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A Uniqueness Set for the Differential Operator $\Delta_z+\lambda^2$

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Introduction

We consider the Laplacian

$$\Delta_z=(\partial/\partial z_1)^2+(\partial/\partial z_2)^2+\cdots+(\partial/\partial z_{d+1})^2$$

in the complex $d+1$ space C^{d+1}. Let

$$M=\{z=(z_1, z_2, \cdots, z_{d+1})\in C^{d+1}; z\neq 0, z^2=0\}$$

be the complex cone defined by the quadratic equation

$$z^2=z_1^2+z_2^2+\cdots+z_{d+1}^2=0 .$$

Suppose λ is an arbitrary complex number. The first named author showed in [13] that the entire function f on C^{d+1} satisfying the differential equation

$$(0.1)\qquad (\Delta_z+\lambda^2)f=0$$

is completely defined by its restriction values on the complex subvariety M. In this sense, we call the cone M a uniqueness set for the differential operator $\Delta_z+\lambda^2$.

We shall show in this paper that this phenomenon occurs locally at the origin. More precisely, we shall prove a semi-local version using the Lie ball.

Let $\tilde{B}(r)$ be the Lie ball of radius r with center at the origin in C^{d+1} (see definition in §1). The space of holomorphic functions on $\tilde{B}(r)$ is denoted by $\mathcal{O}(\tilde{B}(r))$. We shall denote by $\mathcal{O}_\lambda(\tilde{B}(r))$ the subspace of $\mathcal{O}(\tilde{B}(r))$ defined by the differential equation (0.1). Remark that $\mathcal{O}_0(\tilde{B}(r))=\mathcal{O}_\Delta(\tilde{B}(r))$ in our notation in our previous paper [8], [10], etc., and that $\mathcal{O}_0(\tilde{B}(r))$ is the space of harmonic functions on the Lie ball $\tilde{B}(r)$.

Let us consider the space of functions on $M\cap\tilde{B}(r)$:

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(0.2) $$\mathcal{O}(\tilde{B}(r))|_M = \{f|_{M\cap\tilde{B}(r)};\ f\in\mathcal{O}(\tilde{B}(r))\}\ .$$

We may call $\mathcal{O}(\tilde{B}(r))|_M$ the space of holomorphic functions on the truncated complex cone $M\cap\tilde{B}(r)$.

Our main result is that the restriction mapping is a linear topological isomorphism of $\mathcal{O}_\lambda(\tilde{B}(r))$ onto $\mathcal{O}(\tilde{B}(r))|_M$ (Theorem 2.4). The global version in [13] corresponds to the case $r=\infty$.

Our method of proof relies heavily on the properties of spherical harmonics. If $d=1$, the situation becomes very simple and is studied in [7].

The first author described in [14] a uniqueness set for more general linear partial differential operators of the second order with constant coefficients.

The Fourier-Borel transformation P_λ has been studied in [2], [6], [10], [12], [13], etc. We will determine in the last section the inverse image of $\mathcal{O}_\lambda(\tilde{B}(r))$ by the transformation P_λ (Theorem 3.1).

§1. Preliminaries.

Let d be a positive integer and assume $d\geqq 2$. $S=S^d=\{x\in\boldsymbol{R}^{d+1};\ \|x\|=1\}$ denotes the unit sphere in $\boldsymbol{R}^{d+1}$, where $\|x\|^2=x_1^2+x_2^2+\cdots+x_{d+1}^2$. ds denotes the unique $O(d+1)$ invariant measure on S with $\int_S 1ds=1$, where $O(k)$ is the orthogonal group of degree k. $\|\ \ \|_2$ and $\|\ \ \|_\infty$ are the L^2-norm and sup norm on S respectively. $H_{n,d}$ is the space of spherical harmonics of degree n in dimension $d+1$. For spherical harmonics, see Müller [11]. For $S_n\in H_{n,d}$, $\tilde{S}_n$ denotes the unique homogeneous harmonic polynomial of degree n on $\boldsymbol{C}^{d+1}$ such that $\tilde{S}_n|_S=S_n$.

The Lie norm $L(z)$ and the dual Lie norm $L^*(z)$ on $\boldsymbol{C}^{d+1}$ are defined as follows:

$$L(z)=L(x+iy):=[\|x\|^2+\|y\|^2+2\{\|x\|^2\|y\|^2-(x\cdot y)^2\}^{1/2}]^{1/2}\ ,$$

$$\begin{aligned}L^*(z)=L^*(x+iy):&=\sup\{|\xi\cdot z|;\ L(\xi)\leqq 1\}\\ &=(1/\sqrt{2})[\|x\|^2+\|y\|^2+\{(\|x\|^2-\|y\|^2)^2+4(x\cdot y)^2\}^{1/2}]^{1/2}\ ,\end{aligned}$$

where $z,\xi\in\boldsymbol{C}^{d+1}$, and $z\cdot\xi=z_1\xi_1+z_2\xi_2+\cdots+z_{d+1}\xi_{d+1}$, $x, y\in\boldsymbol{R}^{d+1}$, (see Drużkowski [1]). We put

$$\tilde{B}(r):=\{z\in\boldsymbol{C}^{d+1};\ L(z)<r\}\qquad\text{for}\quad 0<r\leqq\infty$$

and

$$\tilde{B}[r]:=\{z\in \boldsymbol{C}^{d+1};\ L(z)\leqq r\}\qquad \text{for}\quad 0\leqq r<\infty\ .$$

Note that $\tilde{B}[0]$ is $\{0\}$. Let us denote by $\mathcal{O}(\tilde{B}(r))$ the space of holomorphic functions on $\tilde{B}(r)$. Then $\mathcal{O}(\tilde{B}(r))$ is an FS space. $\mathcal{O}(\tilde{B}(\infty))=\mathcal{O}(\boldsymbol{C}^{d+1})$ is the space of entire functions on $\boldsymbol{C}^{d+1}$. Let us define

$$\mathcal{O}(\tilde{B}[r]):=\operatorname*{ind\,lim}_{r'>r}\mathcal{O}(\tilde{B}(r'))\ .$$

Then $\mathcal{O}(\tilde{B}[r])$ is a DFS space. For $\lambda\in\boldsymbol{C}$, we put $\mathcal{O}_\lambda(\tilde{B}(r)):=\{f\in\mathcal{O}(\tilde{B}(r));\ (\Delta_z+\lambda^2)f=0\}$ and $\mathcal{O}_\lambda(\tilde{B}[r])=\{f\in\mathcal{O}(\tilde{B}[r]);\ (\Delta_z+\lambda^2)f=0\}$, where $\Delta_z=(\partial/\partial z_1)^2+(\partial/\partial z_2)^2+\cdots+(\partial/\partial z_{d+1})^2$. $P_n(\boldsymbol{C}^{d+1})$ denotes the space of homogeneous polynomials of degree n on $\boldsymbol{C}^{d+1}$.

For $r>0$ we put

$$X_{r,L}:=\{f\in\mathcal{O}(\boldsymbol{C}^{d+1});\ \sup_{z\in \boldsymbol{C}^{d+1}}|f(z)|\exp(-rL(z))<\infty\}\ .$$

Then $X_{r,L}$ is a Banach space with respect to the norm

$$\|f\|_{r,L}=\sup_{z\in \boldsymbol{C}^{d+1}}|f(z)|\exp(-rL(z))\ .$$

Define

$$\operatorname{Exp}(\boldsymbol{C}^{d+1}\colon (r\colon L)):=\operatorname*{proj\,lim}_{r'>r}X_{r',L}\qquad \text{for}\quad 0\leqq r<\infty\ ,$$

$$\operatorname{Exp}(\boldsymbol{C}^{d+1}\colon [r\colon L]):=\operatorname*{ind\,lim}_{r'<r}X_{r',L}\qquad \text{for}\quad 0<r\leqq\infty\ .$$

$\operatorname{Exp}(\boldsymbol{C}^{d+1}\colon (r\colon L))$ is an FS space and $\operatorname{Exp}(\boldsymbol{C}^{d+1}\colon [r\colon L])$ is a DFS space. $\operatorname{Exp}(\boldsymbol{C}^{d+1})=\operatorname{Exp}(\boldsymbol{C}^{d+1}\colon [\infty\colon L])$ is called the space of entire functions of exponential type. $\operatorname{Exp}'(\boldsymbol{C}^{d+1}\colon (r\colon L))$ and $\operatorname{Exp}'(\boldsymbol{C}^{d+1}\colon [r\colon L])$ denote the spaces dual to $\operatorname{Exp}(\boldsymbol{C}^{d+1}\colon (r\colon L))$ and $\operatorname{Exp}(\boldsymbol{C}^{d+1}\colon [r\colon L])$ respectively.

$\tilde{S}=\{z\in\boldsymbol{C}^{d+1};\ z_1^2+z_2^2+\cdots+z_{d+1}^2=1\}$ is the complex sphere. We define $\operatorname{Exp}(\tilde{S}\colon [r\colon L]):=\operatorname{Exp}(\boldsymbol{C}^{d+1}\colon[r\colon L])|_{\tilde{S}}$ and $\operatorname{Exp}(\tilde{S}\colon (r\colon L)):=\operatorname{Exp}(\boldsymbol{C}^{d+1}\colon (r\colon L))|_{\tilde{S}}$. We write $\operatorname{Exp}(\tilde{S}\colon [\infty\colon L])=\operatorname{Exp}(\tilde{S})$ and $\operatorname{Exp}(\tilde{S}\colon (0\colon L))=\operatorname{Exp}(\tilde{S}\colon (0))$. The topology of $\operatorname{Exp}(\tilde{S}\colon [r\colon L])$ is defined to be the quotient topology $\operatorname{Exp}(\boldsymbol{C}^{d+1}\colon [r\colon L])/\mathcal{J}_{\exp[r:L]}(\boldsymbol{C}^{d+1})$, where we put $\mathcal{J}_{\exp[r:L]}(\boldsymbol{C}^{d+1})=\{f\in\operatorname{Exp}(\boldsymbol{C}^{d+1}\colon [r\colon L]);\ f=0 \text{ on } \tilde{S}\}$. We also define the topologies of $\operatorname{Exp}(\tilde{S}\colon (r\colon L))$, $\operatorname{Exp}(\tilde{S})$ and $\operatorname{Exp}(\tilde{S}\colon (0))$ similarly. $\operatorname{Exp}'(\tilde{S}\colon [r\colon L])$, $\operatorname{Exp}'(\tilde{S}\colon (r\colon L))$, $\operatorname{Exp}'(\tilde{S})$ and $\operatorname{Exp}'(\tilde{S}\colon (0))$ denote the spaces dual to $\operatorname{Exp}(\tilde{S}\colon [r\colon L])$, $\operatorname{Exp}(\tilde{S}\colon (r\colon L))$, $\operatorname{Exp}(\tilde{S})$ and $\operatorname{Exp}(\tilde{S}\colon (0))$, respectively.

If f is a function or a functional on S, we denote by $S_n(f;\ )$ the n-th spherical harmonic component of f:

$$S_n(f;s)=N(n,d)\langle f,\ P_{n,d}(\ \cdot s)\rangle\qquad \text{for}\quad s\in S\ ,\tag{1.1}$$

where

$$N(n, d)=\dim H_{n,d}=\frac{(2n+d-1)(n+d-2)!}{n!\,(d-1)!} \tag{1.2}$$

and $P_{n,d}$ is the Legendre polynomial of degree n and of dimension $d+1$. We put $L_n(x)=\|x\|^n P_{n,d}(\alpha\cdot x/\|x\|)$ for fixed $\alpha\in S$. Then L_n is the unique homogeneous harmonic polynomial of degree n with the following properties:

$$L_n(Ax)=L_n(x) \quad \text{for all } A\in O(d+1) \text{ such that } A\alpha=\alpha\ .$$

$$L_n(\alpha)=1\ .$$

We see that $S_n(f;\ )$ belongs to $H_{n,d}$ for $n=0, 1, \cdots$.

Put $\Lambda_+=\{(n, k)\in \boldsymbol{Z}_+^2;\ n\equiv k \pmod 2 \text{ and } n\geqq k\}$, where $\boldsymbol{Z}_+=\{0, 1, 2, \cdots\}$. For any $F\in\mathcal{O}(\tilde{B}(r))$ we can determine uniquely $S_{n,k}(F;\ )\in H_{k,d}$ for every $(n, k)\in\Lambda_+$ in such a way that

$$F(z)=\sum_{(n,k)\in\Lambda_+}(\sqrt{z^2})^{n-k}\tilde{S}_{n,k}(F; z)\ , \tag{1.3}$$

where $z^2=z_1^2+z_2^2+\cdots+z_{d+1}^2$, and the right hand side of (1.3) converges uniformly on every compact set of $\tilde{B}(r)$. The $S_{n,k}(F;\ )$ is called the (n, k)-component of F (see [8] [9]).

Next we consider a complex cone M as follows:

$$M=\{z\in \boldsymbol{C}^{d+1}\backslash\{0\};\ z^2=0\}\ .$$

M is identified with the cotangent bundle of S minus its zero section. $P_n(M)$ denotes the restriction to M of $P_n(\boldsymbol{C}^{d+1})$. We put the subset N of M as follows:

$$N=\{z=x+iy\in M;\ \|x\|=\|y\|=1\}\ ,$$

where $x, y\in \boldsymbol{R}^{d+1}$. The unit cotangent bundle to S is identified with the subset N and we have $N\simeq O(d+1)/O(d-1)$. dN denotes the unique $O(d+1)$-invariant measure on N with $\int_N 1dN(z)=1$. We define the inner product $\langle\varphi, \psi\rangle_N=\int_N \varphi(z)\overline{\psi(z)}dN(z)$ and the norm $\|\varphi\|_{N,2}=\langle\varphi, \varphi\rangle_N^{1/2}$. It is known that for any $f_n, g_n\in H_{n,d}$

$$\langle f_n, g_n\rangle_S=2^{-2n}\frac{n!\,N(n, d)\Gamma((d+1)/2)}{\Gamma(n+(d+1)/2)}\langle\tilde{f}_n, \tilde{g}_n\rangle_N\ , \tag{1.4}$$

where

$$\langle f, g\rangle_S=\int_S f(s)\overline{g(s)}ds\ ,$$

(see, for example [4] [5] [13]). $\|\ \|_{N,\infty}$ denotes the sup norm on N.

§2. Some properties of $\mathcal{O}_\lambda(\tilde{B}(r))$.

THEOREM 2.1. *Let $F\in\mathcal{O}_\lambda(\tilde{B}(r))$ and $S_{n,k}$ be the (n, k)-component of F. Then we have*

$$S_{n,k}=(i\lambda/2)^{n-k}\frac{\Gamma(k+(d+1)/2)}{\Gamma((n-k)/2+1)\Gamma((n+k+d+1)/2)}S_{k,k} \tag{2.1}$$

for $(n, k)\in\Lambda_+$ and

$$\limsup_{n\to\infty}\|S_{n,n}\|_\infty^{1/n}\leq 1/r\ . \tag{2.2}$$

Conversely, if we are given a sequence of spherical harmonics $\{S_{n,k}\}_{(n,k)\in\Lambda_+}$ satisfying (2.1) *and* (2.2) *and if we put for $z\in\tilde{B}(r)$*

$$F(z)=\sum_{(n,k)\in\Lambda_+}(\sqrt{z^2})^{n-k}\tilde{S}_{n,k}(z)\ , \tag{2.3}$$

then the right hand side of (2.3) *converges uniformly and absolutely on every compact set of $\tilde{B}(r)$ and F belongs to $\mathcal{O}_\lambda(\tilde{B}(r))$. Furthermore we have*

$$\tilde{S}_{n,k}(z)=\tilde{S}_{n,k}(F; z)\qquad for\quad (n, k)\in\Lambda_+\ .$$

Remark the case $\lambda=0$ is known (see [9]).

PROOF. By [8] Theorem 3.2 we have

$$\begin{aligned}\Delta_zF(z)&=\sum_{(n,k)\in\Lambda_+}\Delta_z((\sqrt{z^2})^{n-k}\tilde{S}_{n,k}(z))\\&=\sum_{\substack{(n,k)\in\Lambda_+\\ n>k}}(n-k)(n+k+d-1)(\sqrt{z^2})^{n-k-2}\tilde{S}_{n,k}(z)\ .\end{aligned} \tag{2.4}$$

(2.4) gives us, for $0\leq k\leq n-2$ with $n\equiv k\ (\mathrm{mod}\ 2)$,

$$(n-k)(n+k+d-1)S_{n,k}=-\lambda^2S_{n-2,k}\ , \tag{2.5}$$

because $F\in\mathcal{O}_\lambda(\tilde{B}(r))$ and $H_{n,d}\perp H_{m,d}$ if $n\neq m$. (2.1) follows from (2.5). (2.2) follows from [8] Theorem 3.2 (3.33), since $\limsup_{n\to\infty}\|f_n\|_\infty^{1/n}=\limsup_{n\to\infty}\|f_n\|_\infty^{1/n}$ if $f_n\in H_{n,d}$.

Conversely, suppose we are given a sequence $\{S_{n,k}\}$ satisfying (2.1) and (2.2). By (2.2) for any $\varepsilon>0$ there exists a constant C_ε such that

$$\sup_{k\in Z_+}\|S_{k,k}\|_\infty \leqq C_\varepsilon(r-\varepsilon)^{-k}\,. \tag{2.6}$$

By (2.1) and (2.6) we have for $(n, k)\in \Lambda_+$

$$\begin{aligned}\|S_{n,k}\|_\infty &= (|\lambda|/2)^{n-k}\frac{\Gamma(k+(d+1)/2)\|S_{k,k}\|_\infty}{\Gamma((n-k)/2+1)\Gamma((n+k+d+1)/2)} \\ &\leqq (|\lambda|/2)^{n-k}\frac{C_\varepsilon(r-\varepsilon)^{-k}}{\Gamma((n-k)/2+1)}\,.\end{aligned} \tag{2.7}$$

From (2.7) we can see that $\sum_{k=0}^\infty \sum_{l=0}^\infty(\sqrt{z^2})^{2l}\tilde{S}_{k+2l,k}(z)$ converges uniformly and absolutely on every compact set of $\tilde{B}(r)$ since the Šilov boundary of $\tilde{B}(\rho)$ is $\{\rho e^{i\theta}s;\ 0\leqq\theta<2\pi,\ s\in S\}$ (see Hua [3]). So the right hand side of (2.3) converges uniformly and absolutely on every compact set of $\tilde{B}(r)$. Therefore F belongs to $\mathcal{O}(\tilde{B}(r))$ and $S_{n,k}(s)=S_{n,k}(F; s)$ for any $s\in S$. It is easy to show that $\Delta_z F=-\lambda^2 F$. Q.E.D.

REMARK 2.2. In Theorem 2.1 the condition (2.2) can be replaced by the following conditions

$$\limsup_{n\to\infty}\|\tilde{S}_{n,n}\|_{N,2}^{1/n}\leqq 2/r \tag{2.2$'$}$$

or

$$\limsup_{n\to\infty}\|\tilde{S}_{n,n}\|_{N,\infty}^{1/n}\leqq 2/r\ , \tag{2.2$''$}$$

since for any $f_n\in H_{n,d}$ we have

$$\begin{aligned}\limsup_{n\to\infty}\|\tilde{f}_n\|_{N,2}^{1/n} &= 2\limsup_{n\to\infty}\|f_n\|_2^{1/n}=2\limsup_{n\to\infty}\|f_n\|_\infty^{1/n} \\ &= \limsup_{n\to\infty}\|\tilde{f}_n\|_{N,\infty}^{1/n}\ ,\end{aligned}$$

by (1.4) and the fact that $\tilde{B}[2]\supset N$.

We recall the definition of the Bessel function of order ν, $\nu\neq -1, -2, \cdots$:

$$J_\nu(t)=\left(\frac{t}{2}\right)^\nu\sum_{k=0}^\infty\frac{(-1)^k}{k!\,\Gamma(\nu+k+1)}\left(\frac{t}{2}\right)^{2k}. \tag{2.8}$$

COROLLARY 2.3. *Let $F\in\mathcal{O}_\lambda(\tilde{B}(r))$. Then we have for any $z\in\tilde{B}(r)$*

$$F(z)=\sum_{n=0}^\infty \rho_0^{(d-1)/2}\frac{J_{n+(d-1)/2}(\lambda\sqrt{z^2})}{J_{n+(d-1)/2}(\lambda\rho_0)}(\sqrt{z^2})^{-n-(d-1)/2}\tilde{S}_n(F_{\rho_0}; z) \tag{2.9}$$

for every $\rho_0\in C$ such that $0<|\rho_0|<r$ and $J_{n+(d-1)/2}(\lambda\rho_0)\neq 0$ for any $n\in Z_+$.

where

$$S_n(F_{\rho_0}; s) = N(n, d)\int_S F(\rho_0 s')P_{n,d}(s'\cdot s)ds' . \tag{2.10}$$

PROOF. We put $S_{n,k}(F; s) = S_{n,k}(s)$. Then (2.1) gives

$$F(z) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(\sqrt{z^2})^{2l}\left(\frac{i\lambda}{2}\right)^{2l}\frac{\Gamma(k+(d+1)/2)}{l!\,\Gamma(k+l+(d+1)/2)}\tilde{S}_{k,k}(z) . \tag{2.11}$$

By (2.10) and (2.11) and the orthogonality of spherical harmonics we obtain

$$S_k(F_{\rho_0}; s) = S_{k,k}(s)\left\{\sum_{l=0}^{\infty}\rho_0^{2l+k}\left(\frac{i\lambda}{2}\right)^{2l}\frac{\Gamma(k+(d+1)/2)}{l!\,\Gamma(1+k+(d+1)/2)}\right\} \tag{2.12}$$

because the right hand side of (2.11) converges uniformly on the sphere $\rho_0 S$. (2.8) and (2.12) imply

$$\tilde{S}_{k,k}(z) = \rho_0^{-k}\frac{(\lambda\rho_0/2)^{k+(d-1)/2}}{\Gamma(k+(d+1)/2)J_{k+(d-1)/2}(\lambda\rho_0)}\tilde{S}_k(F_{\rho_0}; z) . \tag{2.13}$$

(2.9) follows from (2.8), (2.11) and (2.13). Q.E.D.

Our main theorem in this section is the following

THEOREM 2.4. (i) *The restriction mapping* $F\to F|_M$ *defines the following bijections*:

$$\alpha_\lambda\colon \mathcal{O}_\lambda(\tilde{B}(r)) \longrightarrow \mathcal{O}(\tilde{B}(r))|_M \qquad \textit{for any} \quad \lambda\in \boldsymbol{C} . \tag{2.14}$$

(ii) *If* $f\in\mathcal{O}(\tilde{B}(r))|_M$ *then* $\alpha_\lambda^{-1}f$ *can be expressed as follows*:

$$\alpha_\lambda^{-1}f(z) = \int_N f(\rho z'/2)K_\lambda(z, \bar{z}'/\rho)dN(z') \qquad \textit{for} \quad z\in\tilde{B}(r) , \tag{2.15}$$

where $L(z) < \rho < r$ *and*

$$K_\lambda(z, \xi) = \sum_{n=0}^{\infty}\{N(n, d)\Gamma(n+(d+1)/2)(\lambda\sqrt{z^2}/2)^{-n-(d-1)/2} \times J_{n+(d-1)/2}(\lambda\sqrt{z^2})(z\cdot\xi)^n\} . \tag{2.16}$$

In particular, if $\lambda=0$ *we have a "Poisson" formula*:

$$\alpha_0^{-1}f(z) = \int_N f(\rho z'/2)\frac{(1+\bar{z}'\cdot(z/\rho))}{(1-\bar{z}'\cdot(z/\rho))^d}dN(z') . \tag{2.17}$$

(iii) α_λ *is a linear topological isomorphism of* $\mathcal{O}_\lambda(\tilde{B}(r))$ *onto* $\mathcal{O}(\tilde{B}(r))|_M$

if we equip $\mathcal{O}(\tilde{B}(r))|_M$ *with the topology of uniform convergence on every compact set of* $\tilde{B}(r)\cap M$.

We need the following lemma in order to prove the theorem.

LEMMA 2.5. *For* $F\in\mathcal{O}(\tilde{B}(r))$ *we have for any* $z\in \boldsymbol{C}^{d+1}$

$$\tilde{S}_{n,n}(F;z)=N(n,d)\int_N F(\rho z'/2)\left(\bar{z}'\cdot\frac{z}{\rho}\right)^n dN(z')\,, \tag{2.18}$$

where ρ *is any real number such that* $0<\rho<r$ *and the right hand side of* (2.18) *is independent of* ρ.

PROOF. Since $S_{n,n}(F;\ )\in H_{n,d}$ it is valid for any $s\in S$

$$S_{n,n}(F;s)=N(n,d)\int_S S_{n,n}(s')P_{n,d}(s\cdot s')ds'\,. \tag{2.19}$$

(2.19) and (1.4) give

$$\begin{aligned} S_{n,n}(F;s)&=(N(n,d)/2^n)^2\frac{n!\,\Gamma((d+1)/2)}{\Gamma(n+(d+1)/2)}\\ &\quad\times\int_N \tilde{S}_{n,n}(F;z')\overline{\tilde{P}_{n,d}(z'\cdot s)}dN(z')\\ &=N(n,d)\int_N \tilde{S}_{n,n}(F;z/2)(\bar{z}'\cdot s)^n dN(z')\,, \end{aligned} \tag{2.20}$$

since $\tilde{P}_{n,d}(z\cdot s)=\{2^n\Gamma(n+(d+1)/2)(z\cdot s)^n\}/\{n!\,N(n,d)\Gamma((d+1)/2)\}$ on N. If $z'\in N$, $(\bar{z}'\cdot z)^n$ is a homogeneous harmonic polynomial of degree n in z. Then we have by (2.20)

$$\begin{aligned} \tilde{S}_{n,n}(F;z)&=N(n,d)\int_N \tilde{S}_{n,n}(F;z'/2)(\bar{z}'\cdot z)^n dN(z')\\ &=N(n,d)\int_N \tilde{S}_{n,n}(F;\rho z'/2)\left(\bar{z}'\cdot\frac{z}{\rho}\right)^n dN(z')\,. \end{aligned} \tag{2.21}$$

(2.18) follows from (2.21) because $\sum_{k=0}^{\infty}\tilde{S}_{k,k}(F;\rho z'/2)$ converges to $F(\rho z'/2)$ on N and $P_n(M)\perp P_m(M)$ $(n\neq m)$ on N. (2.21) is independent of ρ and so is (2.18). Q.E.D.

PROOF OF THEOREM 2.4. (i) For any $\lambda\in\boldsymbol{C}$ it is clear that $\mathcal{O}_\lambda(\tilde{B}(r))|_M\subset\mathcal{O}(\tilde{B}(r))|_M$. Let $F\in\mathcal{O}_\lambda(\tilde{B}(r))$. Then for any $z\in M\cap\tilde{B}(r)$ we have

$$F(z)=\sum_{n=0}^{\infty}\tilde{S}_{n,n}(F;z)\,. \tag{2.22}$$

By (2.22) we have for any $z' \in N$

$$F(rz'/4)=\sum_{n=0}^{\infty}(r/4)^n \tilde{S}_{n,n}(F; z') , \tag{2.23}$$

because $(r/4)N \subset \tilde{B}(r) \cap M$. If $\alpha_\lambda(F)=0$ we have $\tilde{S}_{n,n}=0$ on N by (2.23) and the orthogonality of homogeneous polynomials on N. So the spherical harmonic function $S_{n,n}=0$ by (1.4) and $F=0$ by (2.1). Therefore α_λ is injective.

Next for $f \in \mathcal{O}(\tilde{B}(r))$ we define the function F as follows:

$$F(z)=\sum_{(n,k)\in\Lambda_+}(\sqrt{z^2})^{n-k}\tilde{S}_{n,k}(z) ,$$

where

$$\tilde{S}_{n,k}(z)=\frac{\Gamma(k+(d+1)/2)(i\lambda/2)^{n-k}}{\Gamma((n-k)/2+1)\Gamma((n+k+d+1)/2)}\tilde{S}_{k,k}(f; z) .$$

As $f \in \mathcal{O}(\tilde{B}(r))$, $\limsup_{n\to\infty}\|S_{n,n}\|_\infty^{1/n}=\limsup_{n\to\infty}\|S_{n,n}(f; \ )\|_\infty^{1/n}\leqq 1/r$ by [8] Theorem 3.2. Hence by Theorem 2.1 $F \in \mathcal{O}_\lambda(\tilde{B}(r))$ and $F|_M=f|_M$. Therefore α_λ is surjective.

(ii) Suppose $f \in \mathcal{O}(\tilde{B}(r))|_M$. By the proof of surjectivity of α_λ in (i) and (2.18) we obtain for $z \in \tilde{B}(r)$

$$\alpha_\lambda^{-1}f(z)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left\{(\sqrt{z^2})^{2l}(i\lambda/2)^{2l}\frac{\Gamma(k+(d+1)/2)N(k, d)}{l!\,\Gamma(l+k+(d+1)/2)}\right. \\ \left.\times\int_N f(\rho z'/2)\left(\bar{z}'\cdot\frac{z}{\rho}\right)^k dN(z')\right\} . \tag{2.24}$$

Now we have

$$K_\lambda(z, \bar{z}'/\rho)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(\sqrt{z^2})^{2l}(i\lambda/2)^{2l}\frac{\Gamma(k+(d+1)/2)N(k, d)(\bar{z}'\cdot(z/\rho))^k}{l!\,\Gamma(l+k+(d+1)/2)} . \tag{2.25}$$

So we have for $z \in \tilde{B}(\rho)$

$$\alpha_\lambda^{-1}f(z)=\int_N f(\rho z'/2)K_\lambda(z, \bar{z}'/\rho)dN(z') ,$$

since the right hand side of (2.25) converges uniformly and absolutely on $\tilde{B}[\rho-\varepsilon]\times N$ for any $\varepsilon>0$. Hence we get (2.15). In particular, we obtain (2.17) since $K_0(z, \bar{z}'/\rho)=\sum_{k=0}^{\infty}N(k, d)(\bar{z}'\cdot(z/\rho))^k$ and $\sum_{k=0}^{\infty}N(k, d)x^k=(1+x)(1-x)^{-d}$ for $x \in \boldsymbol{C}$, $|x|<1$ (see, for example Müller [11] Lemma 3).

(iii) It is clear that α_λ is continuous. Suppose that $\{f_m\}_{m\in Z_+}\subset \mathcal{O}(\tilde{B}(r))|_M$ and $f_m \to 0$ in the topology of $\mathcal{O}(\tilde{B}(r))|_M$. By (2.15) we have for any ρ' with $0<\rho'<r$

$$(2.26)\qquad \alpha_\lambda^{-1}f_m(z)=\int_N f_m(\rho z'/2)K_\lambda(z, \bar{z}'/\rho)dN(z')$$

if $L(z)\leqq\rho'<\rho<r$. By (2.25), it is true that

$$(2.27)\qquad \sup_{\substack{L(z)\leqq\rho'\\ z'\in N}}|K_\lambda(z, \bar{z}'/\rho)|\leqq \sup_{\substack{L(z)\leqq\rho'\\ z'\in N}}\left\{\sum_{n=0}^{\infty}N(n, d)\left|z\cdot\frac{\bar{z}'}{\rho}\right|^n \exp(|\lambda|^2L(z)^2/4)\right\}.$$

(2.26) and (2.27) give that for any ρ' with $0<\rho'<r$

$$(2.28)\qquad \sup_{L(z)\leqq\rho'}|\alpha_\lambda^{-1}f_m(z)|\leqq \sup_{\substack{L(z)\leqq\rho'\\ z'\in N}}|f_m(\rho z'/2)||K_\lambda(z, \bar{z}'/\rho)|$$

$$\leqq\left(\exp\frac{|\lambda|^2\rho'^2}{4}\right)\frac{1+\rho'/\rho}{(1-\rho'/\rho)^d}\sup_{z\in(\rho/2)N}|f_m(z)|\,.$$

As $(\rho/2)N$ is the compact set of $\tilde{B}(r)\cap M$ we get from (2.28)

$$(2.29)\qquad \sup_{L(z)\leqq\rho'}|\alpha_\lambda^{-1}f_m(z)|\longrightarrow 0\quad (m\longrightarrow\infty)\,.$$

(2.29) means that $\alpha_\lambda^{-1}f_m$ converges to 0 in the topology of $\mathcal{O}_\lambda(\tilde{B}(r))$ as $m\rightarrow\infty$. Therefore α_λ^{-1} is continuous and α_λ is a linear topological isomorphism of $\mathcal{O}_\lambda(\tilde{B}(r))$ onto $\mathcal{O}(\tilde{B}(r))|_M$ by (i). Q.E.D.

COROLLARY 2.6. *If F belongs to $\mathcal{O}_\lambda(\tilde{B}(\rho))$ for $0<\rho<r$ and $F|_M$ belongs to $\mathcal{O}(\tilde{B}(r))|_M$, then F belongs to $\mathcal{O}_\lambda(\tilde{B}(r))$.*

§3. Fourier-Borel transformation.

The Fourier-Borel transformation P_λ for a functional $T\in \mathrm{Exp}'(C^{d+1}: (0; L))$ is defined by

$$(3.1)\qquad P_\lambda T(z)=\langle T_\xi, \exp(i\lambda\xi\cdot z)\rangle\qquad \text{for}\quad z\in C^{d+1}\,,$$

where $\lambda\in C\backslash\{0\}$ is a fixed constant. In this section we will determine the functional space on $\tilde{S}$ whose image by P_λ coincides with $\mathcal{O}_\lambda(\tilde{B}(r))$. Our main theorem in this section is the following

THEOREM 3.1. *The transformation P_λ establishes linear topological isomorphisms*

$$(3.2)\qquad P_\lambda\colon \mathrm{Exp}'(\tilde{S}\colon [|\lambda|r/2\colon L])\xrightarrow{\sim}\mathcal{O}_\lambda(\tilde{B}(r))\quad (0<r\leqq\infty)\,,$$

$$(3.3)\qquad P_\lambda\colon \mathrm{Exp}'(\tilde{S};\, (|\lambda|r/2\colon L))\xrightarrow{\sim}\mathcal{O}_\lambda(\tilde{B}[r])\quad (0\leqq r<\infty)\,.$$

In particular we have the following:

(3.4) $$P_\lambda\colon \operatorname{Exp}'(\tilde{S}\colon (0)) \xrightarrow{\sim} \mathcal{O}_\lambda(\{0\}) .$$

We need the following lemma and theorem in order to prove Theorem 3.1.

Lemma 3.2. *If S_n is the n-th spherical harmonic component of f', then*

(3.5) $$f' \in \operatorname{Exp}'(\tilde{S}\colon [r\colon L]) \Longleftrightarrow \limsup_{n\to\infty}(\|S_n\|_\infty/n!)^{1/n} \leqq 1/r ,$$

(3.6) $$f' \in \operatorname{Exp}'(\tilde{S}\colon (r\colon L)) \Longleftrightarrow \limsup_{n\to\infty}(\|S_n\|_\infty/n!)^{1/n} < 1/r ,$$

(3.7) $$f' \in \operatorname{Exp}'(\tilde{S}\colon (0)) \Longleftrightarrow \limsup_{n\to\infty}(\|S_n\|_\infty/n!)^{1/n} < \infty .$$

Theorem 3.3 (Martineau [6]). *The transformation P_λ establishes linear topological isomorphisms*

(3.8) $$P_\lambda\colon \operatorname{Exp}'(\boldsymbol{C}^{d+1}\colon [r\colon L]) \xrightarrow{\sim} \mathcal{O}(\tilde{B}^*(r/|\lambda|)) ,$$

(3.9) $$P_\lambda\colon \operatorname{Exp}'(\boldsymbol{C}^{d+1}\colon (r\colon L)) \xrightarrow{\sim} \mathcal{O}(\tilde{B}^*[r/|\lambda|]) ,$$

(3.10) $$P_\lambda\colon \operatorname{Exp}'(\boldsymbol{C}^{d+1}\colon (0\colon L)) \xrightarrow{\sim} \mathcal{O}(\{0\}) ,$$

where $\tilde{B}^*(A)=\{z\in \boldsymbol{C}^{d+1};\ L^*(z)<A\}$ and $\tilde{B}^*[A]=\{z\in \boldsymbol{C}^{d+1};\ L^*(z)\leqq A\}$.

Lemma 3.2 can be proved in the same way as in the proof of [10] Theorem 6.1.

Proof of Theorem 3.1. Let f' be in $\operatorname{Exp}'(\tilde{S}\colon [|\lambda| r/2\colon L])$. For any $z\in \tilde{B}^*(r/2)$ $P_\lambda f'(z)$ is well-defined and $P_\lambda f' \in \mathcal{O}_\lambda(\tilde{B}^*(r/2))$ because $\operatorname{Re}|i\lambda\xi\cdot z| \leqq |\lambda|\,|\xi\cdot z| \leqq |\lambda|\,L(\xi)L^*(z)$. Furthermore $P_\lambda f'(z)$ is well-defined for $z\in \tilde{B}(r)\cap M$ since $L^*(z)=L(z)/2$ for $z\in M$. If we put $S_n(s)=S_n(f';s)$, we have for $z\in M\cap \tilde{B}(r)$

(3.11) $$P_\lambda f'(z)=\sum_{n=0}^{\infty}\int_S S_n(s)\frac{(i\lambda z\cdot s)^n}{n!}ds$$

because $(s\cdot z)^n \in H_{n,d}$ if $z\in M$. Here we consider the following function:

(3.12) $$G(z)=\sum_{n=0}^{\infty}(i\lambda/2)^n\frac{\Gamma((d+1)/2)}{\Gamma(n+(d+1)/2)}\tilde{S}_n(z) \qquad \text{for} \quad z\in \boldsymbol{C}^{d+1} .$$

By (3.5)

$$\limsup_{n\to\infty}\left\{\frac{|i\lambda/2|^n\Gamma((d+1)/2)}{\Gamma(n+(d+1)/2)}\|S_n\|_\infty\right\}^{1/n}\leqq 1/r\ ,$$

so G belongs to $\mathcal{O}_0(\tilde{B}(r))$ by (2.1) and (2.2). For $z\in M$ and $\alpha\in S$ it is true that

$$\int_S P_{n,d}(\alpha\cdot s)(s\cdot z)^n ds=\frac{(z\cdot\alpha)^n}{N(n,d)}=\frac{1}{N(n,d)C_{n,n}}\tilde{P}_{n,d}(z\cdot\alpha)\ , \tag{3.13}$$

where $C_{n,n}=2^n\Gamma(n+(d+1)/2)/(N(n,d)\Gamma((d+1)/2)n!)$ is the coefficient of the n-th term of $P_{n,d}$. Since $\{P_{n,d}(\alpha\cdot\ )\}_{\alpha\in S}$ spans $H_{n,d}$, (3.11), (3.12) and (3.13) imply

$$P_\lambda f'=G \qquad \text{on} \quad M\cap\tilde{B}(r)\ . \tag{3.14}$$

By (3.14) and Corollary 2.6 we can see that $P_\lambda f'\in\mathcal{O}_\lambda(\tilde{B}(r))$.

Suppose $P_\lambda f'=0$. Then by (3.12) and (3.14) we have

$$\sum_{n=0}^\infty (i\lambda/2)^n\frac{\Gamma((d+1)/2)}{\Gamma(n+(d+1)/2)}\tilde{S}_n(z)=0 \qquad \text{on} \quad M\cap\tilde{B}(r)\ . \tag{3.15}$$

(3.15) gives $\tilde{S}_n=0$ on N since $P_n(M)\perp P_m(M)$ on N $(n\neq m)$ and $S_n=0$ by (1.4). Therefore $f'=0$ and P_λ is one-to-one.

For $F\in\mathcal{O}_\lambda(\tilde{B}(r))$ put $S_n(s)=\{\Gamma(n+(d+1)/2)/((i\lambda/2)^n\Gamma((d+1)/2))\}\times S_{n,n}(F;s)$. (2.2) and (3.5) imply that $f'=\sum_{n=0}^\infty S_n$ belongs to $\mathrm{Exp}'(\tilde{S}\colon[|\lambda|r/2\colon L])$. By (3.12) and (3.14) we see

$$P_\lambda f'(z)=F(z) \qquad \text{on} \quad M\cap\tilde{B}(r)\ . \tag{3.16}$$

From Theorem 2.4 we conclude that $F=P_\lambda f'$ and P_λ is surjective.

Suppose $\{f'_m\}_{m\in Z_+}\subset\mathrm{Exp}'(\tilde{S}\colon[|\lambda|r/2\colon L])$ and $f'_m\to 0$ in the topology of $\mathrm{Exp}'(\tilde{S}\colon[|\lambda|r/2\colon L])$ $(m\to\infty)$. $\mathrm{Exp}'(\tilde{S}\colon[|\lambda|r/2\colon L])\subset\mathrm{Exp}'(C^{d+1}\colon[|\lambda|r/2\colon L])$ and by (3.8), $P_\lambda f'_m$ converges to 0 on every compact set of $\tilde{B}^*(r/2)$ uniformly when $m\to\infty$. Since $\tilde{B}^*(r/2)\cap M=\tilde{B}(r)\cap M$, $\alpha_\lambda(P_\lambda f'_m)$ converges to 0 in the topology of $\mathcal{O}(\tilde{B}(r))|_M$. Hence $P_\lambda f'_m\to 0$ in the topology of $\mathcal{O}_\lambda(\tilde{B}(r))$ from Theorem 2.4 (iii). Therefore P_λ is a continuous mapping of $\mathrm{Exp}'(\tilde{S}\colon[|\lambda|r/2\colon L])$ onto $\mathcal{O}_\lambda(\tilde{B}(r))$. $\mathrm{Exp}'(\tilde{S}\colon[|\lambda|r/2;L])$ and $\mathcal{O}_\lambda(\tilde{B}(r))$ being FS spaces, P_λ^{-1} is also continuous by the closed graph theorem and we obtain (3.2).

By using (3.6), (3.7), (3.9) and (3.10) we can prove (3.3) and (3.4) similarly. Q.E.D.

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