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Remarks on Perturbations of Function Algebras

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Introduction

Let A be a function algebra on a compact Hausdorff space X. The purpose of this paper is to investigate small perturbations of the algebraic structure of A. In particular, we study the stability of direct sums of function algebras. K. Jarosz ([2], [3]) proved that if two function algebras A, B are both stable, then direct sum $A \oplus B$ of A and B is stable. In this note we deal with direct sums of function algebras $\{A_\lambda\}$ of infinitely many and give a condition under which the direct sum $\oplus_\lambda A_\lambda$ of $\{A_\lambda\}$ is stable (Theorem 1.2). Moreover it is shown that this condition is also a necessary one in order that $\oplus_\lambda A_\lambda$ is stable for $\{A_\lambda\}$ with some conditions (Theorem 1.1).

§1. Definitions and results.

For a function algebra A we write $\mathrm{Ch}\,A$ and ∂_A for the Choquet boundary and the Shilov boundary for A respectively. We consider a function algebra A as a closed subalgebra containing constant functions of the algebra $C(\partial_A)$ of all complex-valued continuous functions on ∂_A with the supremum norm. A closed subset F of ∂_A is called a *p-set* for A if for any open neighborhood U of F there is an $f \in A$ such that $f(s)=\|f\|=1$ $(s \in F)$ and $|f(s)|<1$ $(s \in \partial_A \setminus U)$ (cf. [1]).

Let A be a function algebra. By an *ε-perturbation* of A we mean any multiplication $\times$ defined on the Banach space A such that

$$\|f \times g - fg\| \leqq \varepsilon \|f\|\,\|g\| \qquad (f, g \in A)\,.$$

We call a function algebra A *stable* if there is an $\varepsilon>0$ such that for any ε-perturbation $\times$ of A algebras A and $(A, \times)$ are isomorphic. The stability is equivalent to the following: There is an $\varepsilon_1>0$ such that if T is any linear isomorphism from A onto a function algebra C with

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$T1_A=1_C$ and $\|T\|\,\|T^{-1}\|\leqq 1+\varepsilon_1$, then algebras A and C are isomorphic, where 1_A and 1_C are the identity of A and C respectively. We here put $\varepsilon(A)=\sup\{\varepsilon\geqq 0$: algebras A and C are isomorphic whenever there is a linear isomorphism T from A onto a function algebra C such that $T(1_A)=1_C$, and $\|T\|\,\|T^{-1}\|\leqq 1+\varepsilon\}$.

Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a family of function algebras, where Λ is an index set. $\{A_\lambda\}$ is called *uniformly stable* if $\inf_\lambda \varepsilon(A_\lambda)>0$. If A_λ is a function algebra on a compact Hausdorff space X_λ for each λ, we can define the direct sum $\bigoplus_\lambda A_\lambda$ of $\{A_\lambda\}$ as follows: $\bigoplus_\lambda A_\lambda=\{(f_\lambda)_{\lambda\in\Lambda}$: $f_\lambda\in A_\lambda$ for any λ and there is a $\gamma\in C$ such that for any $\varepsilon>0$ $\sup_{\lambda\in\Lambda-\{\lambda_1,\cdots,\lambda_n\}}\|f_\lambda-\gamma\|_\lambda\leqq\varepsilon$ for some $\lambda_1,\cdots,\lambda_n\in\Lambda\}$, where $\|\cdot\|_\lambda$ is the norm in A_λ.

Let X_0 be the sum of compact Hausdorff spaces X_λ $(\lambda\in\Lambda)$ and $X=X_0\cup\{p\}$ be the one-point compactification of X_0. Then the direct sum $\bigoplus_\lambda A_\lambda$ of $\{A_\lambda\}$ can be regarded as the space of continuous functions f on X such that $f|X_\lambda\in A_\lambda$ for each $\lambda\in\Lambda$.

In this paper we prove the following theorems. Theorem 1.2 shows that the uniform stability of $\{A_\lambda\}_{\lambda\in\Lambda}$ is a sufficient condition for stability of $\bigoplus_\lambda A_\lambda$. It was proved in [6] in the case where Λ is countable and ∂_A is a metric space. In Theorem 1.1 it is shown that the uniform stability is also a necessary condition for stability of $\bigoplus_\lambda A_\lambda$ for $\{A_\lambda\}$ with some conditions.

We here consider A_λ as a function algebra on its Shilov boundary.

THEOREM 1.1. *Let A_λ be a function algebra* $(\lambda\in\Lambda)$. *Suppose that* $\mathrm{Ch}\,A_\lambda$ *is connected for each* $\lambda\in\Lambda$. *If the direct sum* $A=\bigoplus_{\lambda\in\Lambda}A_\lambda$ *of* $\{A_\lambda\}$ *is stable, then* $\{A_\lambda\}$ *is uniformly stable.*

THEOREM 1.2. *Suppose that* $\{A_\lambda\}_{\lambda\in\Lambda}$ *be uniformly stable. Then the direct sum* $A=\bigoplus_{\lambda\in\Lambda}A_\lambda$ *of* $\{A_\lambda\}$ *is stable.*

§2. Proofs of the theorems.

PROOF OF THEOREM 1.1. Since $A=\bigoplus_\lambda A_\lambda$ is stable, there is an $\varepsilon>0$ such that if there is a linear isomorphism T from A onto a function algebra C with $T1_A=1_C$ and $\|T\|\,\|T^{-1}\|\leqq 1+\varepsilon$, then the algebras A and C are isomorphism.

Let A_{λ_0} be any space in $\{A_\lambda\}$ and let T_0 be a linear isomorphism from A_{λ_0} onto a function algebra C_0 on ∂_{C_0} such that $T_0(1_{A_{\lambda_0}})=1_{C_0}$ and $\|T_0\|\,\|T_0^{-1}\|\leqq 1+\varepsilon$. Let $\{A_\lambda\}_{\lambda\in\Lambda_1}$ be the collection of A_λ which is algebraically isomorphic to A_{λ_0}. Then we put $C_\lambda=C_0$ for any $\lambda\in\Lambda_1$, and $C_\mu=A_\mu$ for each $\mu\in\Lambda\backslash\Lambda_1$. A linear isomorphism T from A onto $C=\bigoplus_\lambda C_\lambda$ is defined as follows: $T=$

$\bigoplus_\lambda T_\lambda$, that is, $T((f_\lambda))=(T_\lambda(f_\lambda))$ for $(f_\lambda)\in A$, where $T_\lambda=T_0$ for any $\lambda\in\Lambda_1$, and T_μ is the identity from A_μ to A_μ for any $\mu\in\Lambda\backslash\Lambda_1$. Then we have $T(1_A)=1_C$ and $\|T\|\,\|T^{-1}\|\leqq 1+\varepsilon$. From the first part of the proof, the algebras $A=\bigoplus_\lambda A_\lambda$ and $C=\bigoplus_\lambda C_\lambda$ are isomorphic.

Let U be such the isomorphism from A onto C. If we put $f_{\lambda_0}(x)=1$ for $x\in\partial_{A_{\lambda_0}}$ and $f_{\lambda_0}(x)=0$ for $x\notin\partial_{A_{\lambda_0}}$, then $f_{\lambda_0}\in A$. We here note that there is a homeomorphism φ from $\operatorname{Ch} C_0$ onto $\operatorname{Ch} A_{\lambda_0}$ for a sufficiently small ε ([3], p. 8 (φ)). Hence ∂_{C_0} (= the clorsure of $\varphi^{-1}(\operatorname{Ch} A_{\lambda_0})$) is connected. From this we can assume that ∂_{C_λ} is connected for any $\lambda\in\Lambda$. Now it is not hard to see that $Uf_{\lambda_0}=g_{\mu_1}+g_{\mu_2}+\cdots+g_{\mu_s}$ for some $\mu_1,\mu_2,\cdots,\mu_s\in\Lambda$, where g_μ denotes the characteristic function for ∂_{C_μ} on ∂_C for any $\mu\in\Lambda$. For, since U is an isomorphism from the algebra A onto the algebra C and ∂_{C_λ} is connected for any $\lambda\in\Lambda$, $Uf_{\lambda_0}=1$ or 0 on ∂_{C_λ} and so $Uf_{\lambda_0}(x)$ is always equal to 1 or 0 for $x\in\partial_C-(\partial_{C_{\mu_1}}\cup\partial_{C_{\mu_2}}\cup\cdots\cup\partial_{C_{\mu_s}})$ (for some $\mu_1,\cdots,\mu_s\in\Lambda$). It implies $Uf_{\lambda_0}=g_{\mu_1}+g_{\mu_2}+\cdots+g_{\mu_s}$. We here see that $s=1$. If $s>1$, by putting $h=U^{-1}g_{\mu_1}$ $Uh=g_{\mu_1}=U(f_{\lambda_0}h)$ and so $h=f_{\lambda_0}h$. From this $h(x)=0$ $(x\notin\partial_{A_{\lambda_0}})$, $h=f_{\lambda_0}$ and $Uf_{\lambda_0}=g_{\mu_1}$. This shows that $s=1$ and algebras A_{λ_0} and A_{μ_1} are isomorphic. If $\mu_1\notin\Lambda_1$, then $C_{\mu_1}=A_{\mu_1}$ and so A_{λ_0} and A_{μ_1} are isomorphic. This contradiction shows that $\mu_1\in\Lambda_1$, $C_{\mu_1}=C_0$ and algebras A_{λ_0} and C_0 are isomorphic. It follows that for any $\lambda_0\in\Lambda$, $\varepsilon(A_{\lambda_0})\geqq\varepsilon$ and $\inf_{\lambda\in\Lambda}\varepsilon(A_\lambda)\geqq\varepsilon>0$. This proves the theorem.

PROOF OF THEOREM 1.2. Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be uniformly stable. In order to prove the theorem we must show the existence of $\varepsilon>0$ such that if T is a linear isomorphism from A onto a function algebra C with $T(1_A)=1_C$ and $\|T\|\,\|T^{-1}\|\leqq 1+\varepsilon$, then the algebras A and C are isomorphic.

To show that A and C are algebraically isomorphic, it is sufficient to prove that there is a collection $\{C_\lambda\}_{\lambda\in\Lambda}$ of function algebras such that the algebras A_λ and C_λ are isomorphic for each λ and the algebras C and $\bigoplus_\lambda C_\lambda$ are isomorphic. For, if $\phi_\lambda\colon A_\lambda\to C_\lambda$ is an algebraic isomorphism, then it is also an isometry ([4]). Hence the algebras $\bigoplus_\lambda A_\lambda$ and $\bigoplus_\lambda C_\lambda$ are isomorphic and so are A and C.

Now let T be a linear isomorphism from A onto C such that $T(1_A)=1_C$ and $\|T\|\,\|T^{-1}\|\leqq 1+\varepsilon$ for a sufficiently small ε. Then there is a homeomorphism φ from the Choquet boundary $\operatorname{Ch} C$ for C onto the Choquet boundary $\operatorname{Ch} A$ for A such that

$$(\varphi)\qquad |Tf(s)-f\circ\varphi(s)|\leqq\varepsilon_1\|f\|\qquad (f\in A,\ s\in\operatorname{Ch} C)\ ,$$

where ε_1 tends to zero with ε.

Here we easily see that ∂_{A_λ} $(\lambda\in\Lambda)$ are mutually disjoint and ∂_A is the one-point compactification $\bigcup_\lambda\partial_{A_\lambda}\cup\{p\}$ of $\bigcup_\lambda\partial_{A_\lambda}$.

Moreover we have

$$\mathrm{Ch}\, A = \bigcup_{\lambda\in\Lambda} \mathrm{Ch}\, A_\lambda \cup \{p\}\,.$$

If we put $Y_\lambda=[\varphi^{-1}(\mathrm{Ch}\, A_\lambda)]^-$ $(\lambda\in\Lambda)$ and $q=\varphi^{-1}(p)$, then (φ) tells us the following:

$$(\bigcup_{\mu\neq\lambda} Y_\mu)^- \cap Y_\lambda=\varnothing \qquad (\lambda\in\Lambda)$$

and

$$q\notin Y_\lambda \qquad (\lambda\in\Lambda)\,.$$

We also can prove that the Shilov boundary ∂_C for C is equal to $\cup_{\lambda\in\Lambda} Y_\lambda\cup\{q\}$. It is clear that $\partial_C\supset \cup_{\lambda\in\Lambda} Y_\lambda\cup\{q\}$. For any $x\in\partial_C=(\mathrm{Ch}\, C)^-$, there is a net $\{x_\alpha\}\subset \mathrm{Ch}\, C$ with $x_\alpha\to x$. If there is a $\lambda_0\in\Lambda$ such that for any α there is an $\alpha'>\alpha$ with $x_{\alpha'}\in\varphi^{-1}(\mathrm{Ch}\, A_{\lambda_0})$, then $x\in[\varphi^{-1}(\mathrm{Ch}\, A_{\lambda_0})]^-=Y_{\lambda_0}$. Otherwise, for any $\lambda\in\Lambda$ there is an α_0 such that for any $\alpha>\alpha_0$ $x_\alpha\notin\varphi^{-1}$ $(\mathrm{Ch}\, A_\lambda)$. Since $q\in\mathrm{Ch}\, C$, for any neighborhood $V(q)$ of q in $\mathrm{Ch}\, C$ $\varphi(V(q))$ is a neighborhood of p in $\mathrm{Ch}\, A$. So there are some $\lambda_1,\cdots,\lambda_n\in\Lambda$ such that $\varphi(V(q))\supset \cup_{\lambda\in\Lambda-\{\lambda_1,\cdots,\lambda_n\}} \mathrm{Ch}\, A_\lambda\cup\{p\}$. From this there is an α_1 such that $x_\alpha\in V(q)$ for any $\alpha>\alpha_1$ and $x_\alpha\to q$. This shows that $\partial_C=\cup_{\lambda\in\Lambda} Y_\lambda\cup\{q\}$ and ∂_C is the one-point compactification of the sum of $\{Y_\lambda\}_{\lambda\in\Lambda}$. Since it is shown that $\mathrm{Cl}[\varphi(Y_\lambda\cap\mathrm{Ch}\, C)]$ is a p-set for A, Y_λ is a p-set for C ([3] p. 87), where $\mathrm{Cl}(E)=\{s\in\partial_A: f(s)=0$ whenever $f|E=0,\ f\in A\}$ for $E\subset\partial_A$. Since Y_λ is open and closed, it is a peak set for C. If f_λ is the characteristic function for Y_λ, then $f_\lambda\in C$ for any λ.

Let us put $C_\lambda=C|Y_\lambda$ and $T_\lambda(f)=T(f)|Y_\lambda$ for $f\in A_\lambda$, then it is shown that T_λ is a linear isomorphism from A_λ onto C_λ with $\|T_\lambda\|\leqq\|T\|$ in view of (φ). If we put $h=Tf$ and $g=T_\lambda f$ for $f\in A_\lambda$, then $\|g\|=\|h\|$. By this fact and (φ), we obtain $\|T_\lambda^{-1}\|\leqq(1+\varepsilon_1)\|T^{-1}\|$. And it is simple to check that if e is the identity of $(A_\lambda, \times)$ (cf. [3], p. 8 and p. 22), $\|1-e\|\leqq \varepsilon_1[1+\varepsilon_1(1+\varepsilon)]$ since $T_\lambda e$ is the constant function 1 on Y_λ. By taking a sufficiently small ε, the uniform stability of $\{A_\lambda\}$ guarantees that the algebras A_λ and C_λ are isomorphic for any λ. To complete the proof, it remains only to show that the algebras C and $\bigoplus_\lambda C_\lambda$ are isomorphic.

If we write simply ff_λ for $(ff_\lambda)|Y_\lambda$, $(ff_\lambda)\in\bigoplus_\lambda C_\lambda$ for any $f\in C$. Hence, in order to prove that C and $\bigoplus_\lambda C_\lambda$ are isomorphic, it suffices to show that for any $(g_\lambda f_\lambda)\in\bigoplus_\lambda C_\lambda$ $(g_\lambda\in C_\lambda)$ there is an $h\in C$ such that $g_\lambda f_\lambda=hf_\lambda$ for any λ.

Now, we put $h=\sum_\lambda g_\lambda f_\lambda$. We show that h is the desired one. Since the characteristic function f_λ for Y_λ is in C for any λ, $1_{(\lambda_1,\cdots\lambda_n)}\in C$ if we denote by $1_{(\lambda_1,\dots,\lambda_n)}$ the characteristic function for $\cup_{\lambda\in\Lambda-(\lambda_1,\dots,\lambda_n)} Y_\lambda \cup \{q\}$ (λ_1, $\lambda_2, \cdots, \lambda_n \in \Lambda$). Let $g=(g_\lambda f_\lambda)$ ($g_\lambda \in C_\lambda$) be any function in $\bigoplus_\lambda C_\lambda$. Then there is a $\gamma \in C$ such that for any $\varepsilon>0$ there exist $\lambda_1, \lambda_2, \cdots, \lambda_n \in \Lambda$ satisfying $\|g_\lambda f_\lambda - \gamma\|_\lambda < \varepsilon$ for any $\lambda \in \Lambda-(\lambda_1, \lambda_2, \cdots, \lambda_n)$. Hence if we put $h_{\{\lambda_1,\dots,\lambda_n\}} = \sum_{i=1}^n g_{\lambda_i} f_{\lambda_i} + \gamma 1_{(\lambda_1,\lambda_2,\dots,\lambda_n)}$, then

$$\begin{aligned}\|h-h_{\{\lambda_1,\dots,\lambda_n\}}\| &= \Big\|\sum_{\lambda\in\Lambda-\{\lambda_1,\cdots,\lambda_n\}} g_\lambda f_\lambda - \gamma 1_{(\lambda_1,\lambda_2\cdots,\lambda_n)}\Big\| \\ &= \sup_{\lambda\in\Lambda-\{\lambda_1,\cdots,\lambda_n\}} \|g_\lambda f_\lambda - \gamma\|_\lambda \leqq \varepsilon .\end{aligned}$$

Since $h_{\{\lambda_1,\dots,\lambda_n\}} \in C$, it implies that $h \in C$ and $g_\lambda f_\lambda = h f_\lambda$ for any λ. This completes the proof.

COROLLARY 2.1. *Any direct sum of disc algebras is stable.*

PROOF. Since a disc algebra is stable ([5]), it is clear by Theorem 1.2.

REMARK. Let A be a function algebra on X, $X=\cup_\lambda K_\lambda$ (K_λ: compact) and $K_\lambda \cap K_\mu = \varnothing$ ($\lambda \neq \mu$). In general, even if $\{A|K_\lambda\}$ is a uniformly stable family of function algebras, A is not always stable. Such an example was given by K. Jarosz [3].

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