Tokyo J. Math.
Vol. 10, No. 1, 1987

Generalized Hilbert Transforms in Tempered Distributions

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Introduction

A Hilbert transform H of a function f on real field $\boldsymbol{R}$ is defined as:

$$Hf(x)=\lim_{\substack{\varepsilon\to 0+\\ N\to\infty}} H_{\varepsilon,N}f(x)=\lim_{\substack{\varepsilon\to 0+\\ N\to\infty}}\frac{1}{\pi}\int_{\varepsilon<|t|<N}\frac{f(x-t)}{t}dt \qquad (x\in\boldsymbol{R}).$$

The Hilbert transform H plays an important role in Fourier analysis. The properties of Hilbert transforms in the following Proposition are fundamental.

Let $L^p(\boldsymbol{R})$ be the class of all measurable functions f on $\boldsymbol{R}$ for which

$$\|f\|_{L^p}=\left(\int_{-\infty}^{\infty}|f(x)|^p dt\right)^{1/p}<\infty .$$

PROPOSITION. *Let p be a real number such that $1<p<\infty$. Then*

(i) [*existence*] *for any $f\in L^p(\boldsymbol{R})$,*

$$Hf(x)=\lim_{\substack{\varepsilon\to 0+\\ N\to\infty}} H_{\varepsilon,N}f(x)$$

exists in the topology of $L^p(\boldsymbol{R})$,

(ii) [*boundedness*] *there exists a constant $C>0$ (independent of ε, N and f) such that*

$$\|Hf\|_{L^p}\leqq C\|f\|_{L^p} \quad (\|H_{\varepsilon,N}f\|_{L^p}\leqq C\|f\|_{L^p}) \quad \text{for } all \quad f\in L^p(\boldsymbol{R}),$$

(iii) [*inversion formula*]

$$H(H(f))=-f \quad \text{for } all \quad f\in L^p(\boldsymbol{R}),$$

(iv) [*signum rule*]

$$(Hf)^{\wedge}=-i\,\mathrm{sgn}(x)\hat{f} \quad \text{for } all \quad f\in L^2(\boldsymbol{R}),$$

Received July 30, 1986

where $\hat{f}$ is a Fourier transform of f.

Many mathematicians have tried to define the Hilbert transforms naturally on more general space (see, for example, [2], [3], [4], [7], [8], [9], [11], [12], [14] and [16]).

S. Koizumi ([11], [12]) introduced a generalized Hilbert transform H for $f \in W^2(\boldsymbol{R})$ through

$$Hf(x)=\lim_{\varepsilon\to 0+}\frac{x+i}{\pi}\int_{\varepsilon<|t|}\frac{f(x-t)}{t(x-t+i)}dt$$

where $W^2(\boldsymbol{R})$ (often called Wiener's class) is the class of all measurable functions f for which $f(x)/(1+|x|)\in L^2(\boldsymbol{R})$. And he obtained the similar results in the above Proposition for $W^2(\boldsymbol{R})$ instead of $L^p(\boldsymbol{R})$. Moreover, he studied Hilbert transforms on the class of functions f for which $|f(x)|^p/(1+|x|^\alpha)\in L^1(\boldsymbol{R})$ for some $p\geqq 1$, $\alpha>0$.

Also, H. G. Tillmann ([16]), E. J. Beltrami and M. R. Wohlers ([2]) have studied Hilbert transforms in connection with distribution theory. They showed that the Hilbert transform could be well defined on the space $\mathscr{D}_{L^p}^*$ which is the dual of $\mathscr{D}_{L^p}$, firstly introduced by L. Schwartz (see [15]). The class $\mathscr{D}_{L^p}$ will be studied in the following section as $\mathscr{D}_{L_0^p}(\boldsymbol{R})$. And they obtained the similar results in the above Proposition for $\mathscr{D}_{L^p}$ (or its dual space $\mathscr{D}_{L^p}^*$) instead of $L^p(\boldsymbol{R})$.

In this paper, we generally consider the Hilbert transform on tempered distributions $\mathscr{S}'$ (which includes $\mathscr{D}_{L^p}^*$ and $W^2(\boldsymbol{R})$) and show that it has the suitable properties as in the above Proposition (i)$\sim$(iv).

§ 1. A space $D_{L_k^p}(\boldsymbol{R})$ and its dual space $D_{L_k^p}(\boldsymbol{R})^*$.

Let $\boldsymbol{R}$ be a real field. We denote by $\mathscr{D}'(\boldsymbol{R})$, or simply by $\mathscr{D}'$ (throughout this paper we consider only about one variable functions), the space of distributions. $\mathscr{D}'$ is the strong dual of $\mathscr{D}$, the space of infinitely differentiable functions with compact support in $\boldsymbol{R}$. And we denote a continuous bilinear functional on $\mathscr{D}'\times\mathscr{D}$ by $\langle u, \phi\rangle$ for all $u\in\mathscr{D}'$ and $\phi\in\mathscr{D}$.

$\mathscr{S}$ will denote the space of functions on $\boldsymbol{R}$ having derivatives of all order satisfying $\sup_{x\in\boldsymbol{R}}|x^\beta D^\alpha\phi(x)|<\infty$ for all indicies α and β of non-negative integers, where $D^\alpha=d^\alpha/dx^\alpha$. It is well-known that $\mathscr{S}$ is a Fréchet space with the system of semi-norms $\{\sup_{x\in\boldsymbol{R}}|x^\beta D^\alpha\phi(x)|$: *$\alpha$, β are non-negative integers*$\}$. $\mathscr{S}'$ is the dual space of $\mathscr{S}$, called a space of tempered distributions.

The Fourier transformation $\hat{\phi}$ of a function $\phi\in\mathscr{S}$ is defined by

$$\hat{\phi}(t)=\int_{-\infty}^{\infty} e^{-itx}\phi(x)dx .$$

Since the mapping $\phi\to\hat{\phi}$ of $\mathscr{S}$ onto $\mathscr{S}$ is linear continuous in the topology of $\mathscr{S}$, the Fourier transform $\hat{u}$ of at empered distribution u can be defined as the tempered distribution $\hat{u}$ defined through

$$\langle\hat{u}, \phi\rangle=\langle u, \hat{\phi}\rangle \quad (\phi\in\mathscr{S}) .$$

DEFINITION 1. Let p be a real number such that $1<p<\infty$. And let l and k be non-negative integers. $L^p_{k,l}(\boldsymbol{R})$ denotes the space in $\mathscr{S}'$ of functions on $\boldsymbol{R}$ satisfying

$$q^p_{k,l}(\phi)=\max\{\|x^\alpha D^\beta\phi(x)\|_{L^p}:\ 0\leqq\alpha\leqq k,\ 0\leqq\beta\leqq l\}<\infty$$

where $D^\beta=d^\beta/dx^\beta$ in the sense of distributional derivative. Moreover $C^{(l)}_k(\boldsymbol{R})$ denotes the space of functions on $\boldsymbol{R}$ such that β-th derivative $(0\leqq\beta\leqq l)$ is continuous and

$$\|\phi\|_{C^{(l)}_k}=\max\{\sup_{x\in\boldsymbol{R}}|x^\alpha D^\beta\phi(x)|:\ 0\leqq\alpha\leqq k,\ 0\leqq\beta\leqq l\}<\infty .$$

The following Lemmas 1 and 2 easily follow by the usual arguments of functional analysis.

LEMMA 1. *Let p be a real number such that $1<p<\infty$. And let l and k be non-negative integers. Then,*

(i) *$L^p_{k,l}(\boldsymbol{R})$ is a reflexive Banach space with norm $q^p_{k,l}$,*

(ii) *$\mathscr{S}\subset L^p_{k,l+1}(\boldsymbol{R})\subset L^p_{k,l}(\boldsymbol{R})\subset\mathscr{S}'$ and $\mathscr{S}\subset L^p_{k+1,l}(\boldsymbol{R})\subset L^p_{k,l}(\boldsymbol{R})\subset\mathscr{S}'$ and*

(iii) *each imbedding map in* (ii) *is continuous and $\mathscr{S}$ is a dense set in each space.*

LEMMA 2. *If we define*

$$\hat{q}^p_{k,l}(\phi)=\max\{\|D^\beta x^\alpha\phi(x)\|_{L^p}:\ 0\leqq\alpha\leqq k,\ 0\leqq\beta\leqq l\}$$

$$(\|\phi\|'_{C^{(l)}_k}=\max\{\sup_{x\in\boldsymbol{R}}|D^\beta x^\alpha\phi(x)|:\ 0\leqq\alpha\leqq k,\ 0\leqq\beta\leqq l\}) ,$$

then $q^p_{k,l}$ and $\hat{q}^p_{k,l}$ ($\|\ \|_{C^{(l)}_k}$ and $\|\ \|'_{C^{(l)}_k}$) are equivalent norms in $L^p_{k,l}(\boldsymbol{R})$ ($C^{(l)}_k(\boldsymbol{R})$).

DEFINITION 2. We can, by Lemma 1, define, for $1<p<\infty$ and non-negative integer k,

$$\mathscr{D}_{L^p_k}(\boldsymbol{R})=\lim_{l\to\infty}\text{proj}\,[L^p_{k\,l}(\boldsymbol{R})] .$$

Clearly, $\mathscr{D}_{L^p_k}(\boldsymbol{R})$ is a Fréchet space with the system of countable semi-norms $\{q^p_{k,l}: l=0, 1, 2, \cdots\}$. $\mathscr{D}_{L^p_k}(\boldsymbol{R})^*$ is the dual space of $\mathscr{D}_{L^p_k}(\boldsymbol{R})$.

By Lemma 1 and the properties of the projective limit, the following Lemma 3 immediately follows.

LEMMA 3. *Let p be a real number such that $1<p<\infty$. And let k be a non-negative integer. Then,*

(i) $\mathscr{S}\subset\mathscr{D}_{L^p_{k+1}}(\boldsymbol{R})\subset\mathscr{D}_{L^p_k}(\boldsymbol{R})\subset\mathscr{S}'$

and

(ii) *each imbedding map in* (i) *is continuous and $\mathscr{S}$ is a dense set in each space.*

THEOREM 1. *Let p be a real number such that $1<p<\infty$. And let k be a non-negative integer. Then,*

(i) $\mathscr{D}_{L^p_k}(\boldsymbol{R})^*=\text{lim ind}_{l\to\infty}[L^p_{k,l}(\boldsymbol{R})^*]$

and

(ii) *$\mathscr{D}_{L^p_k}(\boldsymbol{R})$ is a reflexive Fréchet space.*

PROOF. Since Lemma 1 shows that $\{L^p_{k,l}(\boldsymbol{R})^*\}_{l=0}^\infty$ is an increasing sequence of reflexive Banach spaces, $\text{lim ind}_{l\to\infty}[L^p_{k,l}(\boldsymbol{R})^*]$ is a regular inductive limit ([10]). By the properties of inductive limits and projective limits (see, for example, [6], [10] and [13]) and Lemma 1 (i), we get that

$$(1)\quad [\lim_{l\to\infty}\text{ind}\,[L^p_{k,l}(\boldsymbol{R})^*]]^*=\lim_{l\to\infty}\text{proj}\,[L^p_{k,l}(\boldsymbol{R})^{**}]=\lim_{l\to\infty}\text{proj}\,[L^p_{k,l}(\boldsymbol{R})]=\mathscr{D}_{L^p_k}(\boldsymbol{R})\,.$$

Also, we see ([10]) that $\text{lim ind}_{l\to\infty}[L^p_{k,l}(\boldsymbol{R})^*]$ is reflexive, that is

$$(2)\qquad [\lim_{l\to\infty}\text{ind}\,[L^p_{k,l}(\boldsymbol{R})^*]]^{**}=\lim_{l\to\infty}\text{ind}\,[L^p_{k,l}(\boldsymbol{R})^*]\,.$$

Then, we, by (1) and (2), get that

$$[\mathscr{D}_{L^p}(\boldsymbol{R})]^*=[\lim_{l\to\infty}\text{ind}\,[L^p_{k,l}(\boldsymbol{R})^*]]^{**}=\lim_{l\to\infty}\text{ind}\,[L^p_{k,l}(\boldsymbol{R})^*]$$

and

$$[\mathscr{D}_{L^p_k}(\boldsymbol{R})]^{**}=[\lim_{l\to\infty}\text{ind}\,[L^p_{k,l}(\boldsymbol{R})^*]]^{***}=[\lim_{l\to\infty}\text{ind}\,[L^p_{k,l}(\boldsymbol{R})^*]]^*=\mathscr{D}_{L^p_k}(\boldsymbol{R})\,.$$

Therefore, we obtain (i) and (ii). This completes the proof.

LEMMA 4. *Let q be a real number such that $1<q<\infty$. And let k and α be any non-negative integers. Let g be any function in $L^q(\boldsymbol{R})$ and P be any infinitely differentiable function such that*

$$\sup_{x\in R}\left|\frac{D^jP(x)}{(1+x^2)^{k/2}}\right|<\infty \quad \textit{for any non-negative integer } j .$$

Then there exist functions g_j $(j=0, 1, 2, \cdots, \alpha)$ *such that*

$$P(x)D^\alpha g(x)=\sum_{j=0}^{\alpha} D^j g_j(x) \quad \textit{and} \quad \|g_j(x)/(1+x^2)^{k/2}\|_{L^p}<\infty . \tag{3}$$

Proof. We shall prove this lemma by induction. Let $\alpha=0$. Since $P(x)/(1+x^2)^{k/2}$ is bounded, we see that

$$P(x)g(x)/(1+x^2)^{k/2}\in L^q(\boldsymbol{R}) .$$

Then, (3) immediately follows, if we put $g_0(x)=P(x)g(x)$.

Next we prove (3) for $\alpha+1$ under the assumption that (3) is true for α. Since DP is a function having derivatives of all order such that

$$\sup_{x\in R}\left|\frac{D^jDP(x)}{(1+x^2)^{k/2}}\right|<\infty \quad \text{for } \textit{all non-negative integer } j ,$$

there exist g'_j $(j=0, 1, \cdots\alpha)$ such that

$$DP(x)D^\alpha g(x)=\sum_{j=0}^{\alpha} D^j g'_j(x) \quad \text{and} \quad \|g'_j(x)/(1+x^2)^{k/2}\|_{L^q}<\infty .$$

Hence, we, by assumption, see that

$$\begin{aligned} P(x)D^{\alpha+1}g(x) &= D[P(x)D^\alpha g(x)]-[DP(x)][D^\alpha g(x)] \\ &= D\Big(\sum_{j=0}^{\alpha} D^j g_j\Big)-\sum_{j=0}^{\alpha} D^j g'_j \\ &= -g'_0+\sum_{j=1}^{\alpha} D^j(g_{j-1}-g'_j)+D^{\alpha+1}g_\alpha . \end{aligned}$$

Since $\|g'_0(x)/(1+x^2)^{k/2}\|_{L^q}<\infty$, $\|(g_{j-1}-g'_j(x))/(1+x^2)^{k/2}\|_{L^q}<\infty$ $(j=1, 2, \cdots\alpha)$ and $\|g_\alpha(x)/(1+x^2)^{k/2}\|_{L^q}<\infty$, the proof is completed.

Theorem 2. *Let p be a real number such that $1<p<\infty$. And let k be a non-negative integer. Then the following statements are equivalent:*

(i) $u\in \boldsymbol{D}_{L^p_k}(\boldsymbol{R})^*$,

(ii) *there exist functions* u_j $(j=0, 1, \cdots, l)$ *such that*

$$u=\sum_{j=0}^{l} D^j u_j \quad \text{and} \quad \|u_j/(1+x^2)^{k/2}\|_{L^q}<\infty$$

where $1/p+1/q=1$.

Proof. Firstly, we shall prove that (ii) implies (i). Put $C=$

$\max_{0\le j\le l}\|u_j/(1+x^2)^{k/2}\|_{L^q}$. We see, by Lemma 2, that, for any $\phi\in\mathscr{D}$,

$$\begin{aligned}|\langle u,\phi\rangle|&=\Big|\langle\sum_{j=0}^{l}D^j u_j,\phi\rangle\Big|=\Big|\sum_{j=0}^{l}(-1)^{-j}\langle u_j,D^j\phi\rangle\Big|\\&=\Big|\sum_{j=0}^{l}(-1)^{-j}\langle u_j/(1+x^2)^{k/2},(1+x^2)^{k/2}D^j\phi\rangle\Big|\\&\leqq\sum_{j=0}^{l}\|u_j/(1+x^2)^{k/2}\|_{L^q}\|(1+x^2)^{k/2}D^j\phi\|_{L^p}\\&\leqq C\sum_{j=0}^{l}\|(1+x^2)^{k/2}D^j\phi\|_{L^p}\leqq C'q^p_{k,l}(\phi)\end{aligned}$$

which implies that (i) holds.

Next, we shall prove that (i) implies (ii). Assume that $u\in\mathscr{D}_{L^p_k}(\boldsymbol{R})^*$. Then, there exist $M>0$ and non-negative integer m such that, for any $\phi\in\mathscr{D}$,

$$|\langle u,\phi\rangle|\leqq Mq^p_{k,m}(\phi)=M\max\{\|x^\alpha D^\beta\phi(x)\|_{L^p}:0\leqq\alpha\leqq k,\ 0\leqq\beta\leqq m\}<\infty\ .$$

Since $\sup_{x\in R}|[D^j(1+x^2)^{-k/2}](1+x^2)^{k/2}|<\infty$, $(j=0,1,2,\cdots)$, this implies that, for any $\phi\in\mathscr{D}$,

$$\begin{aligned}|\langle u/(1+x^2)^{k/2},\phi\rangle|&=|\langle u,\phi/(1+x^2)^{k/2}\rangle|\\&\leqq M\max_{\substack{0\le\alpha\le k\\0\le\beta\le l}}\|x^\alpha D^\beta(\phi/(1+x^2)^{k/2}\|_{L^p}\\&=M\max_{\substack{0\le\alpha\le k\\0\le\beta\le l}}\Big\|x^\alpha\sum_{j=0}^{\beta}\binom{\beta}{j}D^{\beta-j}(1/(1+x^2)^{k/2})D^j\phi\Big\|_{L^p}\\&\leqq M'\max_{0\le\beta\le l}\|D^\beta\phi\|_{L^p}\ .\end{aligned}$$

Hence we see that

$$u(x)/(1+x^2)^{k/2}\in\mathscr{D}_{L^p_0}(\boldsymbol{R})^*\ .$$

Then, from the theorem of L. Schwartz [15], this implies that there exist functions $g_\alpha(\alpha=0,1,2,\cdots l)(\in L^q(\boldsymbol{R}))$ such that

$$u(x)=(1+x^2)^{k/2}\sum_{\alpha=0}^{l}D^\alpha g_\alpha\ .$$

Putting $P(x)=(1+x^2)^{k/2}$ in Lemma 4, we see that there exist functions $u_{\alpha,j}(\alpha=0,1,\cdots,l$ and $j=0,1,\cdots,\alpha)$ such that

$$u(x)=\sum_{\alpha=0}^{l}\sum_{j=0}^{\alpha}D^j u_{\alpha,j}=\sum_{j=0}^{l}D^j\Big(\sum_{\alpha=j}^{l}u_{\alpha,j}\Big)\quad\text{and}\quad\|u_{\alpha,j}/(1+x^2)^{k/2}\|_{L^q}<\infty\ .$$

This completes the proof.

Though the following Theorem 3 is seemed to be known (for instance see [5], for $p=2$), we mention the proof for the self-consistency as follows.

LEMMA 5. *Let p be a real number such that $1<p<\infty$. And let l and k be non-negative integers. Then,*

(i) $L^p_{k,l+1}(\boldsymbol{R})\subset C^{(l)}_k(\boldsymbol{R})$

and

(ii) $C^{(l)}_{k+1}(\boldsymbol{R})\subset L^p_{k,l}(\boldsymbol{R})$.

Moreover each natural imbedding map in (i) *and* (ii) *is continuous.*

PROOF. By Lemma 2 and the Sobolev imbedding theorem ([1]), we see that, for any $\phi\in L^p_{k,l+1}(\boldsymbol{R})$,

$$\begin{aligned}\|\phi\|_{C^{(l)}_k} &= \max_{\substack{0\leq\alpha\leq k\\0\leq\beta\leq l}}\sup_{x\in\boldsymbol{R}}|x^\alpha D^\beta\phi| \\ &\leqq C\max_{\substack{0\leq\alpha\leq k\\0\leq\beta\leq l}}\sup_{x\in\boldsymbol{R}}|D^\beta x^\alpha\phi| \\ &\leqq C'\max_{\substack{0\leq\alpha\leq k\\0\leq\beta\leq l+1}}\|D^\beta x^\alpha\phi\|_{L^p} \\ &\leqq C'' q^p_{k,l+1}(\phi)\end{aligned}$$

which implies that (i) is true and the natural imbedding map is continuous.

Next we see that, for any $\phi\in C^{(l)}_{k+1}(\boldsymbol{R})$,

$$\begin{aligned}q^p_{k,l}(\phi) &= \max_{\substack{0\leq\alpha\leq k\\0\leq\beta\leq l}}\|x^\alpha D^\beta\phi\|_{L^p} \\ &= \max_{\substack{0\leq\alpha\leq k\\0\leq\beta\leq l}}\left[\int_{-\infty}^{\infty}\left|\frac{(1+x^2)^{1/2}}{(1+x^2)^{1/2}}x^\alpha D^\beta\phi\right|^p dx\right]^{1/p} \\ &\leqq\left[\int_{-\infty}^{\infty}\frac{1}{(1+x^2)^{p/2}}dx\right]^{1/p}[\max_{\substack{0\leq\alpha\leq k\\0\leq\beta\leq l}}\{\sup_{x\in\boldsymbol{R}}|(1+x^2)^{1/2}x^\alpha D^\beta\phi|\}] \\ &\leqq C\,\|\phi\|_{C^{(l)}_{k+1}}\end{aligned}$$

which implies that (ii) is true and the natural imbedding map is continuous. This completes the proof.

THEOREM 3. *Let p be a real number such that $1<p<\infty$. Then,*

(i) $\lim\mathrm{proj}_{k\to\infty}[\mathscr{D}_{L^p_k}(\boldsymbol{R})]=\mathscr{S}$

and

(ii) $\lim\mathrm{ind}_{k\to\infty}[\mathscr{D}_{L^p_k}(\boldsymbol{R})^*]=\mathscr{S}'$.

PROOF. Since

$$\lim_{k\to\infty}\text{proj}\,[\mathscr{D}_{L_k^p}(\boldsymbol{R})]=\lim_{\substack{k\to\infty\\ l\to\infty}}\text{proj}\,[L_{L_{k,l}^p}(\boldsymbol{R})] \quad\text{and}\quad \mathscr{S}=\lim_{\substack{k\to\infty\\ l\to\infty}}\text{proj}\,[C_k^{(l)}(\boldsymbol{R})]\,,$$

we see, by Lemma 5, that (i) is true. Also, since

$$\lim_{k\to\infty}\text{ind}\,[\mathscr{D}_{L_k^p}(\boldsymbol{R})^*]=\lim_{\substack{k\to\infty\\ l\to\infty}}\text{ind}\,[L_{L_{k,l}^p}(\boldsymbol{R})^*] \quad\text{and}\quad \mathscr{S}'=\lim_{\substack{k\to\infty\\ l\to\infty}}\text{ind}\,[C_k^{(l)}(\boldsymbol{R})^*]\,,$$

we see, by Lemma 5, that (ii) is true.

§ 2. Generalized Hilbert transforms in $\mathscr{D}_{L_k^p}(\boldsymbol{R})$.

DEFINITION 3. Let $a=(a_1, \cdots, a_k)$ be a k-tuple of complex numbers such that $\text{Im}[a_j]\neq 0$ ($j=1, 2, \cdots, k$), where $\text{Im}[a_j]$ denotes the imaginary part of a complex number a_j. We define that, for any $\phi\in\mathscr{D}_{L_k^p}(\boldsymbol{R})$,

$$(H_a^{\varepsilon,N}\phi)(x)=\frac{1}{\pi(x-a_1)\cdots(x-a_k)}\int_{\varepsilon<|t|<N}(x-t-a_1)\cdots(x-t-a_k)\frac{\phi(x-t)}{t}dt\,,$$

specially, if $k=0$,

$$(H^{\varepsilon,N}\phi)(x)=\frac{1}{\pi}\int_{\varepsilon<|t|<N}\frac{\phi(x-t)}{t}dt\,.$$

The following lemma easily follows.

LEMMA 6. *Let $a=(a_1, \cdots, a_k)$ be a k-tuple of complex numbers such that $\text{Im}[a_j]\neq 0$ ($j=1, 2, \cdots, k$) (where $\text{Im}[a_j]$ is an imaginary part of a complex number a_j). A mapping T_a: $\mathscr{D}_{L_0^p}(\boldsymbol{R})\to\mathscr{D}_{L_k^p}(\boldsymbol{R})$ such that*

$$T_a\psi(x)=\frac{\psi(x)}{(x-a_1)\cdots(x-a_k)}\qquad\text{for } all\ \psi\in\mathscr{D}_{L_0^p}(\boldsymbol{R})$$

is a bi-continuous surjection.

If a generalized sequence $\{x_\lambda\}_{\lambda\in\Lambda}$ in a Hausdorff topological vector space X converges to x as $\lambda\to\lambda_0$ in the topology of X, we denote it by $(X)\lim_{\lambda\to\lambda_0}x_\lambda=x$.

THEOREM 4. *Let p be a real number such that $1<p<\infty$. Let $a=(a_1, \cdots, a_k)$ be a k-tuple of complex numbers such that $\text{Im}[a_j]\neq 0$ ($j=1, 2, \cdots, k$). Then, for any $\phi\in\mathscr{D}_{L_k^p}(\boldsymbol{R})$, $(\mathscr{D}_{L_k^p})\lim_{\substack{\varepsilon\to 0+\\ N\to\infty}}(H_a^{\varepsilon,N}\phi)$ exists in $\mathscr{D}_{L_k^p}(\boldsymbol{R})$.*

PROOF. Let $k=0$. By Proposition (i), we see that, for any $\phi\in\mathscr{D}_{L_0^p}(\boldsymbol{R})$ and any $0<\varepsilon'<\varepsilon<N<N'<\infty$,

$$\begin{aligned} &q^p_{0,l}(H^{\varepsilon,N}\phi - H^{\varepsilon',N'}\phi) \\ &\leqq \max_{0\leq\beta\leq l}\left\| D^\beta \frac{1}{\pi}\int_{\substack{\varepsilon'<|t|<\varepsilon \\ N<|t|<N'}} \frac{\phi(x-t)}{t}dt\right\|_{L^p} \\ &= \max_{0\leq\beta\leq l}\left\| \frac{1}{\pi}\int_{\substack{\varepsilon'<|t|<\varepsilon \\ N<|t|<N'}} \frac{(D^\beta\phi)(x-t)}{t}dt\right\|_{L^p} \\ &\to 0 \quad \text{as} \quad \varepsilon, \varepsilon' \to 0+ \quad \text{and} \quad N, N' \to \infty . \end{aligned}$$

This implies that $\{H^{\varepsilon,N}\phi\}$ is a Cauchy net as $\varepsilon\to 0+$, $N\to\infty$ in $\mathscr{D}_{L^p_0}(\boldsymbol{R})$. Hence $\lim_{\substack{\varepsilon\to 0+ \\ N\to\infty}}(H^{\varepsilon,N}\phi)$ exists in the topology of $\mathscr{D}_{L^p_0}(\boldsymbol{R})$.

In general case, by the above argument and Lemma 6, we see that, for any $\phi\in\mathscr{D}_{L^p_k}(\boldsymbol{R})$,

$$\begin{aligned} &(\mathscr{D}_{L^p_k})\lim_{\substack{\varepsilon\to 0+ \\ N\to\infty}} H^{\varepsilon,N}_a\phi \\ &= (\mathscr{D}_{L^p_k})\lim_{\substack{\varepsilon\to 0+ \\ N\to\infty}} \frac{1}{\pi(x-a_1)\cdots(x-a_k)}\int_{\varepsilon<|t|<N} \frac{(x-t-a_1)\cdots(x-t-a_k)}{t}\phi(x-t)dt \\ &= (\mathscr{D}_{L^p_k})\lim_{\substack{\varepsilon\to 0+ \\ N\to\infty}} T_a H^{\varepsilon,N}(T_a^{-1}\phi) \\ &= T_a[(\mathscr{D}_{L^p_0})\lim_{\substack{\varepsilon\to 0+ \\ N\to\infty}} H^{\varepsilon,N}(T_a^{-1}\phi)] \qquad \text{(by Lemma 6)} \end{aligned}$$

which exists since $T_a^{-1}\phi\in \boldsymbol{D}_{L^p_0}(\boldsymbol{R})$. This completes the proof.

By this theorem, we can obtain the following definition.

DEFINITION 4. Let $a=(a_1, \cdots, a_k)$ be a k-tuple of complex numbers such that $\mathrm{Im}[a_j]\neq 0$ $(j=1, 2, \cdots, k)$. We define a generalized Hilbert transform H_a: $\mathscr{D}_{L^p_k}(\boldsymbol{R})\to\mathscr{D}_{L^p_k}(\boldsymbol{R})$ such that

$$H_a\phi = (\mathscr{D}_{L^p_k})\lim_{\substack{\varepsilon\to 0+ \\ N\to\infty}} H^{\varepsilon,N}_a\phi \qquad (\phi\in\mathscr{D}_{L^p_k}(\boldsymbol{R})) ,$$

specially, if $k=0$,

$$H\phi = (\mathscr{D}_{L^p_0})\lim_{\substack{\varepsilon\to 0+ \\ N\to\infty}} H^{\varepsilon,N}\phi \qquad (\phi\in\mathscr{D}_{L^p_0}(\boldsymbol{R})) .$$

Note that a generalized Hilbert transform H_a is also represented by $T_a H T_a^{-1}$.

THEOREM 5. *Let p be a real number such that $1<p<\infty$. Let k be a non-negative integer. And let $a=(a_1, \cdots, a_k)$ be a k-tuple of complex numbers such that* $\mathrm{Im}[a_j]\neq 0$ $(j=1, 2, \cdots, k)$. *Then,*

(i) *H_a is a bounded linear operator on $\mathscr{D}_{L^p_k}(\boldsymbol{R})$*

and

(ii) $H_a(H_a\phi)=-\phi \quad (\phi\in\mathscr{D}_{L^p_k}(\boldsymbol{R}))$.

Moreover, H_a: $\mathscr{D}_{L^p_k}(\boldsymbol{R})\to\mathscr{D}_{L^p_k}(\boldsymbol{R})$ *is a bi-continuous surjection such that* $H_a^{-1}=-H_a$.

PROOF. It is sufficient to prove (i) and (ii) for $k=0$ since $H_a=T_aHT_a^{-1}$. Though this thoerem for $k=0$ has been proved in [16], we shall show the proof for the self-consistency.

By the similar way in the Theorem 4, we can easily obtain, from Proposition (ii), that for any $\phi\in\mathscr{D}_{L^p_0}(\boldsymbol{R})$ and any $0<\varepsilon<N<\infty$,

$$\begin{aligned} q^p_{0,l}(H^{\varepsilon,N}\phi) &\leqq \max_{0\leq\beta\leq l}\left\|\frac{1}{\pi}\int_{\varepsilon<|t|<N}\frac{D^\beta\phi(x-t)}{t}dt\right\|_{L^p} \\ &\leqq C\max_{0\leq\beta\leq l}\|D^\beta\phi\|_{L^p} \\ &=Cq^p_{0,l}(\phi) \end{aligned}$$

which implies (i) for $k=0$. Also, by Proposition (iii), (ii) immediately follows since $\mathscr{D}_{L^p_0}(\boldsymbol{R})\subset L^p(\boldsymbol{R})$. This completes the proof.

§3. Generalized Hilbert transforms in $\mathscr{S}'$.

DEFINITION 5. Let p be any $1<p<\infty$ and k be any non-negative integer. Let $a=(a_1,\cdots,a_k)$ be a k-tuple of complex numbers such that $\mathrm{Im}[a_j]\neq 0$ $(j=1,\cdots,k)$. Since the generalized Hilbert transform H_a: $\mathscr{D}_{L^p_k}(\boldsymbol{R})\to\mathscr{D}_{L^p_k}(\boldsymbol{R})$ is linear continuous in the topology of $\mathscr{D}_{L^p_k}(\boldsymbol{R})$, we can define the generalized Hilbert transform H_a^*u of $u\in\mathscr{D}_{L^p_k}(\boldsymbol{R})^*$ as the element of $\mathscr{D}_{L^p_k}(\boldsymbol{R})^*$ defined through

$$\langle H_a^*u,\phi\rangle=\langle u, H_a\phi\rangle \quad (\phi\in\mathscr{D}_{L^p_k}(\boldsymbol{R}))\,.$$

Similarly, $H_a^{\varepsilon,N*}$ is defined as the adjoint operator of $H_a^{\varepsilon,N}$.

Note that the adjoint operator T_a^* of T_a: $\mathscr{D}_{L^p_0}(\boldsymbol{R})\to\mathscr{D}_{L^p_k}(\boldsymbol{R})$ is a bi-continuous linear operator from $\mathscr{D}_{L^p_k}(\boldsymbol{R})^*$ onto $\mathscr{D}_{L^p_0}(\boldsymbol{R})^*$, which is represented by

$$T_a^*u=\frac{u}{(x-a_1)\cdots(x-a_k)} \qquad \text{for } all\ u\in\mathscr{D}_{L^p_k}(\boldsymbol{R})^*\,.$$

The following theorem immediately follows from the property of the adjoint operator and Theorem 5.

THEOREM 6. *It follows that*

(i) H_a^* *is linear continuous in the topology of* $\mathscr{D}_{L^p_k}(\boldsymbol{R})^*$,

(ii) $H_a^*(H_a^* u) = -u \quad (u \in \mathscr{D}_{L_k^p}(\boldsymbol{R})^*)$.
Therefore, $H_a^{*-1} = -H_a^*$.

THEOREM 7. *Let p be a real number such that $1<p<\infty$. Let k be a non-negative integer. And let $a=(a_1, \cdots, a_k)$ be a k-tuple of complex numbers such that* $\mathrm{Im}[a_j] \neq 0$ $(j=1, \cdots, k)$. *Then, for any $u \in \mathscr{D}_{L_k^p}(\boldsymbol{R})^*$,*

$$H_a^* u = (\mathscr{D}_{L_k^p}^*) \lim_{\substack{\varepsilon \to 0+ \\ N \to \infty}} H_a^{\varepsilon, N*} u .$$

PROOF. Firstly we shall prove this theorem in the case that $k=0$. Let u be any element in $\mathscr{D}_{L_0^p}(\boldsymbol{R})^*$. We can easily obtain that $H^{\varepsilon,N} D^j \phi = D_j H^{\varepsilon,N} \phi$ and $HD^j \phi = D^j H \phi$ for any $\phi \in \mathscr{D}_{L_0^p}(\boldsymbol{R})$ $(j=1, 2, \cdots)$. Hence we see by Theorem 2 that, for any $\phi \in \mathscr{D}_{L_0^p}(\boldsymbol{R})$,

$$\begin{aligned}
&|\langle (H^{\varepsilon,N*} - H^*) u, \phi \rangle| \\
&= \left| \left\langle \sum_{j=0}^{l} D^j u_j, (H^{\varepsilon,N} - H) \phi \right\rangle \right| \\
&\leqq \sum_{j=0}^{l} |\langle u_j, (H^{\varepsilon,N} - H) D^j \phi \rangle| \\
&= \sum_{j=0}^{l} |\langle (H^{\varepsilon,N*} - H^*) u_j, D^j \phi \rangle| \\
&\leqq \sum_{j=0}^{l} \|(H^{\varepsilon,N*} - H^*) u_j\|_{L^q} \|D^j \phi\|_{L^p}
\end{aligned}$$

where u_j $(j=1, 2, \cdots, l)$ are defined as in Theorem 2. By Proposition (i), this implies that, for any bounded set $B \subset \mathscr{D}_{L_0^p}(\boldsymbol{R})$

$$\begin{aligned}
&\sup_{\phi \in B} |\langle (H^{\varepsilon,N*} - H^*) u, \phi \rangle| \\
&\leqq C \sum_{j=0}^{l} \|(H^{\varepsilon,N*} - H^*) u_j\|_{L^q} \\
&\to 0 \qquad (as\ \varepsilon \to 0+,\ N \to \infty) .
\end{aligned}$$

Hence we get that, for any $u \in \mathscr{D}_{L_0^p}(\boldsymbol{R})^*$,

$$H^* u = (\mathscr{D}_{L_0^p}^*) \lim_{\substack{\varepsilon \to 0+ \\ N \to \infty}} H^{\varepsilon,N*} u .$$

In general case, we see that, for any $u \in \mathscr{D}_{L_k^p}(\boldsymbol{R})^*$,

$$\begin{aligned}
(\mathscr{D}_{L_k^p}^*) \lim_{\substack{\varepsilon \to 0+ \\ N \to \infty}} H_a^{\varepsilon,N*} u &= (\mathscr{D}_{L_k^p}^*) \lim_{\substack{\varepsilon \to 0+ \\ N \to \infty}} ((T_a^{*-1} H^{\varepsilon,N*} T_a^*) u) \\
&= T_a^{*-1} (\mathscr{D}_{L_0^p}^*) \lim_{\substack{\varepsilon \to 0+ \\ N \to \infty}} ((H^{\varepsilon,N*} T_a^*) u)
\end{aligned}$$

$$=T_a^{*-1}((H^*T_a^*)u)$$
$$=H_a^*u\ .$$

This completes the proof.

THEOREM 8. *Let p be a real number such that $1<p<\infty$. Let k be a non-negative integer. And let $a=(a_1, \cdots, a_k)$ be a k-tuple of complex numbers such that* $\mathrm{Im}[a_j]\neq 0$ $(j=1, \cdots, k)$. *Then, for any $u\in\mathscr{D}_{L_k^p}(\boldsymbol{R})^*$,*

$$\langle (H_a^*u)\hat{}\,, \phi\rangle=\begin{cases}-i\langle \hat{u}, \phi\rangle & \textit{for all } \phi\in\mathscr{D} \textit{ such that } \mathrm{supp}[\phi]\subset(0, \infty)\\ i\langle \hat{u}, \phi\rangle & \textit{for all } \phi\in\mathscr{D} \textit{ such that } \mathrm{supp}[\phi]\subset(-\infty, 0)\end{cases}$$

where $\hat{u}$ is the Fourier transform of u in $\mathscr{S}'$.

PROOF. Let ϕ be any element in $\mathscr{D}$ such that $\mathrm{supp}[\phi]\subset(0, \infty)$. From the properties of Fourier transforms and Proposition (iv), we see that

$$\begin{aligned}\langle (H_a^*u)\hat{}\,, \phi\rangle&=\langle H_a^*u, \hat{\phi}\rangle\\ &=\langle u, T_a^{-1}HT_a\hat{\phi}\rangle\\ &=\langle u, T_a^{-1}H[[(i^{-1}D-a_1)(i^{-1}D-a_2)\cdots(i^{-1}D-a_k)\phi]\hat{}\,]\rangle\\ &=\langle u, T_a^{-1}[-i(i^{-1}D-a_1)(i^{-1}D-a_2)\cdots(i^{-1}D-a_k)\phi]\hat{}\,\rangle\\ &=-i\langle u, T_a^{-1}T_a\hat{\phi}\rangle\\ &=-i\langle u, \hat{\phi}\rangle\\ &=-i\langle \hat{u}, \phi\rangle\ .\end{aligned}$$

In a similar way, we can prove this theorem when ϕ is any element in $\mathscr{D}$ such that $\mathrm{supp}[\phi]\subset(-\infty, 0)$. Hence this completes the proof.

COROLLARY 1. *Let p be a real number such that $1<p<\infty$. Let k, m and n be non-negative integers such that $k\leqq m\leqq n$. And let $a=(a_1, \cdots, a_m)$ and $b=(b_1, \cdots, b_n)$ be respectively m-tuple and n-tuple of complex numbers such that* $\mathrm{Im}[a_j]\neq 0$ $(j=1, 2, \cdots, m)$ *and* $\mathrm{Im}[b_j]\neq 0$ $(j=1, 2, \cdots, n)$. *Then, for any $u\in\mathscr{D}_{L_k^p}(\boldsymbol{R})^*$ $(\subset\mathscr{D}_{L_m^p}(\boldsymbol{R})^*\subset\mathscr{D}_{L_n^p}(\boldsymbol{R})^*)$, $H_a^*u-H_b^*u$ is a polynomial.*

PROOF. By Theorem 8, we see that, for any $\phi\in\mathscr{D}$ with $\mathrm{supp}[\phi]\subset(0, \infty)$,

$$\begin{aligned}\langle (H_a^*u-H_b^*u)\hat{}\,, \phi\rangle&=\langle (H_a^*u)\hat{}\,, \phi\rangle-\langle (H_b^*u)\hat{}\,, \phi\rangle\\ &=-i\langle \hat{u}, \phi\rangle-(-i)\langle \hat{u}, \phi\rangle=0\ .\end{aligned}\tag{4}$$

Similarly we see that, for any $\phi\in\mathscr{D}$ with $\mathrm{supp}[\phi]\subset(-\infty, 0)$,

$$\langle (H_a^*u-H_b^*u)\hat{}\,, \phi\rangle=0\ .\tag{5}$$

By (4) and (5), if follows that $\mathrm{supp}[(H_a^*u-H_b^*u)^\wedge]\subset\{0\}$. This implies that $(H_a^*u-H_b^*u)^\wedge$ is a finite linear combination of a Delta function $\delta(x)$ and its derivatives. Therefore, $H_a^*u-H_b^*u$ is a certain polynomial. This completes that proof.

REMARK. Let u be any element in $\mathscr{S}'$. Since Theorem 3 implies that u belongs to $\mathscr{D}_{L_k^p}(\boldsymbol{R})^*$ for some k, the generalized Hilbert transform of u can be defined by H_a^*u, where $a=(a_1, \cdots, a_k)$ is a k-tuple of complex numbers such that $\mathrm{Im}[a_j]\neq 0$ $(j=1, 2, \cdots, k)$. The above Corollary 1 shows that it is well defined independently of choosing k and a under the identification of the difference of polynomials.

The author wishes to express his sincere thanks to Professor S. Koizumi of Keio University.

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