TOKYO J. MATH.
VOL. 9, NO. 2, 1986

Artin's L-functions and Gassmann Equivalence

Kiyoshi NAGATA

Sophia University

(Communicated by Y. Kawada)

Introduction

R. Perlis, in [3], showed the following theorem.

Theorem. (Cassels & Fröhlich [5, p. 363 ex. 6.4])
Let L be a finite Galois extension of $\boldsymbol{Q}$, let $G=\mathrm{Gal}(L/\boldsymbol{Q})$, and let E and E' be subfields of L corresponding to the subgroups H and H' of G respectively. Then the following conditions are equivalent:
(1) H and H' are Gassmann equivalent in G.
(2) $\zeta_E(s)=\zeta_{E'}(s)$.
(3) The same primes p are ramified in E as in E', and for the non-ramified p the decomposition of p in E and E' is the same.

We shall extend this theorem in case that the base field is not necessarily $\boldsymbol{Q}$. In this case the condition (2) is not sufficient for (1). So we will replace ζ-function by Artin's L-functions for some characters.

T. Funakura, in [4], constructed the isomorphism Φ from the additive group $\mathrm{Ch}(\mathfrak{G})$ to $\mathfrak{L}$ where $\mathrm{Ch}(\mathfrak{G})$ is the character group of $\mathfrak{G}=G(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$ and $\mathfrak{L}=\{L(s, \psi, E/\boldsymbol{Q}): E$ is a subfield of $\bar{\boldsymbol{Q}}$ corresponding to an open normal subgroup $\mathfrak{Y}$ s.t. $\psi\in\mathrm{Ch}(\mathfrak{G}/\mathfrak{Y})\}$. And he showed the equivalent relation between (1) and (2) as a corollary of his theorem. To show that Φ is monomorphic, he used the fact that the L-functions for the irreducible characters of $\mathrm{Gal}(E/\boldsymbol{Q})$ are multiplicativelly independent. We shall show the equivalence of (1) and (2) if the L-functions for the irreducible characters of $\mathrm{Gal}(E/k)$ are multiplicativelly independent.

§1. Preliminaries.

Now let k, K, K' be finite algebraic number fields where K, K' contain k. For a relative normal algebraic extension N over k which contains both K and K', we put $G=\mathrm{Gal}(N/k)$, $H=\mathrm{Gal}(N/K)$, $H'=\mathrm{Gal}(N/K')$. Then

Received June 4, 1985

H and H' are said to be Gassmann equivalent in G when

$$|H \cap c^G| = |H' \cap c^G|$$

for every conjugacy class $c^G = \{\sigma^{-1}c\sigma \mid \sigma \in G\}$ in G. In case $k = \boldsymbol{Q}$, R. Perlis in [3] showed that the condition $\zeta_K(s) = \zeta_{K'}(s)$ holds if and only if H and H' are Gassmann equivalent and when these conditions hold then $|K: \boldsymbol{Q}| = |K': \boldsymbol{Q}|$, $d(K) = d(K')$, the number of real (resp. complex) infinite primes of K and K' coincide. Our purpose in this paper is to show that H and H' are Gassmann equivalent if and only if $L(s, \psi, N/K) = L(s, \psi', N/K')$ for some characters ψ, ψ' even if $k \neq \boldsymbol{Q}$.

Before we start on it, we will refer to the relation between the Gassmann equivalence and the decompositions of ideals of k in K, K' as R. Perlis did.

LEMMA 1. *H and H' are Gassmann equivalent in G if and only if*

$$\textit{coset type } [G \bmod (Z, H)] = \textit{coset type } [G \bmod (Z, H')]$$

for every cyclic subgroup Z of G.

PROOF. See [3], Lemma 1 on p. 344.

Let $\mathfrak{p}$ be a prime of k, we say $A_\mathfrak{p} = (f_1, \cdots, f_g) \in \boldsymbol{N}^g$ the splitting type of $\mathfrak{p}$ in K, if $\mathfrak{p} = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_g^{e_g}$ in K, $N_{K/k}\mathfrak{P}_i = \mathfrak{p}^{f_i}$, $f_i \leqq f_{i+1}$ $(i = 1, \cdots, g-1)$. And for $A = (f_1, \cdots, f_g)$, where g, f_i $(i = 1, \cdots, g)$ are positive integers such that $f_i \leqq f_{i+1}$ $(i = 1, \cdots, g-1)$, we put

$$P_{K/k}(A) = \{\mathfrak{p} \mid \mathfrak{p} \text{ is a prime of } k \text{ s.t. } A = A_\mathfrak{p}\}\,.$$

We note that there are only finite many A for which $P_{K/k}(A)$ is not empty. If $\mathfrak{p}$ is unramified in N, $\mathfrak{p}$ has the same splitting type in both K and K' if and only if

$$\text{coset type } [G \bmod (Z, H)] = \text{coset type } [G \bmod (Z, H')],$$
$$|H| = |H'|, \text{ for the decomposition subgroup } Z \text{ of } \mathfrak{p} \text{ in } G.$$

The notation

$$P_{K/k}(A) \doteq P_{K'/k}(A)$$

will be used to indicate that these two sets differ by at most a finite number of elements.

PROPOSITION 1. *The following conditions are equivalent.*

(1) *$P_{K/k}(A) \doteq P_{K'/k}(A)$ for every A.*

(2) *coset type $[G \bmod (Z, H)]$ = coset type $[G \bmod (Z, H')]$ for every cyclic subgroup Z of G.*

(3) *H and H' are Gassmann equivalent in G.*

Furthermore, when these conditions hold then $[K:\boldsymbol{Q}]=[K':\boldsymbol{Q}]$, the two fields determine same normal closure and the same normal core over k, the number of real (resp. complex) infinite primes of K, K' coincide, and the unit groups are isomorphic $\mathfrak{U}_K \cong \mathfrak{U}_{K'}$.

We can prove this proposition easily by improving what R. Perlis ([3]) did in the proof of Theorem 1.

§2. Main results.

Now we return to L-functions. First we introduce some notations for characters. From now on, χ, ψ and ψ' always denotes charaters of G, H and H' respectively and $\chi|_H$ denotes the restriction of χ to H. $(\ ,\ )_G$ stands for the inner product in G. Let $\{\chi_1, \cdots, \chi_n\}$, $\{\psi_1, \cdots \psi_m\}$, $\{\psi'_1, \cdots \psi'_{m'}\}$ be the sets of normalized irreducible characters of G, H and H' respectively, where χ_1, ψ_1, ψ'_1 are principal characters of each groups.

Let $D(s, \chi, N/k)$ be a function on the complex plain $\boldsymbol{C}$ defined for every Galois extension N/k and for every character of $\mathrm{Gal}(N/k)$ satisfying the following conditions:

$D(s, \chi+\chi', N/k) = D(s, \chi, N/k) \cdot D(s, \chi', N/k)$ for characters χ, χ' of $\mathrm{Gal}(N/k)$,

$D(s, \chi_\psi, N/k) = D(s, \psi, N/K)$ for every intermidiate field K and for every character ψ of $\mathrm{Gal}(N/K)$. We have three examples for D i.e., $A(\chi, N/k)$, $L(s, \chi, N/k)$, $\xi(s, \chi, N/k)$.

LEMMA 2. *The following equation holds.*

$$D(s, \chi_j|_H, N/K) = \prod_{l=1}^{n} D(s, \chi_l, N/k)^{(\chi_l|_H, \chi_j|_H)_H}$$

for $1 \leqq j \leqq n$.

PROOF. If we decompose χ_{ψ_i} $(1 \leqq i \leqq m)$ into the sum of irreducible characters, say

$$\chi_{\psi_i} = \sum_{j=1}^{n} a_{ij} \chi_j \text{ with non-negative integers } a_{ij}\ ,$$

then we have

$$\chi_j|_H = \sum_{i=1}^{m} a_{ij} \psi_i \qquad (1 \leqq j \leqq n)\ ,$$

by Frobenius reciprocity low for characters. And

$$\begin{aligned}D(s, \chi_j|_H, N/K) &= D(s, \chi_{\chi_j|_H}, N/k)\\ &= D\Big(s, \sum_{i=1}^{m} a_{ij}\chi_{\psi_i}, N/k\Big)\\ &= D\Big(s, \sum_{i=1}^{m} a_{ij}\sum_{l=1}^{n} a_{il}\chi_l, N/k\Big)\\ &= D\Big(s, \sum_{l=1}^{n}\Big(\sum_{i=1}^{m} a_{ij}a_{il}\Big)\chi_l, N/k\Big)\\ &= \prod_{l=1}^{n} D(s, \chi_l, N/k)^{\sum_{i=1}^{m} a_{ij}a_{il}}\ .\end{aligned}$$

The exponents

$$\begin{aligned}\sum_{i=1}^{m} a_{il}a_{ij} &= \sum_{i_1=1}^{m}\sum_{i_2=1}^{m} a_{i_1 l}a_{i_2 j}(\psi_{i_1}, \psi_{i_2})_H\\ &= \Big(\sum_{i_1=1}^{m} a_{i_1 l}\psi_{i_1}, \sum_{i_2=1}^{m} a_{i_2 j}\psi_{i_2}\Big)_H\\ &= (\chi_l|_H, \chi_j|_H)_H\ .\end{aligned}$$

This concludes the proof.

We see that $(\chi_l|_H, \chi_j|_H)_H = (\chi_l|_{H'}, \chi_j|_{H'})_{H'}$ if and only if $\sum_{c^G}\chi_l(x)\chi_j(x^{-1})(|H'|\cdot|H\cap c^G| - |H|\cdot|H'\cap c^G|) = 0$, because

$$\begin{aligned}|H|(\chi_l|_H, \chi_j|_H)_H &= \sum_{x\in H}\chi_l(x)\chi_j(x^{-1})\\ &= \sum_{c^G}\sum_{x\in H\cap c^G}\chi_l(x)\chi_j(x^{-1})\\ &= \sum_{c^G}\chi_l(x)\chi_j(x^{-1})|H\cap c^G|,\end{aligned}$$

where c^G ranges over all conjugacy class, and similarly

$$|H'|(\chi_l|_{H'}, \chi_j|_{H'})_{H'} = \sum_{c^G}\chi_l(x)\chi_j(x^{-1})|H'\cap c^G|\ .$$

Since the Gassmann equivalence of H and H' implies $|H| = |H'|$ and $|H\cap c^G| = |H'\cap c^G|$ for every conjugacy class c^G and a character χ of G is decomposed into a sum of χ_j $(1\leq j\leq n)$ with integer coefficiants, we have following lemma.

LEMMA 3. *If H and H' are Gassmann equivalent, then* $(\chi_l|_H, \chi|_H)_H = (\chi_l|_{H'}, \chi|_{H'})_{H'}$ $(1\leq l\leq n)$ *for every character χ of G.*

PROPOSITION 2. *The following conditions are equivalent.*

(1) *H and H' are Gassmann equivalent.*

(2) $(\chi_l|_H, \chi|_H)_H = (\chi_l|_{H'}, \chi|_{H'})_{H'}$ $(1\leq l\leq n)$ *for every character χ of G.*

(3) $(\chi_l|_H, \psi_1)_H = (\chi_l|_{H'}, \psi_1')_{H'}$ $(1\leq l\leq n)$.

(4) $\chi_{\psi_1}=\chi_{\psi'_1}$.

PROOF. (1)⇒(2) Lemma 3.
(2)⇒(3) Since $\chi_1|_H=\psi_1$, $\chi_1|_{H'}=\psi'_1$, this is clear.
(3)⇒(4) As we have done in the proof of Lemma 2, we put

$$\chi_{\psi_i}=\sum_{j=1}^{n} a_{ij}\chi_j \qquad (1\leqq i\leqq m)\ ,$$

$$\chi_{\psi'_i}=\sum_{j=1}^{n} a'_{ij}\chi_j \qquad (1\leqq i\leqq m')\ .$$

Then we have

$$\chi_j|_H=\sum_{i=1}^{m} a_{ij}\psi_i\ ,$$

$$\chi_j|_{H'}=\sum_{i=1}^{m'} a'_{ij}\psi'_i\ , \qquad 1\leqq j\leqq n\ .$$

We calculate $(\chi_{\psi_1}|_H, \psi_1)_H$ and $(\chi_{\psi_1}|_{H'}, \psi'_1)_{H'}$ as follows.

$$\begin{aligned}(\chi_{\psi_1}|_H, \psi_1)_H &=\Big(\sum_{j=1}^{n} a_{1j}\chi_j|_H, \psi_1\Big)_H \\ &=\Big(\sum_{j=1}^{n}\sum_{i=1}^{m} a_{1j}a_{ij}\psi_i, \psi_1\Big)_H \\ &=\sum_{j=1}^{n} a_{1j}^2\ ,\end{aligned}$$

$$\begin{aligned}(\chi_{\psi_1}|_{H'}, \psi'_1)_{H'} &=\Big(\sum_{j=1}^{n} a_{1j}\chi_j|_{H'}, \psi'_1\Big)_{H'} \\ &=\Big(\sum_{j=1}^{n}\sum_{i=1}^{m} a_{1j}a'_{ij}\psi'_i, \psi'_1\Big)_{H'} \\ &=\sum_{j=1}^{n} a_{1j}a'_{1j}\ .\end{aligned}$$

The condition (3) implies $(\chi_{\psi_1}|_H, \psi_1)_H=(\chi_{\psi_1}|_{H'}, \psi'_1)_{H'}$. Therefore

$$\sum_{j=1}^{n}(a_{1j}^2-a_{1j}a'_{1j})=0\ . \qquad \text{(A)}$$

Similarly,

$$\sum_{j=1}^{n}(a'^2_{1j}-a_{1j}a'_{1j})=0\ . \qquad \text{(B)}$$

Making sums of (A) and (B), it holds that

$$\sum_{j=1}^{n}(a_{1j}-a'_{1j})^2=0 ,$$

so $a_{1j}=a'_{1j}$ for every j. This means $\chi_{\psi_1}=\chi_{\psi'_1}$.

(4)$\Rightarrow$(1) For every element σ of G, $\chi_{\psi_1}(\sigma)=\chi_{\psi'_1}(\sigma)$. And

$$\chi_{\psi_1}(\sigma)=\frac{1}{|H|}\sum_{\tau\in G}\psi_1(\tau^{-1}\sigma\tau)$$
$$=\frac{1}{|H|}|H\cap\sigma^G|\cdot|S_G(\sigma)| ,$$

where $S_G(\sigma)$ denotes the stabilzer of σ in G. Thus, $(1/|H|)|H\cap\sigma^G|=(1/|H'|)|H'\cap\sigma^G|$, for every conjugacy class σ^G. If we take the unity element of G as σ, we have $|H|=|H'|$. So we have $|H\cap\sigma^G|=|H'\cap\sigma^G|$, for every σ^G. This completes the proof.

COROLLARY 1. *If $D(s, \chi_l, N/k)$ $(1\leqq l\leqq n)$ are multiplicatively independent, the following conditions are equivalent.*

(1) *H and H' are Gassmann equivalent.*

(2) *$D(s, \chi|_H, N/K)=D(s, \chi|_{H'}, N/K')$, for every character χ of G.*

(3) *$D(s, \psi_1, N/K)=D(s, \psi'_1, N/K')$.*

PROOF. (1)$\Rightarrow$(2) Lemma 2 and Proposition 2.

(2)$\Rightarrow$(3) Trivial.

(3)$\Rightarrow$(1) For $D(s, \psi_1, N/K)=\prod_{l=1}^{n} D(s, \chi_l, N/k)^{(\chi_l|_H,\psi_1)_H}$ and $D(s, \psi'_1, N/K')=\prod_{l=1}^{n} D(s, \chi_l, N/k)^{(\chi_l|_{H'},\psi'_1)_{H'}}$, we have $(\chi_l|_H, \psi_1)_H=(\chi_l|_{H'}, \psi'_1)_{H'}$ $(1\leqq l\leqq n)$. Then H and H' are Gassmann equivalent.

Especially when $k=\boldsymbol{Q}$, it is known that $L(s, \chi_l, N/k)$ $(1\leqq l\leqq n)$ are multiplicatively independent. So we have the following corollary.

COROLLARY 2. *If $k=\boldsymbol{Q}$, the following conditions are equivalent.*

(1) *H and H' are Gassmann equivalent.*

(2) *$L(s, \chi|_H, N/K)=L(s, \chi|_{H'}, N/K')$, for every character χ of G.*

(3) *$\zeta_K(s)=\zeta_{K'}(s)$.*

T. Funakura, in [4], have had the same result.

COROLLARY 3. *Let $L/\boldsymbol{Q}$ be an abelian extention. If $\zeta_K(s)=\zeta_{K'}(s)$, then the following conditions are satisfied.*

(1) *$K\cap L=K'\cap L$, $\mathrm{Gal}(KL/K)\cong\mathrm{Gal}(K'L/K')\cong\mathrm{Gal}(L/k)$, where $k=K\cap L=K'\cap L$.*

(2) *$\{L(s, \psi, KL/K)\}_{\psi\in\widehat{\mathrm{Gal}(KL/K)}}=\{L(s, \psi', K'L/K')\}_{\psi'\in\widehat{\mathrm{Gal}(K'L/K')}}$, where $\hat{A}=\{\psi: A\to \boldsymbol{C}$ homo.$\}$ for an abelian group A.*

PROOF. (1) For proving $\mathrm{Gal}(KL/K)\cong G(K'L/K')$, it is sufficient to show that $K\cap L=K'\cap L$. Since $L/\boldsymbol{Q}$ is abelian, $K\cap L/\boldsymbol{Q}$ is normal. So $K\cap L$ is included in the normal core of K, which is equal to the normal core of K' from Proposition 1. Thus we have $K\cap L\subset K'\cap L$. And we also have $K'\cap L\subset K\cap L$. Hence $K\cap L=K'\cap L$.

(2) Let $N/\boldsymbol{Q}$ be a normal extension such that $N\supset K\cdot K'\cdot L$. A character ψ of $\mathrm{Gal}(KL/K)$ is regarded as a character of $\mathrm{Gal}(L/k)$ and also as the restriction of a character χ of $\mathrm{Gal}(L/\boldsymbol{Q})$, for $\mathrm{Gal}(L/\boldsymbol{Q})$ is abelian. Extending ψ, χ to characters of $G(N/K)$, $\mathrm{Gal}(N/\boldsymbol{Q})$ respectively in a general way, we have $\chi|_H=\psi$. And noting that $L(s, \chi|_H, N/K)=L(s, \chi|_{H'}, N/K')$ for every character of $\mathrm{Gal}(N/\boldsymbol{Q})$, we have

$$\begin{aligned}\{L(s, \psi)\}_{\psi\in\widehat{\mathrm{Gal}(KL/K)}} &=\{L(s, \psi, N/K)|\psi: H\to \boldsymbol{C}^\times \text{ homo.}, \ \psi|_{\bar{H}}=1\}\\ &=\{L(s, \chi|_H, N/K)|\chi: G\to \boldsymbol{C}^\times \text{ homo.}, \ \chi_{\bar{G}}=1\}\\ &=\{L(s, \chi|_{H'}, N/K')|\chi: G\to \boldsymbol{C}^\times \text{ homo.}, \ \chi|_{\bar{G}}=1\}\\ &=\{L(s, \psi', N/K')|\psi': H'\to \boldsymbol{C}^\times \text{ homo.}, \ \psi'|_{\bar{H}'}=1\}\\ &=\{L(s, \psi')\}_{\psi'\in\widehat{\mathrm{Gal}(K'L/K')}},\end{aligned}$$

where $H=\mathrm{Gal}(N/K)$, $H'=\mathrm{Gal}(N/K')$, $\bar{G}=\mathrm{Gal}(N/L)$, $G=\mathrm{Gal}(N/\boldsymbol{Q})$, $\bar{H}=\mathrm{Gal}(N/KL)$ and $\bar{H}'=\mathrm{Gal}(N/K'L)$.

In case $k\neq\boldsymbol{Q}$, $L(s, \chi_l, N/k)$ $(1\leqq l\leqq n)$ are not always multiplicatively independent. But we can show the following theorem considering poles and zeros of them at $s=1$.

THEOREM. *The following conditions are equivalent.*

(1) *H and H' are Gassmann equivalent.*

(2) *$P_{K/k}(A)\doteq P_{K'/k}(A)$ for every $A=(f_1, \cdots, f_g)$.*

(3) *$L(s, \chi|_H, N/K)=L(s, \chi|_{H'}, N/K')$ for every character χ of G.*

(4) $\begin{cases} L(s, \chi_{\psi_1}|_H, N/K)=L(s, \chi_{\psi_1}|_{H'}, N/K'),\\ L(s, \chi_{\psi_1'}|_H, N/K)=L(s, \chi_{\psi_1'}|_{H'}, N/K').\end{cases}$

Furthermore, when these conditions hold then $[K:\boldsymbol{Q}]=[K':\boldsymbol{Q}]$, the number of real (resp. complex) infinite primes of K, K' coincide, the two fields determine the same normal closure and the same normal core over k, the unit groups isomorphic $\mathfrak{U}_K\cong\mathfrak{U}_{K'}$. And it also holds $\Gamma(s, \chi|_H, N/K)=\Gamma(s, \chi_{H'}, N/K')$, $\xi(s, \chi|_H, N/K)=\xi(s, \chi|_{H'}, N/K')$, $A(\chi|_H, N/K)=A(\chi|_{H'}, N/K')$ and $N\mathfrak{f}(\chi|_H, N/K)=N\mathfrak{f}(\chi|_{H'}, N/K')$ for every character χ of G.

PROOF. (1)$\Leftrightarrow$(2) Proposition 1.
(1)$\Rightarrow$(3) Corollary 1 of Proposition 2.
(3)$\Rightarrow$(4) Trivial.

(4)$\Rightarrow$(1) From the Lemma 2, $L(s, \chi|_H, N/K) = \prod_{l=1}^{n} L(s, \chi_l, N/k)^{(\chi_l|_H, \chi|_H)_H}$ for any character χ of G. If $l \neq 1$ $L(s, \chi_l, N/k)$ is expressed as a product of Hecke's L-functions with non-principal character. Then $L(s, \chi_l, N/k)$ has no poles and no zeros at $s=1$. If $l=1$, $L(s, \chi_1, N/k) = \zeta_k(s)$ has a simple pole at $s=1$. Hence $L(s, \chi|_H, N/K)$ has a pole of order $(\chi|_H, \psi_1)_H$ at $s=1$. Similarly $L(s, \chi|_{H'}, N/K')$ has a pole of order $(\chi|_{H'}, \psi_1')_{H'}$ at $s=1$. Thus we have

$$(\chi_{\psi_1}|_H, \psi_1)_H = (\chi_{\psi_1}|_{H'}, \psi_1')_{H'}$$

and

$$(\chi_{\psi_1'}|_H, \psi_1)_H = (\chi_{\psi_1'}|_{H'}, \psi_1')_{H'} ,$$

with which we have shown (1) in the proof of Proposition 2.

The rest is due to Proposition 1 or Corollary 1 of Proposition 2.

References

[1] E. Artin, Zur Theorie der L-Reihen mit allgemeinen Gruppencharakteren, Hamburger Abh., **8** (1930), 292–306.

[2] F. Gassmann, Bemerkungen zu der vorstehenden Arbeit von Hurwitz, Math. Z., **25** (1926), 124–143.

[3] R. Perlis, On the equation $\zeta_K(s)=\zeta_{K'}(s)$, J. Number Theory, **9** (1977), 342–360.

[4] T. Funakura, Characters and Artin's L-functions, Comment. Math. Univ. St. Paul., XXVII-2 (1978), 185–197.

[5] J. W. S. Cassels and A. eds. Fröhlich, Algebraic Number Theory, Academic Press, London, New York, 1967.

Present Address:
Department of Mathematics
Sophia University
Kioicho, Chiyoda-ku, Tokyo 102