ be a one parameter group of unitary operators on X (i.e. $U_t^* = U_{-t}$ for all $t \in \boldsymbol{R}$). Then the Hilbert transform H is a continuous linear operator on X.*

PROOF. A first part of condition (i) in Corollary 1 is satisfied in a Hilbert space from von Neumann's ergodic theorem. And a second part of condition (i) in Corollary 1 is clearly satisfied since $\|U_t\|=1$ for all $t\in \boldsymbol{R}$. Therefore it is sufficient to prove that the condition (ii) in Corollary 1 is satisfied in the Hilbert space X. This is assured in the following lemma.

LEMMA 3. *X and $\{U_t: t\in \boldsymbol{R}\}$ are defined as in Corollary 2. Then, it follows that*

$$\|H_{\varepsilon,N}x\|\leqq\|x\| \quad \text{for all } x\in X \text{ and } 0<\varepsilon<N<\infty .$$

PROOF. We see, from Stone's Theorem, that

$$\begin{aligned}\|H_{\varepsilon,N}x\|^2&=\left\|\frac{1}{\pi}\int_{\varepsilon<|t|<N}\frac{U_t x}{t}dt\right\|^2\\&=\left\|\frac{1}{\pi}\int_{\varepsilon<|t|<N}\frac{1}{t}\left(\int_{-\infty}^{\infty}e^{it\lambda}dE(\lambda)x\right)dt\right\|^2\\&=\left\|\int_{-\infty}^{\infty}g_{\varepsilon,N}(\lambda)dE(\lambda)x\right\|^2\\&=\int_{-\infty}^{\infty}|g_{\varepsilon,N}(\lambda)|^2d\|E(\lambda)x\|^2 ,\end{aligned}$$

where $\{E(\lambda): \lambda\in \boldsymbol{R}\}$ is a spectral family of the one parameter group of unitary operators $\{U_t: t\in \boldsymbol{R}\}$ and

$$g_{\varepsilon,N}(\lambda)=\frac{1}{\pi}\int_{\varepsilon<|t|<N}\frac{e^{i\lambda t}}{t}dt .$$

Since we can easily show that $|g_{\varepsilon,N}(\lambda)|\leqq 1$ for all $\lambda\in \boldsymbol{R}$ and $0<\varepsilon<N<\infty$, we see that

$$\begin{aligned}\|H_{\varepsilon,N}x\|^2&\leqq\int_{-\infty}^{\infty}d\|E(\lambda)x\|^2\\&\leqq\|x\|^2 .\end{aligned}$$

for all $x\in X$ and $0<\varepsilon<N<\infty$. Hence this completes the proof.

§2. Application.

Let (Ω, B, μ) be a σ-finite measure space and let $L^p(\Omega)$ $(1\leqq p<\infty)$ be the set of all p-order integrable functions on Ω with norm $\|\cdot\|_p$. We define $\{T_t: t\in \boldsymbol{R}\}$ as a measure preserving flow on Ω, that is,

(i) for any $t\in \boldsymbol{R}$, T_t is a measure preserving transformation on Ω and T_0 is an identity on Ω,

(ii) $T_tT_s=T_{t+s}$ for all $t, s\in \boldsymbol{R}$,

(iii) a mapping $(t, \omega) \in \boldsymbol{R} \times \Omega \to T_t\omega \in \Omega$ is measurable.
Now we can define a one parameter group of operators $\{U_t: t \in \boldsymbol{R}\}$ on $L^p(\Omega)$ such that for all $f \in L^p(\Omega)$.

$$(U_t f)(\omega) = f(T_t\omega) \quad \text{for all } t \in \boldsymbol{R} \text{ and } \omega \in \Omega .$$

It is well known that $\{U_t: t \in \boldsymbol{R}\}$ satisfies all conditions in Definition 1 and that $U_t^* = U_{-t}$ for all $t \in \boldsymbol{R}$ in $L^2(\Omega)$. Under these preparations, we see the following proposition in [2].

PROPOSITION 1. *There exists a constant $C>0$ (independent of ε, N and λ) such that*

$$\mu\left\{\omega \in \Omega: \left|\frac{1}{\pi}\int_{\varepsilon<|t|<N} \frac{f(T_t\omega)}{t}dt\right| > \lambda\right\} \leqq \frac{C}{\lambda}\|f\|_1$$

for all $0<\varepsilon<N<\infty$, $0<\lambda<\infty$ and all $f \in L^1(\Omega)$.

PROOF. See [2].

Now we have the following generalized M. Riesz's theorem which was first proved by M. Cotlar [1]. Our proof is based on Corollary 1.

THEOREM 4. *Let (Ω, B, μ) be a σ-finite measure space and let $\{T_t: t \in \boldsymbol{R}\}$ be a measure preserving flow on Ω. Let p be any real such that $1<p<\infty$.*
Then, it follows that

(i) *for any $f \in L^p(\Omega)$, $\lim_{\varepsilon\to 0+, N\to\infty} (1/\pi)\int_{\varepsilon<|t|<N} (f(T_t\omega)/t)dt$ (denoted by Hf) exists in the norm topology of $L^p(\Omega)$,*

(ii) *H is a continuous linear operator on $L^p(\Omega)$.*

PROOF. As the previous arguments, we define a one parameter group of operators $\{U_t; t \in \boldsymbol{R}\}$ on $L^p(\Omega)$ such that, for any $f \in L^p(\Omega)$,

$$(U_t f)(\omega) = f(T_t\omega) \quad \text{for all } t \in \boldsymbol{R} \text{ and } \omega \in \Omega .$$

First we see, from von Neumann's and Yoshida's ergodic theorem, that the first part of condition (i) in Corollary 1 is satisfied, that is, $\lim_{T\to\infty} (1/2T)\int_{-T}^{T} U_t x dt$ exists in X for all $x \in X$. And the second part of condition (i) in Corollary 1 is clearly satisfied since $\|U_t f\|_p = \|f\|_p$ for all $f \in L^p(\Omega)$. Therefore it is sufficient to show that the condition (ii) in Corollary 1 is satisfied.

By Proposition 1 and Lemma 3, there exists a constant $C>0$ such that, for any $0<\varepsilon<N<\infty$,

$$\mu\{\omega\in\Omega: |H_{\varepsilon,N}f|>\lambda\}\leqq\frac{C}{\lambda}\|f\|_1 \quad \text{for all } f\in L^1(\Omega)$$

and

$$\|H_{\varepsilon,N}f\|_2\leqq\|f\|_2 \quad \text{for all } f\in L^2(\Omega)\ .$$

This implies, from Marcinkiewicz's interpolation theorem, that, for any $1<p\leqq 2$, there exists a constant $C_p>0$ such that

$$(4)\qquad \|H_{\varepsilon,N}f\|_p\leqq C_p\|f\|_p \quad \text{for all } 0<\varepsilon<N<\infty \text{ and } f\in L^p(\Omega)\ .$$

In the case of $2\leqq p<\infty$, put $q=p/(p-1)$. Then, we see that, for any $0<\varepsilon<N<\infty$, $f\in L^p(\Omega)$ and $g\in L^q(\Omega)$,

$$\begin{aligned}\int_\Omega H_{\varepsilon,N}f\cdot gd\mu&=\int_\Omega\left(\frac{1}{\pi}\int_{\varepsilon<|t|<N}\frac{U_tf}{t}dt\right)gd\mu\\&=\int_\Omega\left(\frac{1}{\pi}\int_{\varepsilon<|t|<N}\frac{f(T_t\omega)}{t}dt\right)\cdot g(\omega)d\mu\\&=\int_{\varepsilon<|t|<N}\frac{1}{\pi t}\left(\int_\Omega f(T_t\omega)g(\omega)d\mu\right)dt\\&=\int_{\varepsilon<|t|<N}\frac{1}{\pi t}\left(\int_\Omega f(\omega)g(T_{-t}\omega)d\mu\right)dt\\&=-\int_\Omega f\left(\frac{1}{\pi}\int_{\varepsilon<|t|<T}\frac{U_tg}{t}dt\right)d\mu\\&=-\int_\Omega f\cdot H_{\varepsilon,N}gd\mu\end{aligned}$$

which implies, by Hörder's inequality and (4), that

$$\begin{aligned}\|H_{\varepsilon,N}f\|_p&=\sup\left\{\left|\int_\Omega H_{\varepsilon,N}f\cdot gd\mu\right|: \|g\|_q\leqq 1\right\}\\&=\sup\left\{\left|\int_\Omega f\cdot H_{\varepsilon,N}gd\mu\right|: \|g\|_q\leqq 1\right\}\\&\leqq\sup\{\|f\|_p\|H_{\varepsilon,N}g\|_q: \|g\|_q\leqq 1\}\\&\leqq C_q\|f\|_p=C_{p/(p-1)}\|f\|_p\ .\end{aligned}$$

It follows, from this and (4), that, for any p such that $1<p<\infty$, there exists $C_p'>0$ such that

$$\|H_{\varepsilon,N}f\|_p\leqq C_p'\|f\|_p \quad \text{for all } 0<\varepsilon<N<\infty \text{ and } f\in L^p(\Omega)\ .$$

This shows that the condition (ii) in Corollary 1 is satisfied. Therefore, by Corollary 1, the proof is completed.

The author wishes to express his sincere thanks to Professor S. Koizumi of Keio university and the referee.

References

[1] M. COTLAR, A unified theory of Hilbert transforms and ergodic theorems, Rev. Mat. Cuyama, **1** (1955), 105–167.
[2] K. PETERSEN, Ergodic Theory, Cambridge University Press, Cambridge, 1983.
[3] K. PETERSEN, Another proof of the existence of the ergodic Hilbert transform, Proc. Amer. Math. Soc., **88** (1983), 39–43.
[4] K. YOSHIDA, Functional Analysis, Springer-Verlag Berlin-Heidelberg-New York, 1971.

Present Address:
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE AND TECHNOLOGY
KEIO UNIVERSITY
HIYOSHI, KOHOKU-KU
YOKOHAMA 223