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On the Asymptotic Behaviors of the Spectrum of Quasi-Elliptic Pseudodifferential Operators on R^n

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Introduction

We consider the asymptotic behaviors of the spectrum of pseudodifferential operators on $\boldsymbol{R}^n$ containing the Schrödinger operator:

$$P(x, D) = -\Delta + V(x) \qquad \text{where} \quad \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} . \tag{0.1}$$

If the potential $V(x)$ is a positive C^∞-function satisfying $\lim_{|x|\to\infty} V(x) = \infty$, then $P(x, D)$ is essentially self-adjoint in $L^2(\boldsymbol{R}^n)$ and its unique self-adjoint extension P is positively definite and has a compact resolvent in $L^2(\boldsymbol{R}^n)$. Therefore the spectrum of P consists only of eigenvalues of finite multiplicity: $\lambda_1 \leqq \lambda_2 \leqq \cdots$, $\lim_{k\to\infty} \lambda_k = +\infty$ with repetition according to multiplicity. Let $N_P(\lambda)$ be the counting function of eigenvalues: $N_P(\lambda) = \mathrm{card}\{j; \lambda_j \leqq \lambda\}$.

In the particular case where $P(x, D)$ is the harmonic oscillator:

$$P(x, D) = -\Delta + V(x) \qquad \text{where} \quad V(x) = |x|^2 ,$$

the asymptotic behavior of $N_P(\lambda)$ is well known (cf. Helffer and Robert [4]). Moreover Helffer and Robert [6] have obtained the asymptotic formula of $N_P(\lambda)$ for a class of quasi-elliptic pseudodifferential operators containing the anharmonic oscillator:

$$P(x, D) = -\Delta + V(x) \qquad \text{where} \quad V(x) = a|x|^{2k} \ (a \text{ real} > 0,\ k \text{ integer} \geqq 2) .$$

They have found not only the first term but also the following several terms of $N_P(\lambda)$.

In this paper, we shall extend the result of [6] on $N_P(\lambda)$ for a class of quasi-elliptic pseudodifferential operators containing, in particular, the one on $\boldsymbol{R}^2$:

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(0.2) $\quad P(x, D)=-\Delta+V(x) \quad$ where $\quad V(x)=x_1^2+x_2^4+ax_2^3 \quad (a \text{ real} >0)$.

In order to obtain the asymptotic behavior of $N_P(\lambda)$, we may essentially examine the asymptotic behavior of

$$I(\mu)=\mathrm{Trace}\left[(2\pi)^{-1}\int e^{-itP}\hat{\rho}(t)e^{it\mu}dt\right]=\sum_{j=1}^{\infty}\rho(\mu-\lambda_j) \tag{0.3}$$

as $\mu\to+\infty$ where ρ is a suitable function belonging to the Schwartz space $\mathscr{S}(\boldsymbol{R}^n)$ (cf. Duistermaat and Guillemin [2]). In order to do so, the authors in [4], [6] and [2] approximate e^{-itP} by the Fourier integral operator for small t. In contrast to this, our method is more direct: First of all, we construct the complex powers P^{-s} ($s\in\boldsymbol{C}$) of P. Then it is well known that if the real part of s is sufficiently large, P^{-s} are of trace class and the trace has a meromorphic extension $Z_P(s)$ in $\boldsymbol{C}$. Then by using the inverse Mellin transformation we have for $\mathrm{Re}\, z>0$,

$$\theta_P(z)=\mathrm{Trace}\, e^{-zP}=\frac{1}{2\pi i}\int_{\mathrm{Re}\, s=c} z^{-s}Z_P(s)\Gamma(s)ds \tag{0.4}$$

where $c>0$ is sufficiently large and $\Gamma(s)$ is the Γ-function. Shifting $c\to-\infty$, we have the asymptotic behavior of $\theta_P(z)$ as $z\to 0$, $\mathrm{Re}\, z>0$. Finally we show that

$$I(\mu)=\lim_{\varepsilon\downarrow 0}(2\pi)^{-1}\int\theta_P(\varepsilon+it)\hat{\rho}(t)e^{i\mu t}dt\ , \tag{0.5}$$

and we can obtain the asymptotic formula of $N_P(\lambda)$ using the one of $\theta_P(\varepsilon+it)$. Consequently, for example, for the operator (0.2), we have:

$$N_P(\lambda)=\frac{2}{21\pi}B\left(\frac{1}{2},\frac{1}{4}\right)\lambda^{7/4}+\frac{a^2}{20\pi}B\left(\frac{1}{2},\frac{3}{4}\right)\lambda^{5/4}+O(\lambda^{7/8})\ , \qquad \lambda\to\infty\ .$$

Here $B(\cdot,\cdot)$ is the Beta function.

§1. Main theorems.

In this section we shall state the main theorems. Let $P(x, D)$ be a pseudodifferential operator with the symbol $p(x,\xi)$:

$$P(x, D)u(x)=(2\pi)^{-n}\int e^{i\langle x,\xi\rangle}p(x,\xi)\hat{u}(\xi)d\xi\ , \qquad u\in\mathscr{S}(\boldsymbol{R}^n) \tag{1.1}$$

where $\langle x,\xi\rangle=\sum_{i=1}^n x_i\xi_i$ and $\hat{u}(\xi)$ is the Fourier transformation of u:

$$\hat{u}(\xi)=\int e^{-i\langle x,\xi\rangle}u(x)dx$$

and $\mathscr{S}(\boldsymbol{R}^n)$ denotes the totality of rapidly decreasing C^∞-functions.

DEFINITION 1.1. Let m be a real number and $(h; k)=(h_1, h_2, \cdots, h_n; k_1, k_2, \cdots, k_n)$ a fixed multi-index such that $h_j, k_j \geqq 1$ for every $j=1, 2, \cdots, n$. Then the space $S^m_{(h;k)}$ is the set of all symbols $p(x, \xi) \in C^\infty(\boldsymbol{R}^n \times \boldsymbol{R}^n)$ satisfying the following:

(1.2) There exists a sequence of functions $\{p_{m-j}(x, \xi)\}_{j=0,1,\cdots}$ where $p_{m-j}(x, \xi)$ are C^∞-functions in $\boldsymbol{R}^{2n}\setminus 0$ and quasi-homogeneous of degree $m-j$ of type $(h; k)$ such that

$$p(x, \xi) \sim \sum_{j=0}^{\infty} p_{m-j}(x, \xi) .$$

Here the quasi-homogeneity of $p_{m-j}(x, \xi)$ of degree $m-j$ of type $(h; k)$ means:

$$p_{m-j}(\lambda^{h_1}x_1, \cdots, \lambda^{h_n}x_n, \lambda^{k_1}\xi_1, \cdots, \lambda^{k_n}\xi_n)=\lambda^{m-j}p_{m-j}(x, \xi)$$

for all $\lambda>0$ and $(x, \xi) \in \boldsymbol{R}^{2n}\setminus 0$.

For brevity of the notations, we put

(1.3) $T=$ the least common multiple of $\{h_1, \cdots, h_n, k_1, \cdots, k_n\}$, $p_j=T/h_j$, $q_j=T/k_j$ and $\lambda(x, \xi)=[1+\sum_{j=1}^n \{|x_j|^{2p_j}+|\xi_j|^{2q_j}\}]^{1/(2T)}$.

Then the meaning of the asymptotic sum of (1.2) is as follows: For every integer $N\geqq 1$ and any multi-indices α, β, there exists a positive constant $C=C_{N,\alpha,\beta}$ such that

$$\left|D_x^\alpha D_\xi^\beta\left[p(x, \xi)-\sum_{j=0}^{N-1} p_{m-j}(x, \xi)\right]\right| \leqq C\lambda(x, \xi)^{m-N}$$

for all $(x, \xi) \in \boldsymbol{R}^{2n}$ such that $\lambda(x, \xi)\geqq 1$. Finally the class of pseudodifferential operators of type (1.1) with symbols in $S^m_{(h;k)}$ is denoted by $\mathrm{OP}S^m_{(h;k)}$.

Throughout this paper we impose the following hypotheses.

(H.1) The order of $P(x, D)$ is positive, i.e., $m>0$.

(H.2) The symbol $p(x, \xi)$ is real valued and $P(x, D)$ is quasi-elliptic, i.e.,

$$p_m(x, \xi)>0 \quad \text{for all} \quad (x, \xi) \in \boldsymbol{R}^{2n}\setminus 0 .$$

(H.3) $P(x, D)$ is formally self-adjoint, i.e., for any $u, v \in \mathscr{S}(\boldsymbol{R}^n)$,

$$(P(x, D)u, v)=(u, P(x, D)v) \qquad \text{where} \quad (u, v)=\int u(x)\overline{v(x)}dx ;$$

If we define an operator P_0 on $L^2(\mathbf{R}^n)$ with definition domain $D(P_0)=\mathscr{S}(\mathbf{R}^n)$ so that $P_0u=P(x, D)u$, $u\in D(P_0)$, it is well known under (H.1)$\sim$(H.3) that P_0 is essentially self-adjoint and the closure P of P_0 has the spectrum consisting only of eigenvalues of finite multiplicity. Moreover P is semi-bounded from below, i.e., there exists a real number C such that for all $u\in\mathscr{S}(\mathbf{R}^n)$, $((P+C)u, u)\geqq 0$ (cf. [3]). Let $\lambda_1\leqq\lambda_2\leqq\cdots$, $\lim_{k\to\infty}\lambda_k=\infty$ be the sequence of eigenvalues with repetition according to multiplicity and let $N_P(\lambda)$ be the counting function as in introduction. In addition to (H.1)$\sim$(H.3), if we assume:

(H.4) $\qquad P$ is positively definite, i.e., $\lambda_1>0$,

we can construct complex powers P^{-s} by the spectral resolution of P and it follows from Seeley [9] that P^{-s} is pseudodifferential operators of order $-m\mathrm{Re}\, s$.

If we define

$$Q=P^{2M/m} \quad \text{where } M=\frac{|h|+|k|}{2n},\ |h|=\sum_{j=1}^{n} h_j,\ |k|=\sum_{j=1}^{n} k_j\ , \tag{1.4}$$

then Q has also the discrete spectrum consisting only of the eigenvalues $\mu_j=\lambda_j^{2M/m}$. By Robert [8] and also Aramaki [1], we have $\mu_j\sim j^{1/n}$ (cf. Remark 3.3). Thus we can define

$$\theta_Q(z)=\mathrm{Trace}\, e^{-zQ}=\sum_{j=1}^{\infty} e^{-z\mu_j} \quad \text{for } \mathrm{Re}\, z>0\ . \tag{1.5}$$

For the asymptotic behavior of $\theta_Q(z)$ as $z\to 0$, we have:

THEOREM 1. *Assume that* $P(x, D)\in \mathrm{OPS}^m_{(h;k)}$ *satisfies* (H.1)$\sim$(H.4). *Let* Q *and* $\theta_Q(z)$ *be as in* (1.4) *and* (1.5). *Then we have*
(i) $\theta_Q(z)$ *is holomorphic in* z *for* $\mathrm{Re}\, z>0$.
(ii) $\theta_Q(z)\sim \sum_{-n+j/(2M)\notin Z_+} \Gamma\left(n-\frac{j}{2M}\right)A_j z^{-n+j/(2M)}+\sum_{-n+j/(2M)=l\in Z_+} B_j z^l\log z+\sum_{l=0}^{\infty} C_l z^l$
as $z\to 0$, $\mathrm{Re}\, z>0$. *Here*

$$A_0=\frac{1}{2M}(2\pi)^{-n}\int_{S(h;k)} p_m(\sigma)^{-(|h|+|k|)/m}d\sigma\ , \tag{1.6}$$

$$A_j=\frac{1}{2M}\sum_{l=1}^{2j}\frac{1}{m^l l!}\prod_{i=0}^{l-1}(|h|+|k|-j+mi)(2\pi)^{-n}$$
$$\times\int_{S(h;k)} \check{d}_{lj}(\sigma)p_m(\sigma)^{(j-|h|-|k|-ml)/m}d\sigma$$

for $j=1, 2, \cdots$, $B_j=A_j(-1)^{l+1}/l!$ *and* C_l *are some constants where*

$$S_{(h;k)}=\{(x,\xi)\in \boldsymbol{R}^{2n};\lambda_0(x,\xi)=1\}\ ,\quad \lambda_0(x,\xi)=\left\{\sum_{j=1}^{n}(|x_j|^{2/h_j}+|\xi_j|^{2/k_j})\right\}^{1/2}$$

and $d\sigma$ *is the Riemannian density on* $S_{(h;k)}$.

We note that the asymptotic sum in (ii) means: For every integer $N>0$, there exist an integer c and a constant $C>0$ such that for all z, $|z|<1$, $\mathrm{Re}\,z>0$,

$$(1.7)\qquad \left|z^{-c}\left\{\theta_Q(z)-\left[\sum_{\substack{-n+j/(2M)\notin \boldsymbol{Z}_+\\ 0\leq j\leq N}}\Gamma\left(n-\frac{j}{2M}\right)A_j z^{-n+j/(2M)}\right.\right.\right.$$
$$\left.\left.\left.+\sum_{\substack{-n+j/(2M)=l\in \boldsymbol{Z}_+\\ 0\leq j\leq N}}B_j z^l\log z+\sum_{l=0}^{N}C_l z^l\right]\right\}\right|\leq C\ .$$

Next we choose $\rho\in\mathscr{S}(\boldsymbol{R})$ satisfying the followings (cf. [3]):

$$(1.8)\qquad \mathrm{supp}\,\hat{\rho}\ \text{is in a neighborhood of 0 and}\ \rho\geq 0,\ \rho(0)>0,\ \hat{\rho}(0)=1\ .$$

Then we have

THEOREM 2. *Assume that* $P(x, D)$ *satisfies* (H.1)$\sim$(H.4) *and let* ρ *be a function satisfying* (1.8). *Then we have*

$$I(\mu)=\int\rho(\mu-\tau)dN_Q(\tau)=\sum_{j=1}^{\infty}\rho(\mu-\mu_j)=\sum_{j=0}^{M_0}\mathrm{Re}A_j\,\mu^{n-1-j/(2M)}+R_n(\mu)$$

where $M_0=\mathrm{Max}\{j\in \boldsymbol{N};\,j<2M\}$ *and*

$$R_n(\mu)=O(\mu^{n-2})\qquad as\quad \mu\to\infty\quad if\quad n\geq 2\ ,$$
$$R_1(\mu)=O(\mu^{-1-\delta})\qquad for\ some\quad \delta>0\quad as\quad \mu\to\infty\ .$$

Finally we can state the result on the asymptotic behavior of $N_P(\lambda)$.

THEOREM 3. *Assume that* $P(x, D)$ *satisfies* (H.1)$\sim$(H.3). *Then we have*

$$N_P(\lambda)=\sum_{j=0}^{M_0}D_j\lambda^{(|h|+|k|-j)/m}+O(\lambda^{(n-1)(|h|+|k|)/(mn)})\qquad as\quad \lambda\to\infty\ ,$$

where

$$D_0=(2\pi)^{-n}\int_{p_m(x,\xi)\leq 1}dxd\xi\ ,$$
$$D_j=\sum_{l=1}^{2j}\frac{1}{m^l l!}\prod_{i=1}^{l}(|h|+|k|-j+mi)(2\pi)^{-n}\int_{p_m(x,\xi)\leq 1}\tilde{d}_{lj}(x,\xi)dxd\xi$$

for $1\leq j\leq M_0$.

$\tilde{d}_{lj}(x, \xi)$ in this theorem are determined later ((2.4)) and we note that they depend only on the symbol of $P(x, D)$. For example, we have

$$\begin{aligned}
&\tilde{d}_{11}(x, \xi)=-p_{m-1}(x, \xi)\,, \qquad \tilde{d}_{21}=0\,,\\
&\tilde{d}_{22}(x, \xi)=p_{m-1}(x, \xi)^2\,, \qquad \tilde{d}_{12}(x, \xi)=-p_{m-2}(x, \xi)\,, \qquad \tilde{d}_{32}=\tilde{d}_{42}=0\,,\\
&\tilde{d}_{33}(x, \xi)=-p_{m-1}(x, \xi)^3\,, \qquad \tilde{d}_{23}(x, \xi)=2p_{m-2}(x, \xi)p_{m-1}(x, \xi)\,,\\
&\tilde{d}_{13}(x, \xi)=-p_{m-3}(x, \xi)\,, \qquad \tilde{d}_{43}=\tilde{d}_{53}=\tilde{d}_{63}=0\,.
\end{aligned}$$

REMARK. For the proof of Theorem 3, without loss of generality, we can assume that $P(x, D)$ satisfies (H.4).

§2. Preliminaries.

In this section we consider the properties of parametrices of $P(x, D)-\zeta$ for some $\zeta\in C$ in order to construct complex powers of P.

By (H.2), there exists a positive constant γ_0 such that $p_m(x, \xi)\geqq\gamma_0$ if $\lambda_0(x, \xi)\geqq 1/2$. Choose a function $\chi\in C^\infty(R^{2n})$ such that

$$\chi(x, \xi)=\begin{cases}1 & \text{if } \lambda_0(x, \xi)\geqq 1\\ 0 & \text{if } \lambda_0(x, \xi)\leqq 1/2\,.\end{cases}$$

For $\zeta\notin[\gamma_0, +\infty)$, we can define:

(2.1) $$b_{\zeta,-m}(x, \xi)=\chi(x, \xi)(p_m(x, \xi)-\zeta)^{-1}\,,$$

and for $j\geqq 1$

(2.2) $$b_{\zeta,-m-j}(x, \xi)=-b_{\zeta,-m}(x, \xi)\sum_{\substack{i+l+\langle\alpha, h+k\rangle=j\\ 0\leqq i<j}}\frac{1}{\alpha!}p_{m-l}(x, \xi)^{(\alpha)}D_x^\alpha b_{\zeta,-m-i}(x, \xi)\,.$$

Then $b_{\zeta,-m-j}(x, \xi)$ is quasi-homogeneous for $\lambda_0(x, \xi)\geqq 1$ of degree $-m-j$ in the sense: If $\rho\geqq 1$ and ζ, $\rho^m\zeta\notin[\gamma_0, +\infty)$, $\lambda_0(x, \xi)\geqq 1$,

$$b_{\rho^m\zeta,-m-j}(\rho^{k_1}x_1, \cdots, \rho^{k_n}x_n, \rho^{k_1}\xi_1, \cdots, \rho^{k_n}\xi_n)=\rho^{-m-j}b_{\zeta,-m-j}(x, \xi)\,.$$

On the other hand, we can also write

(2.3) $$b_{\zeta,-m-j}(x, \xi)=\sum_{l=1}^{2j}d_{lj}(x, \xi)(p_m(x, \xi)-\zeta)^{-l-1} \qquad \text{for } j\geqq 1\,.$$

Here $d_{lj}(x, \xi)$ are independent of ζ and quasi-homogeneous of degree $ml-j$ for $\lambda_0(x, \xi)\geqq 1$.

(2.4) We write the quasi-homogeneous extension of $d_{lj}(x, \xi)$ for $(x, \xi)\neq 0$ by $\check{d}_{lj}(x, \xi)$ and $\mathrm{Re}\,\check{d}_{lj}(x, \xi)$ by $\tilde{d}_{lj}(x, \xi)$.

For every $b_{\zeta,-m-j}$, we have the following estimate.

Lemma 2.1 (cf. Helffer and Robert [7] and [8]). *For every* $j\geqq 0$ *and multi-indices* α, β, *there exists a constant* $C=C_{j,\alpha,\beta}>0$ *such that*

$$(2.5)\qquad \begin{aligned}&|D_x^\alpha D_\xi^\beta b_{\zeta,-m-j}(x,\xi)|\\ &\leqq C\lambda(x,\xi)^{-j-\langle\alpha,h\rangle-\langle\beta,k\rangle}(p_m(x,\xi)+|\zeta|)^{-1}\Big(\frac{|\zeta|}{d(\zeta)}\Big)^{2j+|\alpha|+|\beta|+1}\end{aligned}$$

for all $(x,\xi)\in \boldsymbol{R}^{2n}$, $\zeta\notin[\gamma_0,+\infty)$ *where* $d(\zeta)=\mathrm{dist}(\zeta,[\gamma_0,+\infty))$.

Proof. At first we consider the case $j=0$. We claim:

(2.6) For any multi-indices α, β, we have

$$D_x^\alpha D_\xi^\beta b_{\zeta,-m}(x,\xi)=\sum_{l=0}^{|\alpha|+|\beta|}C_l(x,\xi)(p_m(x,\xi)-\zeta)^{-l-1}$$

where C_l are independent of ζ and satisfy:

(2.7) For every multi-indices γ, δ, there exists a constant $C_{\gamma,\delta}$ independent of ζ such that

$$|D_x^\gamma D_\xi^\delta C_l(x,\xi)|\leqq C_{\gamma,\delta}\lambda(x,\xi)^{ml-\langle\alpha,h\rangle-\langle\beta,k\rangle-\langle\gamma,h\rangle-\langle\delta,k\rangle}\,.$$

In fact, we prove (2.6) by induction on $|\alpha|+|\beta|$. (2.6) is clear for $|\alpha|+|\beta|=0$. We assume that (2.6) is true for $|\alpha|+|\beta|=t$ and let $|\alpha|+|\beta|=t+1$. Without loss of generality, we may assume $\alpha_1\neq 0$ and let $\alpha=(1,0,\cdots,0)+\alpha'$. Then

$$\begin{aligned}D_x^\alpha D_\xi^\beta b_{\zeta,-m}&=\sum_{l=0}^{t}[(D_{x_1}C_l)(p_m-\zeta)^{-l-1}-(l+1)C_l(D_{x_1}p_m)(p_m-\zeta)^{-l-2}]\\&=\sum_{l=0}^{t}(D_{x_1}C_l)(p_m-\zeta)^{-l-1}-\sum_{l=1}^{t+1}lC_{l-1}(D_{x_1}p_m)(p_m-\zeta)^{-l-1}\,.\end{aligned}$$

Obviously $D_{x_1}C_l$ and $C_{l-1}(D_{x_1}p_m)$ satisfy (2.7). Thus (2.6) is proved. Since

$$|(p_m-\zeta)^{-1}|\leqq\frac{|\zeta|}{d(\zeta)}(p_m+|\zeta|)^{-1}\qquad\text{for}\quad \lambda_0(x,\xi)\geqq 1/2\,,$$

we have for some constant C independent of ζ,

$$\begin{aligned}|D_x^\alpha D_\xi^\beta b_{\zeta,-m}|&\leqq C\sum_{l=0}^{|\alpha|+|\beta|}\lambda(x,\xi)^{ml-\langle\alpha,h\rangle-\langle\beta,k\rangle}(p_m+|\zeta|)^{-l-1}\Big(\frac{|\zeta|}{d(\zeta)}\Big)^{l+1}\\&\leqq C\sum_{l=0}^{|\alpha|+|\beta|}\lambda(x,\xi)^{-\langle\alpha,h\rangle-\langle\beta,k\rangle}(p_m+|\zeta|)^{-1}\Big(\frac{|\zeta|}{d(\zeta)}\Big)^{l+1}\,.\end{aligned}$$

If we note that there exists a positive constant $C'>0$ such that $|\zeta|/d(\zeta)\geqq C'$ for all $\zeta\notin[\gamma_0,+\infty)$, we have (2.5) for $j=0$. For general j, we use (2.2) and induction on j. This completes the proof.

Now we define $b_\zeta^{(N)}(x,\xi)=\sum_{j=0}^{N-1}b_{\zeta,-m-j}(x,\xi)$ and write

$$(p-\zeta)\# b_\zeta^{(N)}=1+r_\zeta^{(N)} \tag{2.8}$$

where $\#$ means: $(p-\zeta)\# b_\zeta^{(N)}\sim\sum_\alpha (1/\alpha!)(p-\zeta)^{(\alpha)}D_x^\alpha b_\zeta^{(N)}$.

For $r_\zeta^{(N)}$, we have the following estimate.

LEMMA 2.2 (cf. [7]). *For every $N\geqq 1$ and any multi-indices α, β, there exist an integer $\tilde N>0$ and a positive constant C which are independent of ζ such that*

$$|D_x^\alpha D_\xi^\beta r_\zeta^{(N)}(x,\xi)|\leqq C\lambda(x,\xi)^{m-N-\langle\alpha,h\rangle-\langle\beta,k\rangle}\left(\frac{|\zeta|}{d(\zeta)}\right)^{\tilde N}|\zeta|^{-1} \tag{2.9}$$

for all $(x,\xi)\in \boldsymbol{R}^{2n}$ and $0\neq\zeta\notin[\gamma_0,+\infty)$.

PROOF. Since $p\# b_\zeta^{(N)}=1+\zeta b_\zeta^{(N)}+r_\zeta^{(N)}$ and we can write

$$p\# b_\zeta^{(N)}=\sum_{|\alpha|<N}\frac{1}{\alpha!}p^{(\alpha)}D_x^\alpha b_\zeta^{(N)}+r_{\zeta,N}^{(N)},$$

we have

$$r_\zeta^{(N)}-r_{\zeta,N}^{(N)}=\sum_{|\alpha|<N}\frac{1}{\alpha!}p^{(\alpha)}D_x^\alpha b_\zeta^{(N)}-\zeta b_\zeta^{(N)}-1. \tag{2.10}$$

By the composition formula of two pseudodifferential operators, for every multi-indices α, β, there exist constants $C_{\alpha,\beta}$ and $C'_{\alpha,\beta}$ independent of ζ, and positive integers h_0, k_0 such that

$$\begin{aligned}|\zeta||\lambda(x,\xi)^{N-m+\langle\alpha,h\rangle+\langle\beta,k\rangle}&D_x^\alpha D_\xi^\beta r_{\zeta,N}^{(N)}|\\ \leqq C_{\alpha,\beta}\{&\sum_{|\alpha|\leq h_0,|\beta|\leq k_0}\sup_{(x,\xi)}|\lambda(x,\xi)^{-m+\langle\alpha,h\rangle+\langle\beta,k\rangle}D_x^\alpha D_\xi^\beta p|\}\\ \times\{&\sum_{|\alpha|\leq h_0,|\beta|\leq k_0}|\zeta|\sup_{(x,\xi)}|\lambda(x,\xi)^{m+\langle\alpha,h\rangle+\langle\beta,k\rangle}D_x^\alpha D_\xi^\beta b_\zeta^{(N)}|\}\\ \leqq C'_{\alpha,\beta}&\left(\frac{|\zeta|}{d(\zeta)}\right)^{h_0+k_0+N}.\end{aligned}$$

Thus $r_{\zeta,N}^{(N)}$ satisfies (2.9). If we put $E=r_\zeta^{(N)}-r_{\zeta,N}^{(N)}$, then by (2.10), (2.1) and (2.2), we have

$$\begin{aligned}E&=\sum_{\substack{i,l,|\gamma|\leq N-1\\ i+l+|\gamma|\neq 0}}\frac{1}{\gamma!}p_{m-l}^{(\gamma)}D_x^\gamma b_{\zeta,-m-i}+\sum_{i,|\gamma|\leq N-1}\frac{1}{\gamma!}\left(p-\sum_{l=0}^{N-1}p_{m-l}\right)^{(\gamma)}D_x^\gamma b_{\zeta,-m-i}\\ &=\sum_{\substack{i+l+\langle\gamma,h+k\rangle\geq N\\ i,l,|\gamma|\leq N-1}}\frac{1}{\gamma!}p_{m-l}^{(\gamma)}D_x^\gamma b_{\zeta,-m-i}+\sum_{i,|\gamma|\leq N-1}\frac{1}{\gamma!}\left(p-\sum_{l=0}^{N-1}p_{m-l}\right)^{(\gamma)}D_x^\gamma b_{\zeta,-m-i}.\end{aligned}$$

Therefore by Lemma 2.1, we have for some constants $C_{\beta',\beta''}^{\alpha',\alpha''}$, C and C',

$$|D_x^\alpha D_\xi^\beta E|=\left|\sum_{\substack{\alpha'+\alpha''=\alpha\\ \beta'+\beta''=\beta}}C_{\beta',\beta''}^{\alpha',\alpha''}\left[\sum_{\substack{i+l+\langle\gamma,h+k\rangle\geq N\\ i,l,|\gamma|\leq N-1}}p_{m-l(\alpha')}^{(\gamma+\beta')}b_{\zeta,-m-i(\gamma+\alpha'')}^{(\beta'')}\right.\right.$$

$$+\sum_{i,|\gamma|\leqq N-1}\left(p-\sum_{l=0}^{N-1}p_{m-l}\right)^{(\gamma+\beta')}_{(\alpha')}b^{(\beta'')}_{\zeta,-m-i(\gamma+\alpha'')}\Big]\Big|$$
$$\leqq C\lambda(x,\xi)^{m-N-\langle\alpha,h\rangle-\langle\beta,k\rangle}\left(\frac{|\zeta|}{d(\zeta)}\right)^{2N-1+|\alpha|+|\beta|}|\zeta|^{-1}.$$

This completes the proof.

If necessary, we replace γ_0 with smaller one, so we may assume $\gamma_0\leqq\lambda_1$. Therefore for $\zeta\notin[\gamma_0,+\infty)$, $(P-\zeta)^{-1}$ exists. Since

$$(P-\zeta)b^{(N)}_\zeta(x,D)=I+r^{(N)}_\zeta(x,D)\,, \tag{2.11}$$

we have

$$(P-\zeta)^{-1}=b^{(N)}_\zeta(x,D)-D^{(N)}_\zeta(x,D) \tag{2.12}$$

where

$$D^{(N)}_\zeta(x,D)=(P-\zeta)^{-1}r^{(N)}_\zeta(x,D)\,,\qquad \zeta\notin[\gamma_0,+\infty)\,. \tag{2.13}$$

For the distribution kernels of $r^{(N)}_\zeta(x,D)$ and $D^{(N)}_\zeta(x,D)$, we have the following two lemmas.

Lemma 2.3. *Let $K(r^{(N)}_\zeta)(x,y)$ be the distribution kernel of $r^{(N)}_\zeta(x,D)$. Then for $N\geqq m+(n+1)T$, $K(r^{(N)}_\zeta)(x,y)$ is continuous in $\boldsymbol{R}^{2n}$ and, for some constant C independent of ζ and a positive integer $\tilde{N}$,*

$$|K(r^{(N)}_\zeta)(x,y)|\leqq C(\langle x\rangle\langle y\rangle)^{\{T(n+2)-(N-m)\}/(2T)}\left(\frac{|\zeta|}{d(\zeta)}\right)^{\tilde{N}}|\zeta|^{-1} \tag{2.14}$$

for any $0\neq\zeta\notin[\gamma_0,+\infty)$ and all $(x,y)\in\boldsymbol{R}^{2n}$. Here $\langle x\rangle=\{1+\sum_{j=1}^n x_j^2\}^{1/2}$.

Proof. For every $p\in\boldsymbol{N}$, we have

$$K(r^{(N)}_\zeta)(x,y)=\langle x-y\rangle^{-2p}(2\pi)^{-n}\int e^{i\langle x-y,\xi\rangle}(1-\Delta_\xi)^p r^{(N)}_\zeta(x,\xi)d\xi\,.$$

By Lemma 2.2,

$$|(1-\Delta_\xi)^p r^{(N)}_\zeta(x,\xi)|\leqq C\lambda(x,\xi)^{m-N}\left(\frac{|\zeta|}{d(\zeta)}\right)^{\tilde{N}}|\zeta|^{-1}$$
$$\leqq C(\langle x\rangle+\langle\xi\rangle)^{n+1+(m-N)/T}\langle\xi\rangle^{-(n+1)}\left(\frac{|\zeta|}{d(\zeta)}\right)^{\tilde{N}}|\zeta|^{-1}.$$

Therefore if $N\geqq m+(n+1)T$, $K(r^{(N)}_\zeta)$ is continuous in $\boldsymbol{R}^{2n}$. By Peetre's inequality: $\langle x-y\rangle^{-2p}\leqq C\langle x\rangle^{2p}\langle y\rangle^{-2p}$, we have

$$\begin{aligned}&|K(r_\zeta^{(N)})(x, y)|\\&\leqq C\langle x\rangle^{2p}\langle y\rangle^{-2p}\left(\frac{|\zeta|}{d(\zeta)}\right)^{\tilde N}|\zeta|^{-1}\langle x\rangle^{n+1+(m-N)/T}\int\langle\xi\rangle^{-(n+1)}d\xi\\&\leqq C\langle x\rangle^{2p+n+1+(m-N)/T}\langle y\rangle^{-2p}\left(\frac{|\zeta|}{d(\zeta)}\right)^{\tilde N}|\zeta|^{-1}.\end{aligned}$$

If we put $2p=[(N-m-nT)/(2T)]$, we get the estimate (2.14). This completes the proof.

LEMMA 2.4. *Let $K(D_\zeta^{(N)})(x, y)$ be the distribution kernel of $D_\zeta^{(N)}(x, D)$. Then for $N>(3n+2)T$, $K(D_\zeta^{(N)})(x, y)$ is continuous in R^{2n}. Moreover there exist positive constant C independent of ζ and a positive integer $\tilde N$ such that*

$$|K(D_\zeta^{(N)})(x, y)|\leqq C\langle x\rangle^{-m/(2T)}\langle y\rangle^{\{T(n+2)-(N-m)\}/(2T)}\left(\frac{|\zeta|}{d(\zeta)}\right)^{\tilde N+1}|\zeta|^{-1}. \tag{2.15}$$

PROOF. Let $K_\zeta(x, y)$ be the kernel of $(P-\zeta)^{-1}$. It follows from [8] that for some constant $C>0$, we have $|K_\zeta(x, y)|\leqq C(\langle x\rangle\langle y\rangle)^{-m/(2T)}(|\zeta|/d(\zeta))$. Here we note

$$K(D_\zeta^{(N)})(x, y)=\int K_\zeta(x, z)K(r_\zeta^{(N)})(z, y)dz .$$

Thus (2.15) follows immediately from Lemma 2.3.

§3. Complex powers of P.

Let P be the self-adjoint realization of $P(x, D)\in \mathrm{OP}S^m_{(h;k)}$ satisfying the hypotheses (H.1)$\sim$(H.4) in §1. It is well known that

$$\|(P-\zeta)^{-1}\|_{\mathcal{L}(L^2(R^n))}\leqq\frac{1}{d(\zeta)}\qquad\text{for all}\quad \zeta\notin[\gamma_0, +\infty) .$$

Therefore for $\operatorname{Re} s>0$, we can write

$$P^{-s}=\frac{i}{2\pi}\int_\Gamma\zeta^{-s}(P-\zeta)^{-1}d\zeta \tag{3.1}$$

where Γ is a curve beginning at infinity, passing along the negative real line to a circle $|\zeta|=\varepsilon_0$ $(0<\varepsilon_0<\gamma_0)$, then clockwise about the circle, and back to infinity along the negative real line and ζ^{-s} is defined in $C\setminus R_-=C\setminus\{\zeta\in C; \operatorname{Im}\zeta=0, \operatorname{Re}\zeta\leqq 0\}$ and takes the principal value. For $\operatorname{Re} s\leqq 0$, choose a positive integer k such that $-k+1\geqq\operatorname{Re} s>-k$ and define $P^{-s}=P^kP^{-s-k}$. Then we have

PROPOSITION 3.1 (cf. [8], [1]). *Assume that* $P(x, D)\in \mathrm{OPS}^{m}_{(h;k)}$ *satisfies* (H.1)~(H.4) *and let* P *be the self-adjoint realization of* $P(x, D)$. *Then we have for every* $s\in \boldsymbol{C}$, $P^{-s}\in \mathrm{OPS}^{-m\,\mathrm{Re}\,s}_{(h;k)}$ *and the symbol* $\sigma(P^{-s})$ *has the following asymptotic expansion:*

$$\sigma(P^{-s})\sim\sum_{j=0}^{\infty} p_{s,-m\,\mathrm{Re}\,s-j} \tag{3.2}$$

where

$$p_{s,-m\mathrm{Re}\,s}(x, \xi)=\chi(x, \xi)p_m(x, \xi)^{-s} \quad \text{and}$$
$$p_{s,-m\,\mathrm{Re}\,s-j}(x, \xi)=\sum_{l=1}^{2j}\frac{s(s+1)\cdots(s+l-1)}{l!}d_{lj}(x, \xi)p_m(x, \xi)^{-s-l}$$

for every $j\geqq 1$. *Here* $d_{lj}(x, \xi)$ *are defined by* (2.3).

Note that if $\mathrm{Re}\, s$ is large enough, P^{-s} is of trace class. Moreover we have

PROPOSITION 3.2 (cf. [8]). *Under the same hypotheses as Proposition* 3.1, *we have:*
(i) $\mathrm{Trace}(P^{-s})$ *is holomorphic for* $\mathrm{Re}\, s>(|h|+|k|)/m$ *and extended to a meromorphic function* $Z_P(s)$ *in* $\boldsymbol{C}$.
(ii) *The poles of* $Z_P(s)$ *are simple and belong to a sequence* $\{\hat{s}_j=(|h|+|k|-j)/m\}_{j=0,1,\dots}$ *and the residue at* $\hat{s}_j$ *is as follows:*

$$\mathrm{Res}_P(\hat{s}_0)=\frac{1}{m}(2\pi)^{-n}\int_{S(h;k)} p_m(\sigma)^{-(|h|+|k|)/m}d\sigma \tag{3.3}$$

and for every $j\geqq 1$,

$$\mathrm{Res}_P(\hat{s}_j)=\sum_{l=1}^{2j}\frac{1}{m^{l+1}l!}\prod_{i=0}^{l-1}(|h|+|k|-j+mi)(2\pi)^{-n}$$
$$\times\int_{S(h;k)} d_{lj}(\sigma)p_m(\sigma)^{(j-|h|-|k|-ml)/m}d\sigma\,.$$

(iii) $Z_P(s)$ *is holomorphic at* $s=0$, *i.e.*, $\mathrm{Res}_P(\hat{s}_{|h|+|k|})=0$.

REMARK 3.3. Since $Z_P(s)$ is holomorphic for $\mathrm{Re}\, s>\hat{s}_0$ and $Z_P(s)-(\mathrm{Res}_P(\hat{s}_0))/(s-\hat{s}_0)$ is continuous for $\mathrm{Re}\, s\geqq\hat{s}_0$, it follows from [1] (cf. [8]) that

$$N_P(\lambda)=(2\pi)^{-n}\int_{p_m(x,\xi)\leq 1} dxd\xi\, \lambda^{(|h|+|k|)/m}(1+o(1)) \qquad \text{as} \quad \lambda\to+\infty\,.$$

Now we estimate $Z_P(s)$.

PROPOSITION 3.4. *For any d_1 and d_2 ($d_1<d_2$), there exist positive constants $\tilde{N}$ and C such that*

$$|Z_P(s)|\leqq C(1+|\mathrm{Im}\, s|)^{\tilde{N}}$$

for any $s\in\{s\in \boldsymbol{C};\, d_1\leqq \mathrm{Re}\, s\leqq d_2\}$ excluding neighborhoods of the poles of $Z_P(s)$.

PROOF. Since $Z_{P^k}(s)=Z_P(ks)$, if necessary, replacing P with P^k for large integer k, we may assume $m>2nT$. At first we consider the case $d_1>0$. Then by (3.1) and (3.2), we have

$$P^{-s}=\sum_{j=0}^{N-1} P_{s,-m\mathrm{Re}\, s-j}(x,\, D)+E_s^{(N)}$$

where

$$E_s^{(N)}=\frac{i}{2\pi}\int_\Gamma \zeta^{-s} D_\zeta^{(N)}(x,\, D)d\zeta\, .$$

If $N>(3n+2)T$, it follows from Lemma 2.4 that there exist some positive constants C and $\tilde{N}$ such that

$$\left|\int K(D_\zeta^{(N)})(x,\, x)dx\right|\leqq C\left(\frac{|\zeta|}{d(\zeta)}\right)^{\tilde{N}}|\zeta|^{-1}$$

for all $0\neq\zeta\notin[\gamma_0,\, +\infty)$. Therefore

$$\int K(E_s^{(N)})(x,\, x)dx=\frac{i}{2\pi}\int_\Gamma \zeta^{-s}\int K(D_\zeta^{(N)})(x,\, x)dxd\zeta \tag{3.4}$$

is holomorphic for $\mathrm{Re}\, s>0$. Let $\Gamma_\theta=C_\theta^+ + C_\theta^0 + C_\theta^-$ where $\theta\in(0,\, \pi/2)$ is chosen later:

$$\begin{aligned} C_\theta^+ &: \quad re^{i\theta} \quad (\varepsilon_0\leqq r<+\infty)\\ C_\theta^0 &: \quad \varepsilon_0 e^{-i\phi} \quad (-\theta\leqq\phi\leqq\theta)\\ C_\theta^- &: \quad re^{-i\theta} \quad (\varepsilon_0\leqq r<+\infty)\, . \end{aligned}$$

Since $K(D_\zeta^{(N)})(x,\, x)$ is holomorphic in $\boldsymbol{C}\setminus[\gamma_0,\, +\infty)$, we can replace Γ in the integral of (3.4) with Γ_θ. Thus we have with constants C_1 and C_2 which are independent of s,

$$\begin{aligned}\left|\frac{i}{2\pi}\int_{C_\theta^0}\zeta^{-s}\int K(D_\zeta^{(N)})(x,\, x)dxd\zeta\right| &\leqq C_1\int_{-\theta}^{\theta}\varepsilon_0^{-\mathrm{Re}\, s-1}e^{|\mathrm{Im}\, s|\,|\phi|}d\phi\\ &\leqq 2\theta C_1\varepsilon_0^{-\mathrm{Re}\, s-1}e^{\theta|\mathrm{Im}\, s|}\end{aligned}$$

and

$$\left|\frac{i}{2\pi}\int_{C_\theta^\pm}\zeta^{-s}\int K(D_\zeta^{(N)})(x,x)dxd\zeta\right|\leqq C_2\int_{\varepsilon_0}^{\infty}r^{-\mathrm{Re}\,s-1}dre^{\theta|\mathrm{Im}\,s|}|\sin\theta|^{-\tilde N}$$
$$\leqq C_2(\mathrm{Re}\,s)^{-1}\varepsilon_0^{-\mathrm{Re}\,s}e^{\theta|\mathrm{Im}\,s|}|\sin\theta|^{-\tilde N}\,.$$

Choose $L>0$ sufficiently large such that $|\sin\theta|>|\theta|/2$ for $|\theta|\leqq\pi/L$ and put $\theta=\pi/2$ when $|\mathrm{Im}\,s|\leqq L/\pi$, $\theta=|\mathrm{Im}\,s|^{-1}$ when $|\mathrm{Im}\,s|\geqq L/\pi$. Then there exists a positive constant C such that

$$\left|\int K(E_s^{(N)})(x,x)dx\right|\leqq C(1+|\mathrm{Im}\,s|)^{\tilde N}\,.$$

Next we consider the integral:

$$J_j(s)=(2\pi)^{-n}\int p_{s,-m\,\mathrm{Re}\,s-j}(x,\xi)dxd\xi$$
$$=\sum_{l=1}^{2j}\frac{s(s+1)\cdots(s+l-1)}{l!}(2\pi)^{-n}\int d_{lj}(x,\xi)p_m(x,\xi)^{-s-l}dxd\xi\,.$$

Since

$$\int_{\lambda_0(x,\xi)\leqq1}d_{lj}(x,\xi)p_m(x,\xi)^{-s-l}dxd\xi$$

is an entire function and $|p_m(x,\xi)^{-s-l}|\leqq p_m(x,\xi)^{-\mathrm{Re}\,s-l}$, we have with a positive constant C,

$$\left|\frac{s(s+1)\cdots(s+l-1)}{l!}(2\pi)^{-n}\int_{\lambda_0(x,\xi)\leqq1}d_{lj}(x,\xi)p_m(x,\xi)^{-s-l}dxd\xi\right|$$
$$\leqq C(1+|\mathrm{Im}\,s|)^l \qquad \text{for}\quad d_1\leqq\mathrm{Re}\,s\leqq d_2\,.$$

On the other hand, by the quasi-homogeneity of $d_{lj}(x,\xi)p_m(x,\xi)^{-s-l}$ for $\lambda_0(x,\xi)\geqq1$ and the way of meromorphic extension of Trace(P^{-s}), we have

$$\text{(3.5)}\qquad \frac{s(s+1)\cdots(s+l-1)}{l!}(2\pi)^{-n}\int_{\lambda_0(x,\xi)\geqq1}d_{lj}(x,\xi)p_m(x,\xi)^{-s-l}dxd\xi$$
$$=(2\pi)^{-n}\frac{s(s+1)\cdots(s+l-1)}{l!}\,\frac{1}{ms+j-|h|-|k|}\int_{S(h;k)}d_{lj}(\sigma)p_m(\sigma)^{-s-l}d\sigma\,.$$

Thus there exists a positive constant C such that (3.5) is estimated by $C(1+|\mathrm{Im}\,s|)^l$ for $d_1\leqq\mathrm{Re}\,s\leqq d_2$ excluding neighborhoods of the poles $s=(|h|+|k|-j)/m$.

Now we consider the case $d_2>0$. Then there exists a positive integer l such that $\mathrm{Re}\,s+l\geqq1$ for $s\in\{s\in\boldsymbol{C};\,d_1\leqq\mathrm{Re}\,s\leqq d_2\}$. In this situation, we have

$$P^{-s}=P^lP^{-s-l}=P^l\left\{\sum_{j=0}^{N-1}P_{s+l,-m(\mathrm{Re}\,s+l)-j}(x,D)+E_{s+l}^{(N)}\right\}\,,$$

where

$$P^l E_{s+l}^{(N)} = \frac{i}{2\pi} \int_\Gamma \zeta^{-s-l} P^l D_\zeta^{(N)} d\zeta .$$

Here we note: $P^l D_\zeta^{(N)} = P^l (P-\zeta)^{-1} r_\zeta^{(N)}(x, D) = (P-\zeta)^{-1} P^l r_\zeta^{(N)}(x, D)$. By the composition formula of pseudodifferential operators, for every multi-indices α, β, there exists a positive constant C such that

$$|D_x^\alpha D_\xi^\beta \sigma(P^l r_\zeta^{(N)}(x, D))| \leq C\Big(\frac{|\zeta|}{d(\zeta)}\Big)^{\tilde{N}} |\zeta|^{-1} \lambda(x, \xi)^{m(l+1)-N-\langle\alpha,h\rangle-\langle\beta,k\rangle} .$$

By the proof of Lemma 2.3, if $N > m(l+1)+(n+1)T$, we have with a constant C,

$$|K(P^l r_\zeta^{(N)})(x, y)| \leq C\Big(\frac{|\zeta|}{d(\zeta)}\Big)^{\tilde{N}} |\zeta|^{-1} (\langle x\rangle\langle y\rangle)^{-\{N-m(l+1)-T(n+2)\}/(2T)}$$

for all $0 \neq \zeta \notin [\gamma_0, +\infty)$. Thus by the same arguments as the case $d_1 > 0$, we have with a constant C and $\tilde{N}$,

$$\Big|\int K(P^l E_{s+l}^{(N)})(x, x) dx\Big| \leq C(1+|\operatorname{Im} s|)^{\tilde{N}}$$

for $d_1 \leq \operatorname{Re} s \leq d_2$. On the other hand, we have

$$P^l\Big(\sum_{j=0}^{N-1} p_{s+l,-m(\operatorname{Re} s+l)-j}(x, D)\Big)$$
$$= \sum_{i+j+\langle\alpha,h+k\rangle \leq N-1} \frac{1}{\alpha!} p_{l,ml-i}{}^{(\alpha)} D_x^\alpha p_{s+l,-m(\operatorname{Re} s+l)-j} + \tilde{r}_s^{(N)} .$$

For the first sum, we can use the same arguments as the case $d_1 > 0$ and for the remainder term $\tilde{r}_s^{(N)}$ we use the composition formula. This completes the proof.

§4. Proof of Theorem 1.

Since $Q = P^{2M/m}$, it follows from Proposition 3.2 that the poles of $Z_Q(s)$ are simple and belong to a sequence $\{s_j = n - j/(2M)\}_{j=0,1,\cdots}$ and

$$\operatorname{Res}_Q(s_0) = A_0 = \frac{1}{2M}(2\pi)^{-n} \int_{S(h;k)} p_m(\sigma)^{-(|h|+|k|)/m} d\sigma ,$$

$$\operatorname{Res}_Q(s_j) = A_j = \frac{1}{2M} \sum_{l=1}^{2j} \frac{1}{m^l l!} \prod_{i=1}^{l-1} (|h|+|k|-j+mi)(2\pi)^{-n}$$
$$\times \int_{S(h;k)} d_{lj}(\sigma) p_m(\sigma)^{(j-|h|-|k|-ml)/m} d\sigma$$

for $j \geqq 1$. Moreover $\mu_j = \lambda_j^{2M/m}$ are eigenvalues of Q. Since $\mu_j \sim j^{1/n}$ by Remark 3.3, we can define a holomorphic function for $\operatorname{Re} z > 0$:

$$\theta_Q(z) = \operatorname{Trace} e^{-zQ} = \sum_{j=1}^{\infty} e^{-z\mu_j} . \tag{4.1}$$

At first we must study a property of Γ-function. For $\operatorname{Re} s > 0$, let $\Gamma(s)$ be the function

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$$

and, for $\operatorname{Re} s \leqq 0$, we choose a positive integer k such that $\operatorname{Re} s + k > 0$ and define as usual

$$\Gamma(s) = \frac{1}{s(s+1)\cdots(s+k-1)} \Gamma(s+k) .$$

Then we have:

Lemma 4.1. *For any $c \in (-k, -k+1)$ ($k \geqq 0$ integer), there exists a positive constant $L = L(c)$ such that for all ε, $0 < |\varepsilon| < \pi/2$ and $\sigma \in \boldsymbol{R}$,*

$$|\Gamma(c + i\sigma)| \leqq \frac{L}{(1+|\sigma|)^{k-1}} e^{-\varepsilon\sigma} .$$

Proof. At first we prove the case $k=0$, i.e., $c \in (0, 1)$. Making the change of the variable $t \to te^{i\varepsilon}$, we can write

$$\Gamma(c + i\sigma) = e^{-\varepsilon(\sigma - ic)} \int_0^\infty t^{c+i\sigma-1} e^{-(i \sin \varepsilon + \cos \varepsilon)t} dt .$$

Here we estimate the integral. Since $c \in (0, 1)$ and $\cos \varepsilon > 0$, we have

$$\int_0^1 |t^{c-1+i\sigma} e^{-(i \sin \varepsilon + \cos \varepsilon)t}| dt \leqq \int_0^1 t^{c-1} dt = \frac{1}{c} .$$

Choose ε_0 so that $0 < \varepsilon_0 < \pi/2$. If $0 < |\varepsilon| \leqq \varepsilon_0$,

$$I_1 = \left| \int_1^\infty t^{c+i\sigma-1} e^{-(i \sin \varepsilon + \cos \varepsilon)t} dt \right| \leqq \int_1^\infty t^{c-1} e^{-t \cos \varepsilon} dt .$$

Since

$$\int_1^\infty t^{c-1} e^{-t \cos \varepsilon} dt \leqq (\cos \varepsilon)^{-c} \Gamma(c) ,$$

we have $I_1 \leqq (\cos \varepsilon_0)^{-c} \Gamma(c)$.

Next if $\varepsilon_0 < |\varepsilon| < \pi/2$, we have, by the integration by parts,

$$\begin{aligned}
I_2 &= \left| \int_1^\infty t^{c-1+i\sigma} e^{-t\cos\varepsilon} e^{-it\sin\varepsilon} dt \right| \\
&= \left| \frac{1}{-i\sin\varepsilon} [t^{c-1+i\sigma} e^{-t\cos\varepsilon} e^{-it\sin\varepsilon}]_1^\infty \right. \\
&\quad \left. + \frac{1}{i\sin\varepsilon} \int_1^\infty \{(c-1+i\sigma)t^{c-2+i\sigma} - (\cos\varepsilon)t^{c-1+i\sigma}\} e^{-t\cos\varepsilon} e^{-it\sin\varepsilon} dt \right| \\
&\leqq \frac{1}{|\sin\varepsilon|} + \frac{1+|\sigma|}{|\sin\varepsilon|} \int_1^\infty t^{c-2} dt + \frac{\cos\varepsilon}{|\sin\varepsilon|} \int_1^\infty t^{c-1} e^{-t\cos\varepsilon} dt \\
&\leqq \frac{1}{\sin\varepsilon_0} + \frac{1+|\sigma|}{(\sin\varepsilon_0)(1-c)} + \frac{\Gamma(c)}{\sin\varepsilon_0} (\cos\varepsilon_0)^{1-c} .
\end{aligned}$$

Thus we get the conclusion of this lemma for the case $k=0$.

For general case $c \in (-k, -k+1)$, we have evidently, with a constant L,

$$|\Gamma(c+i\sigma)| \leqq \frac{|\Gamma(c+k+i\sigma)|}{(c+i\sigma)(c+1+i\sigma)\cdots(c+k-1+i\sigma)} \leqq \frac{Le^{-\varepsilon\sigma}}{(1+|\sigma|)^{k-1}} .$$

This completes the proof.

Now by the inverse Mellin transformation, if $c>0$ is large enough, we have

$$\theta_Q(z) = \frac{1}{2\pi i} \int_{\mathrm{Re}\, s=c} z^{-s} Z_Q(s) \Gamma(s) ds . \tag{4.2}$$

On the other hand if $c<0$ and $|c|$ is large enough, we have:

Lemma 4.2. *Let $\tilde{N}$ be as in Proposition 3.4. For any $c<-\tilde{N}-2-k$ ($k=0, 1, \cdots$), $c \notin -\boldsymbol{N}$, there exists a positive constant C such that for any z, $0<|\arg z|<\pi/2$,*

$$\left| z^{c+k} \left(\frac{d}{dz}\right)^k \frac{1}{2\pi i} \int_{\mathrm{Re}\, s=c} z^{-s} Z_Q(s) \Gamma(s) ds \right| \leqq C .$$

Proof. Since $Z_Q(s) = Z_P(2Ms/m)$, it follows from Proposition 3.4 that there exists a positive constant C_1 such that $|Z_Q(c+i\sigma)| \leqq C_1(1+|\sigma|)^{\tilde{N}}$. From Lemma 4.1, it follows that if $c<-\tilde{N}-2-k$ and $c \notin -\boldsymbol{N}$, we have with another positive constant C,

$$|Z_Q(c+i\sigma)\Gamma(c+i\sigma)| \leqq C(1+|\sigma|)^{-k-2} e^{-\varepsilon\sigma} \qquad \text{for any } \varepsilon \ (0<\varepsilon<\pi/2) .$$

Therefore we have

$$\left| z^{c+k} \left(\frac{d}{dz}\right)^k \frac{1}{2\pi i} \int_{\mathrm{Re}\, s=c} z^{-s} Z_Q(s) \Gamma(s) ds \right| \leqq \frac{C}{2\pi} \int_{-\infty}^\infty e^{\sigma \arg z - \varepsilon|\sigma|} (1+|\sigma|)^{-2} d\sigma .$$

Noting that we can put $\varepsilon=|\arg z|$, the proof is complete.

PROOF OF THEOREM 1. $Z_Q(s)\Gamma(s)$ is a meromorphic function in $\boldsymbol{C}$. Moreover if $s_j=n-j/(2M)\notin -\boldsymbol{Z}_+=\{0, -1, -2, \cdots\}$, $Z_Q(s)\Gamma(s)$ has a simple pole at $s=s_j$ and the residue is $A_j\Gamma(n-j/(2M))$ and if $s_j=n-j/(2M)=-l$ for some $l\in\boldsymbol{Z}_+$, $Z_Q(s)\Gamma(s)$ has a double pole at $s=s_j$ and the coefficient of $(z+l)^{-2}$ is equal to $-B_j$. Thus taking the residue theorem into consideration, we can shift the path of the integration in (4.2) by letting $c\to-\infty$. Here we note that by Lemma 3.4 and Lemma 4.1, $Z_Q(s)\Gamma(s)$ is rapidly decreasing in all vertical strips, excluding neighborhoods of poles. By the Cauchy theorem, we have for a small $\delta>0$,

$$\frac{1}{2\pi i}\int_{|s-s_j|=\delta}\frac{z^{-s}}{s-s_j}dsA_j\Gamma\Big(n-\frac{j}{2M}\Big)=z^{-s_j}A_j\Gamma\Big(n-\frac{j}{2M}\Big)$$

and

$$\frac{1}{2\pi i}\int_{|s-s_j|=\delta}\frac{z^{-s}}{(s-s_j)^2}ds(-B_j)=z^lB_j\log z\ .$$

Finally if we apply Lemma 4.2, this completes the proof of Theorem 1.

§5. Proof of Theorem 2.

Let ρ be a function as in (1.8). By the Lebesgue theorem,

$$\begin{aligned}I(\mu)&=\int\rho(\mu-\tau)dN_Q(\tau)\\&=\lim_{\varepsilon\downarrow0}\int e^{-\varepsilon\tau}\rho(\mu-\tau)dN_Q(\tau)\\&=\lim_{\varepsilon\downarrow0}\sum_{j=1}^{\infty}e^{-\varepsilon\mu_j}\rho(\mu-\mu_j)\\&=\lim_{\varepsilon\downarrow0}\sum_{j=1}^{\infty}(2\pi)^{-1}\int e^{-(\varepsilon+it)\mu_j}\hat\rho(t)e^{it\mu}dt\\&=\lim_{\varepsilon\downarrow0}(2\pi)^{-1}\int\theta_Q(\varepsilon+it)\hat\rho(t)e^{it\mu}dt\ .\end{aligned}$$

We want to obtain the asymptotic behavior of $I(\mu)$ as $\mu\to+\infty$ modulo $O(\mu^{n-2})$ if $n\geqq2$ and $O(\mu^{-1-\delta})$ for some $\delta>0$ if $n=1$. By virtue of Theorem 1 it suffices to study the asymptotic behavior as $\mu\to+\infty$ of the following functions:

$$\text{(5.1)}\qquad I_{1j}(\mu)=\lim_{\varepsilon\downarrow0}(2\pi)^{-1}\int(\varepsilon+it)^{-n+j/(2M)}\hat\rho(t)e^{it\mu}dt\qquad\Big(-n+\frac{j}{2M}\notin\boldsymbol{Z}_+\Big)$$

(5.2) $$I_{2l}(\mu)=\lim_{\varepsilon\downarrow 0}(2\pi)^{-1}\int(\varepsilon+it)^l\log(\varepsilon+it)\hat{\rho}(t)e^{i\mu t}dt \qquad (l\in \boldsymbol{Z}_+)$$

(5.3) $$I_{3l}(\mu)=\lim_{\varepsilon\downarrow 0}(2\pi)^{-1}\int(\varepsilon+it)^l\hat{\rho}(t)e^{i\mu t}dt \qquad (l\in \boldsymbol{Z}_+)$$

(5.4) $$R_c(\mu)=\lim_{\varepsilon\downarrow 0}(2\pi)^{-1}\int F_c(\varepsilon+it)\hat{\rho}(t)e^{i\mu t}dt$$

where

$$F_c(\varepsilon+it)=\frac{1}{2\pi i}\int_{\mathrm{Re}\, s=c}(\varepsilon+it)^{-s}Z_Q(s)\Gamma(s)ds\ .$$

At first we consider $I_{2l}(\mu)$ and $I_{3l}(\mu)$.

LEMMA 5.1. *For every integer $l\geqq 0$, we have:*
(i) $I_{2l}(\mu)=O(\mu^{-l-1})$ *as* $\mu\to+\infty$,
(ii) *For any integer* $N\geqq 0$, $I_{3l}(\mu)=O(\mu^{-N})$ *as* $\mu\to+\infty$.

PROOF. (i) At first we consider the case $l=0$. In this case we have

$$I_{20}(\mu)=\frac{-1}{i\mu}\int\left\{\frac{\hat{\rho}(t)}{\varepsilon+it}+\log(\varepsilon+it)\hat{\rho}'(t)\right\}e^{i\mu t}dt\ .$$

If we define a function $a_\varepsilon(\tau)$ such that $a_\varepsilon(\tau)=e^{-\varepsilon\tau}$ if $\tau>0$ and $=0$ if $\tau\leqq 0$, we have

$$\begin{aligned}(2\pi)^{-1}\int\frac{\hat{\rho}(t)}{\varepsilon+it}e^{i\mu t}dt&=a_\varepsilon*\rho(\mu)\\ &=\int_{-\infty}^{\mu}e^{-\varepsilon(\mu-\tau)}\rho(\tau)d\tau\leqq\int_{-\infty}^{\infty}\rho(\tau)d\tau=1\ .\end{aligned}$$

On the other hand, since $\mathrm{supp}\,\hat{\rho}$ is compact and for any $\alpha\in(0,1)$

$$|\log(\varepsilon+it)|\leqq|\varepsilon+it|^{-\alpha}+\frac{\pi}{2}\qquad\text{for all } |\varepsilon|<1,\ t\in\mathrm{supp}\,\hat{\rho}\ ,$$

we have

$$\left|\lim_{\varepsilon\downarrow 0}\int\log(\varepsilon+it)\hat{\rho}'(t)e^{i\mu t}dt\right|\leqq\int\left(|t|^{-\alpha}+\frac{\pi}{2}\right)|\hat{\rho}'(t)|dt\ .$$

Thus we have $I_{20}(\mu)=O(\mu^{-1})$ as $\mu\to+\infty$. Next we consider the case $l>0$. Since we have

$$\int(\varepsilon+it)^l\log(\varepsilon+it)\hat{\rho}(t)e^{i\mu t}dt$$

$$=\frac{-1}{i\mu}\int[li(\varepsilon+it)^{l-1}\log(\varepsilon+it)\hat{\rho}(t)+i(\varepsilon+it)^{l-1}\hat{\rho}(t)$$
$$+(\varepsilon+it)^{l}\log(\varepsilon+it)\hat{\rho}'(t)]e^{i\mu t}dt\ ,$$

by induction on l and the fact:

$$\lim_{\varepsilon\downarrow 0}\int(\varepsilon+it)^{l}\log(\varepsilon+it)\hat{\rho}^{(N)}(t)e^{i\mu t}dt \text{ is bounded as } \mu\to+\infty\ ,$$

we can obtain $I_{2l}(\mu)=O(\mu^{-l-1})$ as $\mu\to+\infty$. (ii) is clear. This completes the proof.

Secondly, in order to study $I_{1j}(\mu)$, we need the following three lemmas.

LEMMA 5.2. *Let* $0<\alpha<1$ *and* ρ *be as in* (1.8). *Then we have*

$$\int(\varepsilon+it)^{\alpha-2}\hat{\rho}(t)e^{i\mu t}dt=\frac{-\mu}{\alpha-1}\int(\varepsilon+it)^{\alpha-1}e^{i\mu t}dt+R(\mu,\varepsilon)$$

where $\lim_{\varepsilon\downarrow 0}R(\mu,\varepsilon)$ *exists and is of* $O(1)$ *as* $\mu\to+\infty$.

PROOF. Let $\operatorname{supp}\hat{\rho}\subset(-a,a)$. Then we have the following decomposition:

$$\int(\varepsilon+it)^{\alpha-2}\hat{\rho}(t)e^{i\mu t}dt=\int(\varepsilon+it)^{\alpha-2}e^{i\mu t}dt$$
$$+\int_{|t|\leqq a}(\varepsilon+it)^{\alpha-2}(\hat{\rho}(t)-1)e^{i\mu t}dt-\int_{|t|\geqq a}(\varepsilon+it)^{\alpha-2}e^{i\mu t}dt\ .$$

The first integral is equal to

$$\frac{-\mu}{\alpha-1}\int(\varepsilon+it)^{\alpha-1}e^{i\mu t}dt\ .$$

Since $\hat{\rho}(t)-1=\hat{\rho}(t)-\hat{\rho}(0)=t\hat{\rho}'(\theta t)$ for some $\theta\in(0,1)$, it follows that $|(\varepsilon+it)^{\alpha-2}(\hat{\rho}(t)-1)|\leqq C|t|^{\alpha-1}$ where C is independent of μ and ε. So the second integral is of $O(1)$ as $\mu\to+\infty$ uniformly in ε. As to the third integral, since we have $|(\varepsilon+it)^{\alpha-2}e^{i\mu t}|\leqq|t|^{\alpha-2}$, it is also of $O(1)$ as $\mu\to+\infty$ uniformly in ε. This completes the proof.

LEMMA 5.3. *Let* $0<\alpha<1$. *Then we have*

$$\lim_{\varepsilon\downarrow 0}\int(\varepsilon+it)^{\alpha-1}e^{i\mu t}dt=2\sin(\pi\alpha)\Gamma(\alpha)\mu^{-\alpha}\ .$$

PROOF. First of all we claim that

$$\lim_{\varepsilon\downarrow 0}\int(\varepsilon+it)^{\alpha-1}e^{i\mu t}dt \tag{5.5}$$

exists and is equal to

$$\int (it)^{\alpha-1} e^{i\mu t} dt .$$

In fact, by the mean value theorem, we have

$$(\varepsilon+it)^{\alpha-1}=(it)^{\alpha-1}+\varepsilon(\alpha-1)\int_0^1 (\varepsilon\theta+it)^{\alpha-2} d\theta .$$

Here we note

$$\varepsilon\int_{|t|\geq 1}\left|\int_0^1 (\varepsilon\theta+it)^{\alpha-2} d\theta\right| dt \leqq \varepsilon\int_{|t|\geq 1} t^{\alpha-2} dt \to 0 \qquad \text{as} \quad \varepsilon\downarrow 0$$

and for any $\delta\in(0, 1)$,

$$\varepsilon\int_{|t|\leq 1}\left|\int_0^1 (\varepsilon\theta+it)^{\alpha-2} d\theta\right| dt \leqq \varepsilon\int_{|t|\leq 1}\int_0^1 (\varepsilon\theta)^{\delta-1} d\theta\, |t|^{\alpha-1-\delta} dt .$$

If we choose δ so that $0<\alpha-\delta<1$, the last integral converges to 0 as $\varepsilon\downarrow 0$. Thus we obtain (5.5).

Next it is well known that for $\alpha\in(0, 1)$,

$$\int_0^\infty t^{\alpha-1}\cos(\mu t) dt=\Gamma(\alpha)\cos\frac{\pi\alpha}{2}\mu^{-\alpha} ,$$

$$\int_0^\infty t^{\alpha-1}\sin(\mu t) dt=\Gamma(\alpha)\sin\frac{\pi\alpha}{2}\mu^{-\alpha} .$$

Moreover we note

$$(it)^{\alpha-1}=\begin{cases} t^{\alpha-1}e^{(\alpha-1)\pi i/2} & \text{if} \quad t>0 , \\ |t|^{\alpha-1}e^{-(\alpha-1)\pi i/2} & \text{if} \quad t<0 . \end{cases}$$

This completes the proof.

LEMMA 5.4. *Let $\alpha\in[0, 1)$ and $j\geqq 0$ integer and ρ be a function as in* (1.8). *Then we have*

$$\lim_{\varepsilon\downarrow 0}\int (\varepsilon+it)^{\alpha\pm j}\hat{\rho}(t)e^{i\mu t} dt=O(\mu^{\mp j-1-\alpha}) \qquad \textit{as} \quad \mu\to+\infty .$$

PROOF. By the integration by parts, we have for $j\geqq 2$,

$$\int (\varepsilon+it)^{\alpha-j}\hat{\rho}(t)e^{i\mu t} dt=\int\frac{-1}{(\alpha-j+1)i}(\varepsilon+it)^{\alpha-(j-1)}\{\hat{\rho}'(t)+i\mu\hat{\rho}(t)\}e^{i\mu t} dt .$$

Therefore repeating this procedure, we are reduced to the following equality:

$$\lim_{\varepsilon\downarrow 0}\int(\varepsilon+it)^{\alpha-2}\hat{\rho}(t)e^{i\mu t}dt=O(\mu^{1-\alpha}) \qquad \text{as} \quad \mu\to+\infty\,. \tag{5.6}$$

For $\alpha\in(0, 1)$, this equality follows from Lemmas 5.2 and 5.3 and for $\alpha=0$, (5.6) follows from the integration by parts and the arguments in the beginning of the proof of Lemma 5.1.

Next we have for $j\geqq-1$,

$$\begin{aligned}&\int(\varepsilon+it)^{\alpha+j}\hat{\rho}(t)e^{i\mu t}dt\\ &=\int\frac{-1}{i\mu}\{(\alpha+j)(\varepsilon+it)^{\alpha+j-1}i\hat{\rho}(t)+(\varepsilon+it)^{\alpha+j}\hat{\rho}'(t)\}e^{i\mu t}dt\,.\end{aligned}$$

Thus we are also reduced to (5.6). This completes the proof.

Finally we study $R_c(\mu)$.

Lemma 5.5. *Let $\tilde{N}$ be the number as in Proposition 3.4. If $c<-\tilde{N}-4$, $c\notin-\boldsymbol{Z}_+$, then $R_c(\mu)=O(\mu^{-2})$ as $\mu\to+\infty$.*

Proof. By the integration by parts, we have

$$\begin{aligned}&\int F(\varepsilon+it)\hat{\rho}(t)e^{i\mu t}dt\\ &=\frac{-1}{\mu^2}\int\{-F''(\varepsilon+it)\hat{\rho}(t)+2iF'(\varepsilon+it)\hat{\rho}'(t)+F(\varepsilon+it)\hat{\rho}''(t)\}e^{i\mu t}dt\,.\end{aligned}$$

Since we can apply Lemma 4.2 and $-c-2>0$, we have

$$\begin{aligned}\left|\lim_{\varepsilon\downarrow 0}\int F''(\varepsilon+it)\hat{\rho}(t)e^{i\mu t}dt\right|&\leqq C\lim_{\varepsilon\downarrow 0}\int|\varepsilon+it|^{-c-2}\hat{\rho}(t)dt\\ &=C\int|t|^{-c-2}\hat{\rho}(t)dt<\infty\,.\end{aligned}$$

The other terms are similarly estimated and this completes the proof.

Proof of Theorem 2. By the arguments in the beginning of this section, it follows that for any $c<-\tilde{N}-4$, $c\notin-\boldsymbol{Z}_+$, there exists a positive integer N such that

$$\begin{aligned}I(\mu)=&\sum_{\substack{-n+j/(2M)\notin \boldsymbol{Z}_+\\ j\leqq N}}\Gamma\Big(n-\frac{j}{2M}\Big)I_{1j}(\mu)\\ &+\sum_{\substack{-n+j/(2M)=l\in \boldsymbol{Z}_+\\ j\leqq N}}B_jI_{2l}(\mu)+\sum_{l=0}^{N}C_lI_{3l}(\mu)+R_c(\mu)\,.\end{aligned}$$

But by Lemma 5.1 (ii) and Lemma 5.5, $I_{3l}(\mu)=O(\mu^{-2})$ and $R_e(\mu)=O(\mu^{-2})$ as $\mu\to+\infty$. Put $\delta=\mathrm{Min}\{(j-2kM)/(2M);\ j>2kM,\ j=1,2,\cdots,N,\ k=1,2,\cdots\}$. Then it follows from Lemma 5.1 (i) that when $n\geqq 2$, $I_{2l}(\mu)=O(\mu^{n-2})$ as $\mu\to+\infty$. By Proposition 3.2, $B_{2M}=0$ for $n=1$. Therefore when $n=1$, $I_{2l}(\mu)=O(\mu^{-2})$ as $\mu\to+\infty$. If $j\geqq 2M$ and $-n+j/(2M)\notin \boldsymbol{Z}_+$, we can choose an integer $k\geqq 1$ such that $2kM<j<2(k+1)M$. Therefore by Lemma 5.4,

$$\begin{aligned}I_{1j}(\mu)&=\lim_{\varepsilon\downarrow 0}(2\pi)^{-n}\int(\varepsilon+it)^{-n+k+(j-2kM)/(2M)}\hat{\rho}(t)e^{i\mu t}dt\\&=O(\mu^{n-k-1-(j-2kM)/(2M)})=O(\mu^{n-2-(j-2kM)/(2M)})\qquad\text{as}\quad \mu\to\infty\,.\end{aligned}$$

Thus if we write

$$I(\mu)=\sum_{j=0}^{M_0}\Gamma\Big(n-\frac{j}{2M}\Big)A_jI_{1j}(\mu)+r_n^{(1)}(\mu)\,,$$

we have

$$r_n^{(1)}(\mu)=O(\mu^{n-2})\ \text{if}\ n\geqq 2\quad\text{and}\quad r_1^{(1)}(\mu)=O(\mu^{-1-\delta})\qquad\text{as}\ \mu\to+\infty\,. \tag{5.7}$$

Finally we compute $I_{1j}(\mu)$ for $j=0,1,\cdots,M_0$. By Lemma 5.4 and by induction on j, we have

$$\begin{aligned}&\int(\varepsilon+it)^{-n+j/(2M)}\hat{\rho}(t)e^{i\mu t}dt\\&=\frac{\mu^{n-1}}{(n-1-j/(2M))\cdots(1-j/(2M))}\int(\varepsilon+it)^{-1+j/(2M)}\hat{\rho}(t)e^{i\mu t}dt+r_n^{(1)}(\mu,\varepsilon)\,,\end{aligned}$$

where $r_n^{(1)}(\mu,\varepsilon)=O(\mu^{n-2})$ uniformly in ε as $\mu\to+\infty$ if $n\geqq 2$ and $r_1^{(1)}(\mu,\varepsilon)=0$. Here we have

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}(2\pi)^{-1}\int(\varepsilon+it)^{-1}\hat{\rho}(t)e^{i\mu t}dt&=\lim_{\varepsilon\downarrow 0}a*\rho(\mu)\\&=\int_{-\infty}^{\mu}\rho(\tau)d\tau=\hat{\rho}(0)+\int_{\mu}^{+\infty}\rho(\tau)d\tau=1+O(\mu^{-N})\end{aligned}$$

for any $N>0$ as $\mu\to+\infty$. By Lemmas 5.2 and 5.3, we have for $1\leqq j<M_0$,

$$\lim_{\varepsilon\downarrow 0}(2\pi)^{-1}\int(\varepsilon+it)^{-1+j/(2M)}\hat{\rho}(t)e^{i\mu t}dt=\frac{1}{\pi}\sin\frac{j\pi}{2M}\Gamma\Big(\frac{j}{2M}\Big)\mu^{-j/(2M)}\,.$$

Thus it follows that

$$I(\mu)=\Gamma(n)A_0\frac{\mu^{n-1}}{(n-1)!}+\sum_{j=1}^{M_0}\Gamma\Big(n-\frac{j}{2M}\Big)_jA\frac{1}{\pi}\sin\frac{j\pi}{2M}$$

$$\times\Gamma\Big(\frac{j}{2M}\Big)\frac{(-1)^{n-1}}{(1-n+j/(2M))\cdots(-1+j/(2M))}\mu^{n-1-j/(2M)}+r_n(\mu)$$

where $r_n(\mu)$ has the same property as (5.7). We note

$$\Gamma\Big(n-\frac{j}{2M}\Big)=\Big(n-1-\frac{j}{2M}\Big)\cdots\Big(1-\frac{j}{2M}\Big)\Gamma\Big(1-\frac{j}{2M}\Big)$$

and for $1\leqq j<2M$, $\Gamma(1-j/(2M))\Gamma(j/(2M))=\pi/\sin(j\pi/(2M))$. This completes the proof of Theorem 2.

§6. Proof of Theorem 3.

Since the proof of Theorem 3 is essentially due to [3], we shall give only an outline of the proof. First of all, we quote the following two lemmas.

LEMMA 6.1 (cf. [3; Lemma 4.2.8]). *Under the hypotheses of Theorem 2, there exists a constant $\gamma>0$ such that for all $K>0$ and μ,*

$$\int_{|\mu-\tau|\leqq K} dN_Q(\tau)\leqq\gamma(1+K)^n(1+|\mu|)^{n-1}\,. \tag{6.1}$$

LEMMA 6.2 (cf. [3; Lemma 4.2.9]). *Under the hypotheses of the above lemma, for all $\varepsilon>0$ there exists a constant $K>0$ such that*

$$\int\rho(\mu-\tau)dN_Q(\tau)\leqq\varepsilon\mu^{n-1}\qquad \text{for all}\quad \mu\geqq 1\,, \tag{6.2}$$

$$\int_{\tau>\lambda+K}\Big(\int_{-\infty}^{\lambda}\rho(\mu-\tau)d\mu\Big)dN_Q(\tau)\leqq\varepsilon\lambda^{n-1}\qquad \text{for all}\quad \lambda\geqq 1\,, \tag{6.3}$$

$$\int_{\tau<\lambda-K}\Big(\int_{\lambda}^{\infty}\rho(\mu-\tau)d\mu\Big)dN_Q(\tau)\leqq\varepsilon\lambda^{n-1}\qquad \text{for all}\quad \lambda\geqq 1\,. \tag{6.4}$$

Now we give the proof of Theorem 3. Since the support of the measure dN_Q is contained in $[0,+\infty)$ and ρ is rapidly decreasing, there exists a constant $C>0$ such that

$$\int_{-\infty}^{-1} I(\mu)d\mu\leqq C\,. \tag{6.5}$$

By integrating the asymptotic formula of Theorem 2 from -1 to λ and taking (6.5) into consideration, we have

$$\int_{-\infty}^{\lambda} I(\mu)d\mu=\sum_{j=0}^{M_0}\frac{A_j}{n-j/(2M)}\lambda^{n-j/(2M)}+O(\lambda^{n-1})\qquad \text{as}\quad \lambda\to+\infty\,. \tag{6.6}$$

On the other hand we rewrite the left hand side of (6.6) in the form $A+B+C$ where

$$A=\int_{\tau>\lambda+K}\left(\int_{-\infty}^{\lambda}\rho(\mu-\tau)d\mu\right)dN_Q(\tau)\ ,$$

$$B=\int_{\tau<\lambda-K}\left(\int_{-\infty}^{\lambda}\rho(\mu-\tau)d\mu\right)dN_Q(\tau)\ ,$$

$$C=\int_{|\tau-\lambda|\leq K}\left(\int_{-\infty}^{\lambda}\rho(\mu-\tau)d\mu\right)dN_Q(\tau)\ .$$

It is clear from (6.3) that A is of $O(\lambda^{n-1})$ as $\lambda\to+\infty$. Next, since $\hat{\rho}(0)=1$, we have

$$B=\int_{\tau<\lambda-K}dN_Q(\tau)-\int_{\tau<\lambda-K}\left(\int_{\lambda}^{\infty}\rho(\mu-\tau)d\mu\right)dN_Q(\tau)$$

$$=N_Q(\lambda)-\int_{\lambda-K\leq\tau\leq\lambda}dN_Q(\tau)-\int_{\tau<\lambda-K}\left(\int_{\lambda}^{\infty}\rho(\mu-\tau)d\mu\right)dN_Q(\tau)\ .$$

It follows from (6.1) and (6.4) that the second term and third term in the last integral are of $O(\lambda^{n-1})$ as $\lambda\to+\infty$. Since the term C is estimated by

$$\text{const.}\int_{|\tau-\lambda|\leq K}dN_Q(\tau)\ ,$$

it follows from (6.1) that C is also of $O(\lambda^{n-1})$ as $\lambda\to\infty$.

Therefore we have

$$N_Q(\lambda)=\sum_{j=0}^{M_0}\frac{A_j}{n-j/(2M)}\lambda^{n-j/(2M)}+O(\lambda^{n-1})\qquad\text{as}\quad\lambda\to+\infty\ .$$

Since $N_P(\lambda)=N_Q(\lambda^{2M/m})$, we have

$$N_P(\lambda)=\sum_{j=0}^{M_0}\frac{A_j}{n-j/(2M)}\lambda^{(2Mn-j)/m}+O(\lambda^{2M(n-1)/m})\qquad\text{as}\quad\lambda\to+\infty\ .$$

This completes the proof of Theorem 3.

REMARK 6.3. When $h_j=k_j=1$ for $j=1, 2, \cdots, n$, the same result is in [4; Theorem 3]. When $h_1=h_2=\cdots=h_n=h$, $k_1=k_2=\cdots=k_n=k$, see [6; Theorem 3] and also [5].

§7. Examples.

(1) Let $P(x, D)=D_x^2+x^4+ax^3$ on $\boldsymbol{R}$ ($a\in\boldsymbol{R}$). Then if we put $p_4(x, \xi)=\xi^2+x^4$, $p_3(x, \xi)=ax^3$, $P(x, D)\in\mathrm{OP}S^4_{(1;1)}$. We note $M=3/2$, hence $M_0=2$. By Theorem 3, we see

$$N_P(\lambda)=\frac{1}{3\pi}B\left(\frac{1}{4}, \frac{1}{2}\right)\lambda^{3/4}+\frac{3a^2}{32\pi}B\left(\frac{3}{4}, \frac{1}{2}\right)\lambda^{1/4}+O(1)\qquad\text{as}\quad\lambda\to\infty$$

where $B(\cdot, \cdot)$ denotes the Beta function. This operator is treated in [5].

(2) Let $P(x, D)=D_{x_1}^2+D_{x_2}^2+x_1^2+x_2^4+ax_2^3+bx_1+cx_2^2+dx_2+e$ on $\boldsymbol{R}^2$ $(a, b, c, d, e \in \boldsymbol{R})$. Then if we put $p_4(x, \xi)=\xi_1^2+\xi_2^2+x_1^2+x_2^4$, $p_3(x, \xi)=ax_2^3$, $p_2(x, \xi)=bx_1+cx_2^2$, $p_1(x, \xi)=dx_2$, $p_0(x, \xi)=e$, we see $P(x, D)\in \mathrm{OP}S^4_{(2,1;2,2)}$. We note $M=7/4$ and hence $M_0=3$. By Theorem 3, we see

$$N_P(\lambda)=\frac{2}{21\pi}B\left(\frac{1}{2}, \frac{1}{4}\right)\lambda^{7/4}+\frac{2c+a^2}{20\pi}B\left(\frac{1}{2}, \frac{3}{4}\right)\lambda^{5/4}+O(\lambda^{7/8}) \quad \text{as } \lambda\to\infty .$$

(3) Let $P(x, D)=D_{x_1}^2+D_{x_2}^4+x_1^2+x_2^4+ax_2^3+bD_{x_2}^3+cx_1+dD_{x_1}+ex_2^2+fD_{x_2}^2+R_0(x, D)$ on $\boldsymbol{R}^2$ where a, b, c, d, e, f are real numbers and $R_0(x, D)$ is a polynomial in x_2 and D_{x_2} of order 1 with real constant coefficients. Then if we put $p_4(x, \xi)=\xi_1^2+\xi_2^4+x_1^2+x_2^4$, $p_3(x, \xi)=ax_2^3+b\xi_2^3$, $p_2(x, \xi)=cx_1+d\xi_1+ex_2^2+f\xi_2^2$, we can regard $P(x, D)\in \mathrm{OP}S^4_{(2,1;2,1)}$. In this case we see $M=3/2$, hence $M_0=2$. By Theorem 3, we see

$$N_P(\lambda)=\frac{1}{12\pi}B\left(\frac{1}{4}, \frac{1}{4}\right)\lambda^{3/2}+\left[(a^2+b^2)\frac{3\times 7^2}{2^{10}\sqrt{2}}-(e+f)\frac{1}{8\sqrt{2}}\right]\lambda+O(\lambda^{3/4})$$

as $\lambda\to\infty$.

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