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Complex Hypersurfaces in an Indefinite Complex Space Form

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Introduction

Let $M_s^n(c)$ be an n $(n\geqq 2)$-dimensional indefinite complex space form of constant holomorphic curvature c and of index $2s$. Recently Romero [5] proved that an indefinite complex hypersurface with parallel Ricci tensor in $M_{s+a}^{n+1}(c)$ $(c\neq 0)$ is Einstein. The purpose of this paper is to study an indefinite complex hypersurface M in $M_{s+a}^{n+1}(c)$ satisfying the condition

$$(*)\qquad R(X, Y)S=0\ ,$$

for any vector fields X and Y of M, where R denotes the curvature tensor, S is the Ricci tensor and $R(X, Y)$ operates on the tensor algebra as a derivation. We shall prove the following

THEOREM. *Let M be a complex hypersurface of index $2s$ in $M_{s+a}^{n+1}(c)$ $(n\geqq 2)$. If $c\neq 0$ and M satisfies the condition $(*)$, then M is Einstein.*

In the last section it is shown that there exist many examples of Einstein complex hypersurfaces in an indefinite complex Euclidean space different from those given by Romero [3].

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§1. Complex hypersurfaces in an indefinite complex space form.

Let M be a complex m-dimensional indefinite Kaehlerian manifold. Then M is equipped with an almost complex structure J which is

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parallel, that is, $\nabla J=0$, and an indefinite Riemannian metric g which is J-Hermitian:

$$g(JX, JY)=g(X, Y)\ , \quad \text{for any vector fields } X \text{ and } Y .$$

The pair (g, J) is called an *indefinite Kaehlerian structure* of M. It follows that J is integrable and the index of g is an even number $2s$ $(0\leqq s\leqq m)$. A holomorphic plane spanned by u and Ju is non-degenerate if and only if it contains some v such that $g(v, v)\neq 0$. The manifold M is said to be of *constant holomorphic sectional curvature* c, if all non-degenerate holomorphic planes have the same constant sectional curvature c. A complete, simply connected and connected indefinite Kaehlerian manifold M is called an *indefinite complex space form*, which is denoted by $M_s^m(c)$, provided that it is of complex dimension m, of index $2s$ and of constant holomorphic sectional curvature c. There are three kinds of types about indefinite complex space forms [1], an indefinite complex projective space $P_s^m C$, an indefinite complex Euclidean space C_s^m or an indefinite hyperbolic space $H_s^m C$, according as c is positive, zero or negative.

Let $\bar{M}=M_{s+a}^{n+1}(c)$ be an indefinite complex space form, where $a=0$ or 1 and let M be an n-dimensional complex hypersurface of index $2s$ in $\bar{M}$. Let $(\bar{g}, \bar{J})$ be an indefinite Kaehlerian structure of $\bar{M}$ and (g, J) be an indefinite Kaehlerian structure of M induced from $(\bar{g}, \bar{J})$. We choose a local field $\{E_I\}=\{E_A, E_{A^*}\}$, where $E_{A^*}=\bar{J}E_A$, of orthonormal frames defined on a neighborhood of $\bar{M}$ in such a way that, restricted to M, $\{E_i\}=\{E_a, E_{a^*}\}$ is tangent to M, and $\{E_0, E_{0^*}\}$ is normal to M. They satisfy

$$g(E_0, E_0)=g(E_{0^*}, E_{0^*})=\varepsilon=1 \quad \text{or} \quad -1\ ,$$

according as $a=0$ or 1. The range of indices are as follows:

$$\begin{aligned}
&A, B, \cdots=0, 1, \cdots, n\ ,\\
&a, b, \cdots=1, 2, \cdots, n\ ,\\
&I, J, \cdots=0, 1, \cdots, n, 0^*, 1^*, \cdots, n^*\ ,\\
&i, j, \cdots=1, \cdots, n, 1^*, \cdots, n^*\ .
\end{aligned}$$

Let $\{\bar{w}_I\}=\{\bar{w}_A, \bar{w}_{A^*}\}$ be the local field of dual frames on $\bar{M}$ with respect to the frame field $\{E_I\}$ chosen above. Namely they satisfy

$$\bar{w}_I(E_J)=\varepsilon_I\delta_{IJ}\ . \tag{1.1}$$

Then the indefinite Kaehlerian metric $\bar{g}$ can be expressed locally as

$$\bar{g}=\sum \varepsilon_I\bar{w}_I\otimes\bar{w}_I\ .$$

Associated with the frame field $\{E_I\}$, there exist linear forms $\bar{w}_{IJ}$ on $\bar{M}$ and the structure equations of $\bar{M}$ can be given by

$$(1.2)\qquad \begin{cases} d\bar{w}_I+\sum \varepsilon_J\bar{w}_{IJ}\wedge\bar{w}_J=0\ , \\ \bar{w}_{IJ}+\bar{w}_{JI}=0\ , \\ d\bar{w}_{IJ}+\sum \varepsilon_K\bar{w}_{IK}\wedge\bar{w}_{KJ}=\bar{\Omega}_{IJ}\ , \\ \bar{\Omega}_{IJ}=-\sum(\varepsilon_K\varepsilon_L\bar{R}_{IJKL}/2)\bar{w}_K\wedge\bar{w}_L\ , \end{cases}$$

where $\varepsilon_I\bar{w}_{IJ}$ are connection forms on $\bar{M}$ relative to $\{E_I\}$ and $\bar{\Omega}_{IJ}$ denote the curvature forms on $\bar{M}$, and $\bar{R}_{IJKL}$ are the components of the Riemannian curvature tensor $\bar{R}$ of $\bar{M}$. They satisfy

$$\begin{cases} \bar{w}_{0b}=\bar{w}_{0^*b^*}\ , & \bar{w}_{0b^*}=\bar{w}_{b0^*}\ , \\ \bar{w}_{ab}=\bar{w}_{a^*b^*}\ , & \bar{w}_{ab^*}=\bar{w}_{ba^*}\ . \end{cases}$$

Since the almost complex structure $\bar{J}$ satisfies

$$\bar{J}=\sum \varepsilon_I\varepsilon_J\bar{J}_{IJ}E_I\otimes\bar{w}_J\ ,$$

the equation $\bar{J}^2=-\mathrm{id.}$ is equivalent to

$$(1.3)\qquad \sum \varepsilon_K\bar{J}_{IK}\bar{J}_{KJ}=-\varepsilon_I\delta_{IJ}\ ,\qquad \bar{J}_{IJ}+\bar{J}_{JI}=0\ .$$

Since $\bar{M}$ is of constant holomorphic sectional curvature c, the Riemannian curvature tensor is given by (cf. [1])

$$(1.4)\qquad \bar{R}_{IJKL}=c\{\varepsilon_I\varepsilon_J(\delta_{IL}\delta_{JK}-\delta_{IK}\delta_{JL})+\bar{J}_{IL}\bar{J}_{JK}-\bar{J}_{IK}\bar{J}_{JL}-2\bar{J}_{IJ}\bar{J}_{KL}\}/4\ .$$

The restriction of these forms $\bar{w}_I$ and $\bar{w}_{IJ}$ to M are simply denoted by w_I and w_{IJ} without bar, respectively. Hence we have $w_0=0$ and $w_{0^*}=0$. The metric on M induced from the indefinite Riemannian metric $\bar{g}$ on $\bar{M}$ is given as $g=\sum\varepsilon_i w_i\otimes w_i$. Hence $\{E_i\}$ is a local field of orthonormal frames on M with respect to the metric, and $w_1, \cdots, w_n$ are the canonical forms on M. In terms of the canonical forms w_i and the connection forms w_{ij}, the structure equations of the hypersurface M are given as follows:

$$(1.5)\qquad \begin{cases} dw_i+\sum \varepsilon_j w_{ij}\wedge w_j=0\ , \\ w_{ij}+w_{ji}=0\ , \\ \Omega_{ij}=\bar{\Omega}_{ij}-\varepsilon(w_{i0}\wedge w_{0j}+w_{i0^*}\wedge w_{0^*j})\ , \\ \Omega_{ij}=-(\sum \varepsilon_k\varepsilon_l R_{ijkl}/2)w_k\wedge w_l \end{cases}$$

where $\varepsilon=\varepsilon_0=\varepsilon_{0^*}$, and Ω_{ij} (resp. R_{ijkl}) denotes the curvature form (resp. the components of the curvature tensor R) on M. The components J_{ij}

of the almost complex structure J on M satisfy

$$\sum \varepsilon_k J_{ik} J_{kj} = -\varepsilon_i \delta_{ij} , \tag{1.6}$$

by means of (1.3). It follows from $w_0=0$ and $w_{0^*}=0$ that

$$\sum \varepsilon_j w_{0j} \wedge w_j = 0 ,$$
$$\sum \varepsilon_j w_{0^*j} \wedge w_j = 0 .$$

By Cartan's lemma, we see

$$\begin{cases} w_{0i} = \sum \varepsilon_j h_{ij} w_j , \\ w_{0^*i} = \sum \varepsilon_j h^*_{ij} w_j , \\ h_{ij} = h_{ji} , \quad h^*_{ij} = h^*_{ji} . \end{cases} \tag{1.7}$$

Then the quadratic form

$$\varepsilon \sum \varepsilon_i \varepsilon_j (h_{ij} w_i \otimes w_j \otimes E_0 + h^*_{ij} w_i \otimes w_j \otimes E_{0^*})$$

is called the *second fundamental form* of M. Accordingly, by means of the above structure equations of M and $\bar{M}$ the equation of Gauss is obtained as

$$\begin{aligned} R_{ijkl} = & c\{\varepsilon_i \varepsilon_j (\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) + J_{il}J_{jk} - J_{ik}J_{jl} - 2J_{ij}J_{kl}\}/4 \\ & + \varepsilon(h_{il}h_{jk} - h_{ik}h_{jl} + h^*_{il}h^*_{jk} - h^*_{ik}h^*_{jl}) . \end{aligned} \tag{1.8}$$

For any point x in M, let $T_x(M)$ and $T_x(\bar{M})$ be tangent spaces at x to M and $\bar{M}$. Then $T_x(M)$ is by definition a non-degenerate subspace of $T_x(\bar{M})$ and a direct sum decomposition $T_x(\bar{M}) = T_x(M) + N_x(M)$ is given, where $N_x(M)$ is also non-degenerate and $\dim N_x(M) = 2$, which is called the *normal space* of M at x. Let $\mathfrak{X}(M)$ and $\mathfrak{X}^\perp(M)$ be the submodules of $\mathfrak{X}(\bar{M})$ consisting of all vector fields tangent to M and normal to M, respectively. By ∇ and $\bar{\nabla}$ the Levi-Civita connections of (M, g) and $(\bar{M}, \bar{g})$ are denoted. Then the second fundamental form α is given by

$$\bar{\nabla}_X Y = \nabla_X Y + \alpha(X, Y) , \qquad X, Y \in \mathfrak{X}(M) ,$$

and the shape operator A_ξ of M relative to the normal vector field ξ in $\mathfrak{X}^\perp(M)$ is given by

$$g(A_\xi X, Y) = \bar{g}(\alpha(X, Y), \xi) .$$

A_ξ is the self-adjoint endomorphism of $\mathfrak{X}(M)$ and A_{E_0} and $A_{E_0^*}$ are simply denoted by A and A^* for any orthonormal frame field $\{E_I\}$. It satisfies

$$(1.9)\qquad \begin{cases} \alpha(X, Y)=\alpha(Y, X) , \\ \alpha(JX, Y)=\alpha(X, JY)=\bar{J}\alpha(X, Y) , \end{cases}$$

$$(1.10)\qquad \begin{cases} h_{ij}=\bar{g}(\alpha(E_i, E_j), E_0)=g(AE_i, E_j) , \\ h^*_{ij}=\bar{g}(\alpha(E_i, E_j), E_{0*})=g(A^*E_i, E_j) , \end{cases}$$

and furthermore

$$(1.11)\qquad \begin{cases} A^*=JA , \quad A=-JA^* , \\ h^*_{ij}=\sum \varepsilon_k J_{ik}h_{kj} , \\ h_{ij}=-\sum \varepsilon_k J_{ik}h^*_{kj} , \end{cases}$$

and

$$(1.12)\qquad \begin{cases} AJ+JA=0 , \quad A^*J+JA^*=0 , \\ \sum \varepsilon_k h^*_{ik}h^*_{kj}=\sum \varepsilon_k h_{ik}h_{kj} , \\ \sum \varepsilon_k h_{ik}h^*_{kj}=-\sum \varepsilon_k h^*_{ik}h_{kj} . \end{cases}$$

The Ricci tensor S of M is given by

$$(1.13)\qquad S_{ij}=(n+1)c\varepsilon_i\delta_{ij}/2-2\varepsilon \sum \varepsilon_k h_{ik}h_{kj} .$$

§2. Proof of the theorem.

In this section, let M be an n-dimensional ($n\geqq 2$) indefinite complex hypersurface in $M^{n+1}_{s+a}(c)$. Assume that $c\neq 0$ and M satisfies the condition (*). Then this condition is written as

$$(2.1)\qquad \sum \varepsilon_l(R_{ijkl}S_{lm}+R_{ijml}S_{kl})=0 .$$

For the sake of brevity, a tensor h^m_{ij} and a function h_m on M for any integer m ($\geqq 2$) are introduced as follows:

$$(2.2)\qquad \begin{cases} h^m_{ij}=\sum \varepsilon_{i_1}\cdots\varepsilon_{i_{m-1}}h_{ii_1}h_{i_1i_2}\cdots h_{i_{m-1}j} , \\ h_m=\sum \varepsilon_i h^m_{ii} . \end{cases}$$

By means of (1.8) and (1.13), (2.1) is reduced to

$$(2.3)\qquad \begin{aligned} &c\sum \varepsilon_l[\varepsilon_j h^2_{il}\delta_{jk}-\varepsilon_i\delta_{ik}h^2_{jl}+\varepsilon_j h^2_{jk}\delta_{jl}-\varepsilon_i\delta_{il}h^2_{jk}+\sum \varepsilon_r\{(J_{ir}J_{jk}\\ &\quad -J_{ik}J_{jr}-2J_{ij}J_{kr})h^2_{rl}+(J_{ir}J_{jl}-J_{il}J_{jr}-2J_{ij}J_{lr})h^2_{rk}\}]/4\\ &\quad +\varepsilon[h^3_{il}h_{jk}-h_{ik}h^3_{jl}+h^3_{ik}h_{jl}-h_{il}h^3_{jk}+\sum \varepsilon_r\{(h^*_{ir}h^*_{jk}\\ &\quad -h^*_{ik}h^*_{jr})h^2_{rl}+(h^*_{ir}h^*_{jl}-h^*_{il}h^*_{jr})h^2_{rk}\}]=0 . \end{aligned}$$

By summing up this result with respect to i and l, it follows that

$$c(h^2_{jk}-h_2\delta_j\delta_{jk}/2n)=0\ ,$$

by virtue of (1.11) and (1.12), which yields that $h^2_{ij}=h_2\varepsilon_i\delta_{ij}/2$ when $c\neq 0$. This implies that M is Einstein provided that $n\geqq 2$. Consequently the proof of the theorem is complete.

REMARK. This property is an extension of a theorem of Ryan [6] in the case of complex hypersurfaces in an indefinite complex space form. The proof is slightly different.

Assume that the ambient space is an indefinite complex Euclidean space. Multiplying $\varepsilon_i\varepsilon_l h^{2m-1}_{il}$ for any integer m to (2.3) and summing up this result for i and l, we obtain

$$h_{2m}h^3_{jk}=h_{2m+2}h_{jk}\ ,$$

which implies that

$$h^3_{jk}=fh_{jk} \quad \text{for a function } f \text{ on } M\ , \tag{2.4}$$

if the set of points on M at which the function h_2 is zero is of measure zero. Under this hypothesis, it follows from (1.11) that the equation (2.3) is equivalent to (2.4).

A complex hypersurface M of index $2s$ in C^{n+1}_{s+a} is said to be *cylindrical* if M is a product manifold of C^{n-1}_t and a complex curve in C^2_r orthogonal to C^{n-1}_t in C^{n+1}_{s+a} $(r+t=s)$. It is evident that a cylinder M of index $2s$ in C^{n+1}_{s+a} satisfies the condition $(*)$, but it is not Einstein.

REMARK. (1) Romero [3] showed that there exist complete complex hypersurfaces in C^{2n+1}_n which are Ricci-flat. These satisfy the condition $(*)$ and are not cylindrical. Other examples will be given in the next section.

(2) In a definite case, Takahashi [7] proved that the cylindrical hypersurface is the only complete complex hypersurfaces in C^{n+1} satisfying the condition $(*)$ except for C^n. However, as shown in Remark (1), the property can not be extended in an indefinite complex Euclidean space. It is not known that whether or not there exist complex hypersurfaces satisfying $(*)$ in C^{n+1}_s which are not Einstein and not cylindrical.

Next a complex hypersurface with parallel Ricci tensor in an indefinite complex Euclidean space will be investigated. The components h_{ijk} and h^*_{ijk} of the covariant derivative of the second fundamental form are defined by

$$\sum \varepsilon_k h_{ijk} w_k = dh_{ij} - \sum \varepsilon_k (h_{kj} w_{ki} + h_{ik} w_{kj}) + \varepsilon h^*_{ij} w \ ,$$
$$\sum \varepsilon_k h^*_{ijk} w_k = dh^*_{ij} - \sum \varepsilon_k (h^*_{kj} w_{ki} + h^*_{ik} w_{kj}) - \varepsilon h_{ij} w \ ,$$

where $w = w_{00^*}$. Restricting the third equation of the structure equations of M to the hypersurface, we have

$$dw_{0i} + \sum \varepsilon_j w_{0j} \wedge w_{ji} + \varepsilon w_{00^*} \wedge w_{0^*i} = \bar{\Omega}_{0i} \ ,$$

from which together with $w_{0i} = \sum \varepsilon_j h_{ij} w_j$ it follows

$$\sum \varepsilon_j \varepsilon_k h_{ijk} w_j \wedge w_k = 0 \ , \qquad \sum \varepsilon_j \varepsilon_k h^*_{ijk} w_j \wedge w_k = 0 \ .$$

This means that

$$h_{ijk} = h_{ikj} \ , \qquad h^*_{ijk} = h^*_{ikj} \ .$$

On the other hand, since the hypersurface M has parallel Ricci tensor, it follows that

$$\sum \varepsilon_r h_{ijr} h_{rk} = 0 \ . \tag{2.5}$$

PROPOSITION 2.1. *Let M be a complex hypersurface of index 0 with parallel Ricci tensor in C_1^{n+1}. Then M is totally geodesic.*

PROOF. The component h_{ijkl} of the covariant derivative $\nabla^2\alpha$ of $\nabla\alpha$ is defined by

$$\sum \varepsilon_l h_{ijkl} w_l = dh_{ijk} - \sum \varepsilon_l (h_{ljk} w_{li} + h_{ilk} w_{lj} + h_{ijl} w_{lk}) + \varepsilon h^*_{ijk} w \ .$$

Differentiating $\sum \varepsilon_k h_{ijk} w_k$ exteriorly, we obtain

$$\sum \varepsilon_k \varepsilon_l h_{ijkl} w_l \wedge w_k = \sum \varepsilon_k \varepsilon_r \varepsilon_s ((R_{kirs} h_{kj} + R_{kjrs} h_{ik})/2$$
$$- h^*_{ij} \bar{R}_{00^*rs} + \varepsilon h^*_{ij} h_{kr} h^*_{ks}) w_r \wedge w_s \ ,$$

and hence

$$h_{ijkl} - h_{ijlk} = -\sum \varepsilon_r (R_{lkir} h_{rj} + R_{lkjr} h_{ir} + 2h^*_{ij} h_{kr} h^*_{rl}) \ .$$

Substituting (1.8) into the result above and making use of

$$\sum \varepsilon_r \varepsilon_l h_{ir} (h_{rjkl} - h_{rjlk}) h_{lm} = 0 \ ,$$

we have

$$h^2_{hk} h^3_{jm} - h^3_{hm} h^2_{jk} + h^4_{hm} h_{jk} - h^3_{hk} h^2_{jm}$$
$$+ \sum \varepsilon_r \varepsilon_s (-h_{hs} h^*_{sk} h^*_{jr} h^2_{rm} + h^2_{hs} h^*_{sm} h^*_{kr} h_{rj} - h^3_{hs} h^*_{sm} h^*_{jk}$$
$$- h^2_{hr} h^*_{rk} h^*_{js} h_{sm} - 2h_{hr} h^*_{rj} h^2_{ks} h^*_{sm}) = 0 \ .$$

Summing up the relation with respect to m and h, we have

$$4h^5_{ij}+h_4h_{ij}=0$$

and hence

$$4h_6+h_2h_4=0 .$$

Since the functions h_2, h_4 and h_6 are all non-negative, h_6 must vanish identically. This implies that M is totally geodesic.

REMARK. (1) Here the complete different method from that of the proof of a theorem due to Nomizu and Smyth [2] in the complex Euclidean space C^{n+1} is used.

(2) Let M be an indefinite complex hypersurface with parallel Ricci tensor of C^{n+1}_{s+1}. Then the fact $h^3_{ij}=0$ is proved by Romero (personal communication) and the authors independently. Their method of the proof is dependent on the complex version which is different from Romero's one.

§3. Examples.

This section is devoted to investigating some examples of Einstein complex hypersurfaces in C^{2n+1}_s. Let h_j be holomorphic functions of C. In this section, the range of indices are given as follows:

$$i, j, \cdots=1, \cdots, n ,$$
$$a, b, \cdots=1, \cdots, s ,$$
$$x, y, \cdots=s+1, \cdots, n ,$$
$$A, B, \cdots=1, \cdots, 2n .$$

For the complex coordinate system (z_A, z_{2n+1}) of C^{2n+1}_s, let $M=M^{2n}_s(h_j; c_j)$ be the complex hypersurface in C^{2n+1}_s given by the equation

$$z_{2n+1}=\sum h_j(z_j+c_jz_{j^*}) , \qquad j^*=n+j$$

for any complex number c_j. Then $M^{2n}_s(h_j; c_j)$ is a family of complex hypersurfaces in C^{2n+1}_s.

REMARK. $M^{2n}_n(z^p; 1)$ for any integer p ($\geqq 2$) is a complete complex hypersurface given by Romero [3], which is Ricci-flat but not flat and of index $2n$.

For the simplicity, the calculation from the standpoint of the complex version is used. For an isometric and holomorphic imbedding of C^{2n}_s into

C_s^{2n+1} defined by

$$f(z_A)=f(z_A,\ z_{2n+1})\ ,\quad z_{2n+1}=\sum h_j(z_j+c_jz_{j^*})\ ,$$

it is easily seen that $M=f(C_s^{2n})$ is a complete complex hypersurface in C_s^{2n+1} and the natural basis of the tangent space $T_z(M)$ of M at any point $z=(z_A,\ z_{2n+1})$ is given as follows:

$$f_A=(0,\ \cdots,\ \overset{A}{1},\ 0,\ \cdots,\ 0,\ h'_A)\ , \tag{3.1}$$

where $\partial h_j/\partial z_j=h'_j$, $\partial h_j/\partial z_{j^*}=h'_{j^*}=c_jh'_j$. Then

$$\xi_z=(\bar h'_a\ ,\ -\bar h'_x\ ,\ -\bar c_a\bar h'_a\ ,\ -c_x\bar h'_x\ ,\ 1)$$

is a normal vector to M at z. Let g be the usual Kaehlerian flat metric of index $2s$ on C_s^{2n+1}. By the same g is denoted an indefinite Kaehlerian metric induced from the Kaehlerian flat metric in the ambient space. Since ξ_z satisfies

$$g(\xi_z,\ \xi_z)=1+\sum(|c_a|^2-1)|h'_a|^2+\sum(|c_x|^2+1)|h'_x|^2\ ,$$

the normal vector field ξ is space-like and M is of index $2s$ in C_s^{2n+1} provided that c_a satisfies $|c_a|\geqq 1$ for any a. Furthermore, it is shown that $M_s^{2n}(h_j,\ c_j)$ is a graph of a holomorphic function of C^{2n}, which means that it is holomorphically diffeomorphic to C^{2n}. Thus we have

THEOREM 3.1. *$M_s^{2n}(h_j;\ c_j)$ is a complete connected complex hypersurface of index $2s$ in C_s^{2n+1} if $|c_a|\geqq 1$ for any a. Furthermore it is holomorphically diffeomorphic to C^{2n}.*

By setting $\xi'_z=\xi_z/|\xi_z|$, where $|\xi_z|=g(\xi_z,\ \bar\xi_z)^{1/2}$, ξ' is a unit normal vector field on M. Since the covariant derivatives of the vector field f_A in the direction of f_B are given as follows;

$$\begin{cases} f_{ij}=(0,\ \cdots,\ 0,\ \delta_{ij}h''_i)\ , \\ f_{ij^*}=f_{i^*j}=(0,\ \cdots,\ 0,\ c_i\delta_{ij}h''_i)\ , \\ f_{i^*j^*}=(0,\ \cdots,\ 0,\ c_i^2\delta_{ij}h''_i)\ , \end{cases} \tag{3.2}$$

where $h''_j=\partial h'_j/\partial z_j$, the shape operator A associated with the unit normal ξ' satisfies

$$\begin{aligned} &g(Af_i,\ f_j)=\delta_{ij}h''_i/|\xi|=h_{ij}\ , \\ &g(Af_i,\ f_{j^*})=c_i\delta_{ij}h''_i/|\xi|=h_{ij^*}\ , \\ &g(Af_{i^*},\ f_{j^*})=c_i^2\delta_{ij}h''_i/|\xi|=h_{i^*j^*}\ , \end{aligned} \tag{3.3}$$

where h_{ij}, h_{i^*j} and $h_{i^*j^*}$ denote the components of the second fundamental form of M derived from the unit normal ξ' relative to the natural frame $\{f_A\}$. These formulas and the Gauss equation give an information about the isometric structure for each hypersurface.

PROPOSITION 3.2. *Under the same assumption of Theorem 3.1, two indefinite hypersurfaces $M_s^{2n}(h_j; c_j)$ and $M_s^{2n}(\tilde{h}_j; \tilde{c}_j)$ are congruent to each other if and only if $c_j=\tilde{c}_j$, $h_j'=\tilde{h}_j'$ and $h_j''=\tilde{h}_j''$ for any j up to an order.*

On the other hand, it is easily seen by (3.3) that we have

$$Af_{j*}=\bar{c}_j Af_j \tag{3.4}$$

and it follows from the straightforward calculation that the coefficients of

$$Af_i=\sum \bar{\beta}_{ij}f_j+\sum \bar{\gamma}_{ij}Af_j$$

satisfy the following relationships:

$$\begin{cases}\bar{c}_b\beta_{ib}+\gamma_{ib}=0 & \text{for any } b\,, \\ \bar{c}_y\beta_{iy}-\gamma_{iy}=0 & \text{for any } y\,,\end{cases} \tag{3.5}$$

and for any fixed indices a and x

$$\begin{aligned}&(1+(|c_a|^2-1)|h_a'|^2)\beta_{ia}+\sum_{b\neq a}(|c_b|^2-1)\bar{h}_b'h_a'\beta_{ib}\\ &\qquad -\sum_y(|c_y|^2+1)\bar{h}_y'h_a'\beta_{iy}=-\delta_{ia}h_a''/|\xi|\,,\\ &(1+(|c_x|^2+1)|h_x'|^2)\beta_{ix}-\sum_b(|c_b|^2-1)\bar{h}_b'h_x'\beta_{ib}\\ &\qquad +\sum_{y\neq x}(|c_y|^2+1)\bar{h}_y'h_x'\beta_{iy}=\delta_{ix}h_x''/|\xi|\,.\end{aligned} \tag{3.6}$$

By giving attention to these equations, the following property is valid.

THEOREM 3.3. *If all functions h_x are linear and if $|c_a|=1$, then $M_s^{2n}(h_j; c_j)$ is Ricci-flat. In particular, it is not flat provided that there is an index a such that h_a is not linear.*

PROOF. Under the assumption the second equation of (3.6) is a homogeneous system of linear equations with constant coefficients and the matrix of the coefficients is regular. Accordingly it is easily seen that we have

$$\beta_{ia}=-\delta_{ia}h_a''/|\xi|\,,\qquad \beta_{ix}=0\,,$$

which yield that

$$(3.7)\qquad \begin{cases} Af_a = h''_a(-f_a + \bar{c}_a f_{a*})/|\xi| , \\ Af_x = 0 . \end{cases}$$

Let $u=(u_A, u_{2n+1})$ in C_s^{2n+1} be a tangent vector to M at z. Then it is expressed as a linear combination

$$u = \sum u_A f_A , \qquad u = \sum (u_j + c_j u_{j*}) h'_j ,$$

and moreover we have $Au = \sum u_A A f_A$, which yields together with (3.4) and (3.7) that

$$(3.8)\qquad Au = \sum (\bar{u}_a + \bar{c}_a \bar{u}_{a*}) h''_a (-f_a + \bar{c}_a f_{a*})/|\xi| .$$

Let P_a be the tensor field of type (1, 1) defined by

$$P_a u = (0, \cdots, 0, -(\bar{u}_a + \bar{c}_a \bar{u}_{a*}), 0, \cdots, 0, \bar{c}_a(\bar{u}_a + \bar{c}_a \bar{u}_{a*}), 0, \cdots, 0)$$

where $u=(u_A, u_{2n+1})$ denotes any tangent vector to M at z. Then (3.8) means that the shape operator A can be decomposed into

$$A = \sum A_a(z) P_a , \qquad A_a(z) = h''_a(z)/|\xi| ,$$

and moreover it follows that operation P_a satisfies the following properties:

(a) P_a is the self-adjoint operator of the tangent space of M,

(b) $P_a \circ P_b = 0$ for any a and b.

This implies $\bar{A} \circ A = 0$, from which it turns out that M is Ricci-flat. Since A does not vanish identically, the Gauss equation implies that M is not flat.

In particular, if $s=n$ and $c_i=1$ for any i, then M satisfies the assumption of the above theorem. Thus one finds the following

COROLLARY 3.4. *$M_n^{2n}(h_j; 1)$ is a complex hypersurface of index $2n$ of C_n^{2n+1} and it is Ricci-flat.*

Now, for any integer p ($\geqq 2$), let $M_p(c_j)$ be an indefinite complete hypersurface of C_n^{2n+1} defined by the equation

$$z_{2n+1} = \sum (z_j - c_j z_{j*})^p , \qquad |c_j| = 1 .$$

Romero [3] studied the case $c_j=1$ for each j, which is denoted by M_p. Then the normal vector is unit and hence, by taking account of (3.2), the covariant derivatives of the vector fields f_{AB} in the direction of f_A are given as follows:

$$f_{ijk}=(0, \cdots, 0, \delta_{ij}\delta_{ik}h_i''') ,$$
$$\cdots\cdots\cdots\cdots$$
$$f_{i^*j^*k^*}=(0, \cdots, 0, c_i^3\delta_{ij}\delta_{ik}h_i''') ,$$

from which it follows that for the components h_{ABC} of the covariant derivatives of the second fundamental form we have

$$\text{(3.9)}\qquad \begin{cases} h_{ijk}=\delta_{ij}\delta_{ik}h_i''' , \\ h_{i^*jk}=h_{ij^*k}=h_{ijk^*}=c_i\delta_{ij}\delta_{ik}h_i''' , \\ h_{i^*j^*k}=h_{i^*jk^*}=h_{ij^*k^*}=c_i^2\delta_{ij}\delta_{ik}h_i''' , \\ h_{i^*j^*k^*}=c_i^3\delta_{ij}\delta_{ik}h_i''' . \end{cases}$$

By means of Theorem 3.1 and Theorem 3.3 it is seen that $M_p(c_j)$ is a complete hypersurface of index $2n$ of C_n^{2n+1}, which is Ricci-flat but not flat. Furthermore it follows from (3.9) that the second fundamental form is parallel provided that $p=2$ and also

$$\text{(3.10)}\qquad \sum \xi_A h_{ijA}h_{Ak}=(\xi_i+|c_i|^2)\delta_{ij}\delta_{ik}h_i'''\bar{h}_k''\neq 0$$

provided that $p\geqq 3$. This means that $M_p(c_j)$ ($p\geqq 3$) is not locally symmetric because of the Gauss equation. Thus one finds

THEOREM 3.5. *$M_2(c_j)$ is locally symmetric and $M_p(c_j)$ is not locally symmetric if $p\geqq 3$.*

About the homogeneity of these examples $M=M_s^{2n}(h_j, c_j)$ with respect to the induced Kaehlerian metric, one finds

THEOREM 3.6. *If each function h_j satisfies $h_j''(0)=0$, then $M_s^{2n}(h_j, c_j)$ is not homogeneous with respect to the induced indefinite Kaehlerian metric.*

PROOF. For the point z_0 in M such that $z_j=-c_j z_{j^*}$ for all j, the Gauss equation and (3.3) imply $R(z_0)=0$. It means that M is not homogeneous; otherwise we have $R=0$ at every point. But it is impossible. because M is not flat.

Bibliography

[1] M. BARROS and A. ROMERO, Indefinite Kähler manifolds, Math. Ann., **261** (1982), 55-62.
[2] K. NOMIZU and B. SMYTH, Differential geometry of complex hypersurfaces II, J. Math. Soc. Japan, **20** (1968), 498-521.
[3] A. ROMERO, Some examples of indefinite complete Einstein hypersurfaces, Proc. Amer. Math. Soc., **98** (1986), 283-286.

[4] A. ROMERO, On a certain class of complete complex Einstein hypersurfaces in indefinite complex space forms, Math. Z., **192** (1986), 627-635.
[5] A. ROMERO, Differential geometry of complex hypersurfaces in an indefinite complex space form, Lecture Note, Univ. de Granada, 1986.
[6] P. J. RYAN, A class of complex hypersurfaces, Colloq. Math., **26** (1972), 175-182.
[7] T. TAKAHASHI, Complex hypersurfaces with $RS=0$ in C^{n+1}, Tôhoku Math. J., **25** (1973), 527-533.
[8] J. A. WOLF, Spaces of Constant Curvature, McGraw-Hill, 1967.

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