TOKYO J. MATH.
VOL. 10, NO. 2, 1987

On the Value of Dedekind Sums and Eta-Invariants for 3-Dimensional Lens Spaces

Kiyoshi KATASE

Gakushuin University

Introduction

Let $[x]$ denote the greatest integer less than or equal to the real number x, and p and q be relatively prime positive integers. It is a difficult problem to determine the exact value of the Dedekind sum $\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^n$ except the trivial case: $\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]=\frac{1}{2}(p-1)(q-1)$. In general, it is impossible to evaluate the sum for $n\geqq 2$ as a polynomial on p and q. However, in case of $n=2$, we can express the sum as a finite sum with fewer terms using the sequence of remainders in the Euclidean algorithm for calculating the $\gcd(p, q)$ (see T. M. Apostol [1], p. 73).

On the other hand, instead of evaluating the sum, the reciprocity formulas which can be used as an aid in calculating the Dedekind sums have been studied. The only results we have known are:

$$(1)\qquad 6p\sum_{h=1}^{p-1}\left[\frac{hq}{p}\right]^2+6q\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2=(p-1)(q-1)(2p-1)(2q-1)$$

and

$$(2)\qquad 4p(p-1)\sum_{h=1}^{p-1}\left[\frac{hq}{p}\right]^3+4q(q-1)\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^3=(p-1)^2(q-1)^2(2pq-p-q+1)\,.$$

(See, for example, L. Carlitz [4] (1.7).)

There are some ways to prove these formulas (see, for example, T. M. Apostol and T. H. Vu [2], L. Carlitz [4], H. Rademacher and A. Whiteman [11], (3.5) and D. Zagier [12]). In the preceding paper [9], we introduced a lemma which was derived, for example, from Rademacher and Whiteman's method and is available to reduce almost all types of the generalized Dedekind sums to the sums of fewer terms. So, in this paper, we shall apply the lemma to evaluate the sum $\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2$.

Received January 13, 1987

We show, in §1, a recursive formula and we evaluate the sum $\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2$ as a polynomial in §2. The result is Theorem 3:

$$\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2=\frac{1}{3}(p-1)^2(q-1)-\frac{p}{6}(p-q)+\frac{p}{12}-\frac{1}{6}+\frac{1}{4}(-1)^{n-1}p$$
$$+\frac{p}{6}\sum_{i=0}^{n}(-1)^i\left[\frac{q_{i-1}}{q_i}\right]+\frac{1}{6}(-1)^n a_{n-1}\ ,$$

where the integers q_i are defined inductively by $q_{-1}=p$, $q_0=q$, $q_i=q_{i-2}-\left[\frac{q_{i-2}}{q_{i-1}}\right]q_{i-1}$ $(i=1, 2, \cdots)$ and $q_n=1$ and a positive integer a_{n-1} $\left(<\frac{p}{2}\right)$ is represented by a polynomial on $\left[\frac{q_{i-1}}{q_i}\right]$ $(i=0, 1, \cdots, n-1)$ which is also determined uniquely by the equation $qa_{n-1}\equiv(-1)^n$ modulo p.

We represent the eta-invariant $\eta(p; q)$ for the 3-dimensional lens space $L(p; q)$ by the sum $\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2$ in §3 and, in §4, we prove Theorem 5 which states its complete invariantness for the isometric class when p is a prime number or $p=kp'$ where p' is a prime number and $k=2$ or 3.

As for the case where p is a general composite number, we study, in §5, to what extent the eta-invariant $\eta(p; q)$ is available taking the length of the sequence of remainders in the Euclidean algorithm for calculating the gcd(p, q) into account. Our main result is Theorem 12 which states, by clarifying when exceptional cases will occur, that the eta-invariant is "almost complete".

§1. Recursive formula.

Let p, q be relatively prime positive integers and put

$$\gamma_k=kp-\left[\frac{kp}{q}\right]q$$

for $k=1, \cdots, q-1$. Then

$$\left[\frac{(k+1)p}{q}\right]-\left[\frac{kp}{q}\right]=\begin{cases}\left[\frac{p}{q}\right]+1 & \text{if } \gamma_k+\gamma_1\geqq q\ ,\\ \left[\frac{p}{q}\right] & \text{otherwise}\ ,\end{cases}$$

for each integer k $(1\leqq k\leqq q-1)$.

Since

$$\sum_{k=1}^{q-1}\left\{\left[\frac{(k+1)p}{q}\right]-\left[\frac{kp}{q}\right]\right\}=(q-1)\left[\frac{p}{q}\right]+\#\{k;\ \gamma_k+\gamma_1\geqq q\}$$

and the left-hand side is equal to $p-\left[\frac{p}{q}\right]$, there exist $p-\left[\frac{p}{q}\right]q=\gamma_1$ such integers that satisfy $\gamma_k+\gamma_1\geqq q$. We put these integers as

$$k_1<k_2<\cdots<k_{\gamma_1-1}<k_{\gamma_1}\ .$$

Then we obtain the following

LEMMA 1. *These integers are represented as follows:*

$$k_i=\left[\frac{iq}{\gamma_1}\right]\qquad for\ i=1,\ \cdots,\ \gamma_1-1$$

and

$$k_{\gamma_1}=q-1\ .$$

PROOF. It follows from the inequalities

$$\frac{iq}{\gamma_1}-1<\left[\frac{iq}{\gamma_1}\right]<\frac{iq}{\gamma_1}$$

that

$$i-\frac{\gamma_1}{q}<\left[\frac{iq}{\gamma_1}\right]\frac{\gamma_1}{q}<i$$

and since $\gamma_1<q$, we get $\left[\left[\frac{iq}{\gamma_1}\right]\frac{\gamma_1}{q}\right]=i-1$ for each $i=1,\cdots,\gamma_1-1$. Thus

$$\begin{aligned}\gamma_{\left[\frac{iq}{\gamma_1}\right]}+\gamma_1&=\left[\frac{iq}{\gamma_1}\right]p-\left[\left[\frac{iq}{\gamma_1}\right]\frac{p}{q}\right]q+\gamma_1\\&=\left[\frac{iq}{\gamma_1}\right]p-\left[\left[\frac{iq}{\gamma_1}\right]\frac{\gamma_1}{q}+\left[\frac{iq}{\gamma_1}\right]\left[\frac{p}{q}\right]\right]q+\gamma_1\\&=\left[\frac{iq}{\gamma_1}\right]p-\left(i-1+\left[\frac{iq}{\gamma_1}\right]\left[\frac{p}{q}\right]\right)q+\gamma_1\\&=q-\left\{iq-\left[\frac{iq}{\gamma_1}\right]\left(p-\left[\frac{p}{q}\right]q\right)\right\}+\gamma_1\\&=q-\left(iq-\left[\frac{iq}{\gamma_1}\right]\gamma_1\right)+\gamma_1\\&\geqq q+1\ ,\end{aligned}$$

and we conclude that $\left[\frac{iq}{\gamma_1}\right]=k_i$ $(1\leqq i\leqq\gamma_1-1)$. The last equation $k_{\gamma_1}=q-1$ follows from the fact that $\gamma_{q-1}+\gamma_1=q$ and $k_{\gamma_1-1}<q-1$.

Applying Lemma 1, we get

$$\sum_{k=1}^{q-1}k^2\left\{\left[\frac{(k+1)p}{q}\right]-\left[\frac{kp}{q}\right]\right\}=\left[\frac{p}{q}\right]\sum_{k=1}^{q-1}k^2+\sum_{i=1}^{\gamma_1}k_i^2$$
$$=\frac{1}{6}q(q-1)(2q-1)\left[\frac{p}{q}\right]+\sum_{i=1}^{\gamma_1-1}\left[\frac{iq}{\gamma_1}\right]^2+(q-1)^2\,.$$

On the other hand, since the left-hand side is equal to

$$\sum_{k=1}^{q-1}\left\{(k+1)^2\left[\frac{(k+1)p}{q}\right]-k^2\left[\frac{kp}{q}\right]\right\}-2\sum_{k=1}^{q-1}(k+1)\left[\frac{(k+1)p}{q}\right]+\sum_{k=1}^{q-1}\left[\frac{(k+1)p}{q}\right]$$
$$=p(q-1)^2-2\sum_{k=1}^{q-1}k\left[\frac{kp}{q}\right]+\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right],$$

we get

$$(3)\qquad 2\sum_{k=1}^{q-1}k\left[\frac{kp}{q}\right]+\sum_{i=1}^{\gamma_1-1}\left[\frac{iq}{\gamma_1}\right]^2=\frac{1}{2}(p-1)(q-1)(2q-1)-\frac{1}{6}q(q-1)(2q-1)\left[\frac{p}{q}\right].$$

Now, since the numbers $\gamma_k=kp-\left[\frac{kp}{q}\right]q$ for $k=1,\cdots,q-1$ are simply the numbers $1,\cdots,q-1$ in some order, we get

$$(4)\qquad p^2\sum_{k=1}^{q-1}k^2-2pq\sum_{k=1}^{q-1}k\left[\frac{kp}{q}\right]+q^2\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2=\frac{1}{6}q(q-1)(2q-1)\,.$$

Combining the equations (3) and (4) we obtain the equation:

$$(5)\qquad \frac{1}{p}\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2+\frac{1}{q}\sum_{i=1}^{\gamma_1-1}\left[\frac{iq}{\gamma_1}\right]^2$$
$$=\frac{1}{6pq}(p-1)(2p-1)(q-1)(2q-1)-\frac{1}{6}(q-1)(2q-1)\left[\frac{p}{q}\right].$$

Note that the equation (5) coincides with the reciprocity formula (1) when $p<q$. On the other hand, $\gamma_1<q$ when $p>q$ and hence the equation (5) means that the sum $\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2$ is reduced to the sum $\sum_{i=1}^{\gamma_1-1}\left[\frac{iq}{\gamma_1}\right]^2$ which has definitely fewer terms. Thus we assume $1\leqq q<p$ in the rest of this paper and let us regard the equation (5) as a recursive formula.

§ 2. Calculation.

Let

$$q_{-1}=p\ ,\quad q_0=q\ ,\quad q_1=\gamma_1=p-\left[\frac{p}{q}\right]q$$

and

$$q_{i+2}=q_i-\left[\frac{q_i}{q_{i+1}}\right]q_{i+1}\qquad (i=0,1,\cdots)\ .$$

Then it follows easily that

$$q=q_0>q_1>q_2>\cdots$$

and

$$q_n=1$$

for some n ($\geqq 1$). Note that n is, in most cases, small enough as compared with q.

We also get, as in § 1, the following recursive formulas:

$$\begin{aligned}
&\frac{1}{q_i}\sum_{k=1}^{q_{i+1}-1}\left[\frac{kq_i}{q_{i+1}}\right]^2+\frac{1}{q_{i+1}}\sum_{j=1}^{q_{i+2}-1}\left[\frac{jq_{i+1}}{q_{i+2}}\right]^2\\
&=\frac{1}{6q_iq_{i+1}}(q_i-1)(2q_i-1)(q_{i+1}-1)(2q_{i+1}-1)\\
&\quad-\frac{1}{6}(q_{i+1}-1)(2q_{i+1}-1)\left[\frac{q_i}{q_{i+1}}\right]\\
&=\frac{1}{6q_iq_{i+1}}(q_i-1)^2(q_{i+1}-1)^2+\frac{1}{6q_i}(q_i-1)^2(q_{i+1}-1)\\
&\quad+\frac{1}{6q_{i+1}}(q_{i+1}-1)^2(q_{i+2}-1)+\frac{1}{6}(q_{i+1}-1)(q_{i+2}-1)
\end{aligned}$$

for $i=-1, 0, \cdots, n-2$.

Multiplying $(-1)^{i+1}$ to the both sides and then adding them from $i=-1$ to $n-2$, we get

$$\begin{aligned}
&\frac{1}{p}\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2+(-1)^{n-1}\frac{1}{q_{n-1}}\cdot\sum_{j=1}^{q_n-1}\left[\frac{jq_{n-1}}{q_n}\right]^2\\
&=\sum_{i=-1}^{n-2}\frac{(-1)^{i+1}}{6q_iq_{i+1}}(q_i-1)^2(q_{i+1}-1)^2+\frac{1}{6p}(p-1)^2(q-1)\\
&\quad+\frac{(-1)^{n-1}}{6q_{n-1}}(q_{n-1}-1)^2(q_n-1)+\sum_{i=0}^{n-1}\frac{(-1)^i}{6}(q_i-1)(q_{i+1}-1)\ .
\end{aligned}$$

Substituting $q_n=1$, this equation reduces to

$$\frac{1}{p}\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2=\frac{1}{6p}(p-1)^2(q-1)+\sum_{i=-1}^{n-2}\frac{(-1)^{i+1}}{6q_iq_{i+1}}(q_i-1)^2(q_{i+1}-1)^2$$
$$+\sum_{i=0}^{n-2}\frac{(-1)^i}{6}(q_i-1)(q_{i+1}-1)\ .$$

Thus we have expressed the sum $\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2$ as a sum of fewer terms.

Next we are going to express the sum $\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2$ in some polynomial form for it has an integral value. Since

$$\frac{1}{6q_iq_{i+1}}(q_i-1)^2(q_{i+1}-1)^2=\frac{1}{6}(q_i-1)(q_{i+1}-1)-\frac{1}{6q_i}(q_i-1)(q_{i+1}-1)$$
$$-\frac{1}{6q_{i+1}}(q_i-1)(q_{i+1}-1)+\frac{1}{6}-\frac{1}{6q_i}-\frac{1}{6q_{i+1}}+\frac{1}{6q_iq_{i+1}}\ ,$$

we get

$$\frac{1}{p}\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2=\frac{1}{6p}(p-1)^2(q-1)+\frac{1}{6}(p-1)(q-1)$$
$$+\sum_{i=-1}^{n-2}(-1)^i\left\{\frac{1}{6q_i}(q_i-1)(q_{i+1}-1)+\frac{1}{6q_{i+1}}(q_i-1)(q_{i+1}-1)\right\}$$
$$+\frac{1}{6}\cdot\frac{1-(-1)^n}{2}-\frac{1}{6p}-\frac{(-1)^{n-1}}{6q_{n-1}}+\frac{1}{6}\sum_{i=-1}^{n-2}\frac{(-1)^{i+1}}{q_iq_{i+1}}$$
$$=\frac{1}{3p}(p-1)^2(q-1)-\sum_{i=0}^{n-2}(-1)^i\left\{\frac{1}{6q_i}(q_{i-1}-1)(q_i-1)-\frac{1}{6q_i}(q_i-1)(q_{i+1}-1)\right\}$$
$$-\frac{(-1)^{n-1}}{6q_{n-1}}(q_{n-2}-1)(q_{n-1}-1)+\frac{1}{6}\cdot\frac{1-(-1)^n}{2}-\frac{1}{6p}+\frac{1}{6}\sum_{i=-1}^{n-1}\frac{(-1)^{i+1}}{q_iq_{i+1}}\ .$$

Now, the sum of the second term on the right-hand side of the above equation is reduced to

$$\sum_{i=0}^{n-2}(-1)^i\left\{\frac{1}{6}(q_{i-1}-q_{i+1})-\frac{1}{6}\left[\frac{q_{i-1}}{q_i}\right]\right\}$$
$$=\frac{1}{6}\{p-q-(-1)^{n-3}q_{n-2}-(-1)^{n-2}q_{n-1}\}-\frac{1}{6}\sum_{i=0}^{n-2}(-1)^i\left[\frac{q_{i-1}}{q_i}\right]$$

and as for the sum of the last term we have the following lemma which is proved easily by induction on h.

LEMMA 2. *Let $a_{-1}=1$, $a_0=\left[\frac{p}{q}\right]$ and a_j $(j=1, 2, \cdots, n-1)$ be the positive integers defined inductively by the following relation:*

$$p=a_j q_j+a_{j-1}q_{j+1}\,. \tag{6}$$

Then we get

$$\sum_{i=-1}^{h}\frac{(-1)^{i+1}}{q_i q_{i+1}}=\frac{(-1)^{h+1}a_h}{pq_{h+1}} \qquad (h=-1,\,0,\,\cdots,\,n-1)\,.$$

Note that $a_{-1}\leqq a_0<a_1<\cdots<a_{n-1}<\frac{p}{2}$ and also note that

$$\begin{aligned}(-1)^n q a_{n-1}&=1+pq\sum_{i=0}^{n-1}\frac{(-1)^{i+1}}{q_i q_{i+1}}\\ &\equiv 1 \qquad \text{modulo } p\,,\end{aligned} \tag{7}$$

since $q\sum_{i=0}^{n-1}\frac{(-1)^{i+1}}{q_i q_{i+1}}=(-1)^n\bar{a}_{n-1}$ is an integer where $\bar{a}_{n-1}$ is defined inductively by $\bar{a}_0=1$, $\bar{a}_1=\left[\frac{q}{q_1}\right]$ and the equation

$$q=\bar{a}_j q_j+\bar{a}_{j-1}q_{j+1}\,, \qquad j=1,\,2,\,\cdots,\,n-1\,.$$

Hence we have only to express a_{n-1} in some polynomial form and an induction argument using the recursive equation

$$a_{j+1}=a_{j-1}+a_j\left[\frac{q_j}{q_{j+1}}\right] \qquad (j=0,\,1,\,\cdots,\,n-2)$$

yields

$$\begin{aligned}a_{n-1}=\sum\left[\frac{p}{q}\right]\left[\frac{q}{q_1}\right]\cdots\widehat{\left[\frac{q_{i_1-1}}{q_{i_1}}\right]}\widehat{\left[\frac{q_{i_1}}{q_{i_1+1}}\right]}\cdots\widehat{\left[\frac{q_{i_k-1}}{q_{i_k}}\right]}\widehat{\left[\frac{q_{i_k}}{q_{i_k+1}}\right]}\cdots\left[\frac{q_{n-2}}{q_{n-1}}\right]\\ +\left[\frac{p}{q}\right]\left[\frac{q}{q_1}\right]\cdots\left[\frac{q_{n-2}}{q_{n-1}}\right]+\frac{1+(-1)^n}{2}\,,\end{aligned}$$

where $\hat{x}$ denotes deleting x and the sum is taken over all integers satisfying $1\leqq k\leqq\left[\frac{n}{2}\right]$, $0\leqq i_1,\,\cdots,\,i_k\leqq n-2$ and $2\leqq i_j-i_{j-1}$.

Thus we get

$$\begin{aligned}\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2=&\frac{1}{3}(p-1)^2(q-1)-\frac{p}{6}\{p-q-(-1)^{n-3}q_{n-2}-(-1)^{n-2}q_{n-1}\}\\ &+\frac{p}{6}\sum_{i=0}^{n-2}(-1)^i\left[\frac{q_{i-1}}{q_i}\right]-\frac{p}{6}(-1)^{n-1}q_{n-2}+\frac{p}{6}(-1)^{n-1}\\ &+\frac{p}{6}(-1)^{n-1}\left[\frac{q_{n-2}}{q_{n-1}}\right]+\frac{p}{6}\cdot\frac{1-(-1)^n}{2}-\frac{1}{6}+\frac{1}{6}(-1)^n a_{n-1}\end{aligned}$$

and hence we obtain the following

THEOREM 3. *$\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2$ can be evaluated as a polynomial on p, q and $\left[\frac{q_{i-1}}{q_i}\right]$ $(i=0, 1, \cdots, n-1)$:*

$$(8)\qquad \sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2=\frac{1}{3}(p-1)^2(q-1)-\frac{p}{6}(p-q)+\frac{p}{12}-\frac{1}{6}+\frac{1}{4}(-1)^{n-1}p$$
$$+\frac{p}{6}\sum_{i=0}^{n}(-1)^i\left[\frac{q_{i-1}}{q_i}\right]+\frac{1}{6}(-1)^n a_{n-1}\,.$$

§ 3. Eta-invariant for $L(p; q)$.

Let C^2 be the space of pairs (z_0, z_1) of complex numbers with the standard flat Kähler metric $ds^2=dz_0\cdot d\bar{z}_0+dz_1\cdot d\bar{z}_1$. Let p and q be relatively prime integers satisfying $1\leqq q<p$. Put $z=\exp\frac{2\pi\sqrt{-1}}{p}$ and define an isometry g of C^2 by

$$g\cdot(z_0, z_1)=(zz_0, z^q z_1)\,.$$

Then g generates a cyclic subgroup $G=\{g^k\}_{k=0,1,\cdots,p-1}$ of the unitary group $U(2)$ and the elements g^k act on the unit sphere

$$S^3=\{(z_0, z_1)\in C^2;\ z_0\bar{z}_0+z_1\bar{z}_1=1\}$$

without fixed point. The sphere S^3 is the universal covering manifold of the differentiable manifold S^3/G with the covering projection φ: $S^3\to S^3/G$ and S^3/G has a unique riemannian metric so that φ gives a local isometry of S^3 onto S^3/G. This riemannian manifold S^3/G is called a lens space and denoted by $L(p; q)$.

The eta-invariant for $L(p; q)$ is given by the explicit formula:

$$\eta(p; q)=-\frac{1}{p}\sum_{j=1}^{p-1}\cot\frac{j\pi}{p}\cot\frac{jq\pi}{p}$$

(see M. F. Atiyah, V. K. Patodi and I. M. Singer [3]). Note that this eta-invariant corresponds to the trivial one-dimensional unitary representation of Z/pZ.

Although the result is well-known, we first rewrite $\eta(p; q)$ in the sum $\sum_{k=1}^{p-1}\left[\frac{kp}{q}\right]^2$ by the most elementary method (cf. Zagier [12]).

Put $\zeta=\exp\frac{2j\pi\sqrt{-1}}{p}$, then

$$\sqrt{-1}\cot\frac{j\pi}{p}=\frac{1+\zeta}{1-\zeta}=-1-\frac{2}{\zeta-1}\,.$$

Since $\sum_{k=1}^{p-1}\zeta^k=-1$ and $\sum_{k=1}^{p-1}k\zeta^k=\frac{p}{\zeta-1}$, we get

$$\sqrt{-1}\cot\frac{j\pi}{p}=\sum_{k=1}^{p-1}\zeta^k-\frac{2}{p}\sum_{k=1}^{p-1}k\zeta^k=\sum_{k=1}^{p-1}\left(1-\frac{2k}{p}\right)\zeta^k$$

so that

$$\begin{aligned}\eta(p;q)&=\frac{1}{p}\sum_{j=1}^{p-1}\left\{\sum_{h=1}^{p-1}\left(1-\frac{2h}{p}\right)\zeta^h\right\}\left\{\sum_{k=1}^{p-1}\left(1-\frac{2k}{p}\right)\zeta^{kq}\right\}\\&=\frac{1}{p}\sum_{j=1}^{p-1}\left\{\sum_{h,k=1}^{p-1}\left(1-\frac{2h}{p}\right)\left(1-\frac{2k}{p}\right)\zeta^{h+kq}\right\}\,.\end{aligned}$$

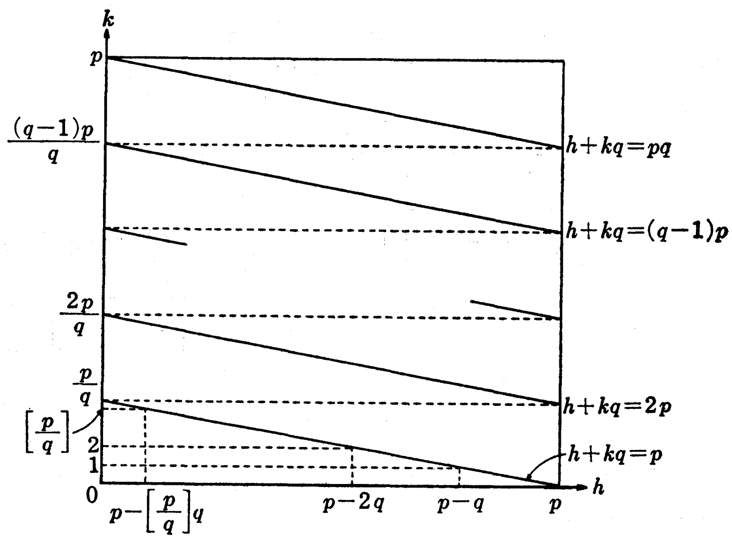
Now we know that

$$\sum_{j=1}^{p-1}\zeta^{h+kq}=\begin{cases}p-1 & \text{if } p\mid h+kq\\ -1 & \text{if } p\nmid h+kq\,,\end{cases}$$

and hence we get

$$\begin{aligned}\eta(p;q)&=\sum_{\substack{h,k=1\\p\mid h+kq}}^{p-1}\left(1-\frac{2h}{p}\right)\left(1-\frac{2k}{p}\right)-\frac{1}{p}\sum_{h,k=1}^{p-1}\left(1-\frac{2h}{p}\right)\left(1-\frac{2k}{p}\right)\\&=\sum_{\substack{h,k=1\\p\mid h+kq}}^{p-1}\left\{1-\frac{2}{p}(h+k)+\frac{4}{p^2}hk\right\}\,.\end{aligned}$$

Since h $(1\leqq h\leqq p-1)$ is uniquely determined according as k moves from 1 to $p-1$, we obtain the following equations:



$$(9)\qquad \sum_{\substack{h,k=1\\ p\mid h+kq}}^{p-1} 1 = p-1\,,$$

$$(10)\qquad \sum_{\substack{h,k=1\\ p\mid h+kq}}^{p-1} h = \sum_{\substack{h,k=1\\ p\mid h+kq}}^{p-1} k = \frac{p(p-1)}{2}\,.$$

To calculate $\sum_{p\mid h+kq} {}_{h,k=1}^{p-1} hk$, we must clarify the lattice points (h, k) satisfying $h+kq=mp$ for some integer m $(1\leqq m\leqq q)$. (See the figure on the previous page.) The result is as follows:

$$\begin{aligned}(h, k) &= (p-q, 1),\ (p-2q, 2),\ \cdots,\ \left(p-\left[\frac{p}{q}\right]q, \left[\frac{p}{q}\right]\right),\\ &\left(2p-\left(\left[\frac{p}{q}\right]+1\right)q, \left[\frac{p}{q}\right]+1\right),\ \cdots,\ \left(2p-\left[\frac{2p}{q}\right]q, \left[\frac{2p}{q}\right]\right),\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ &\left(qp-\left(\left[\frac{(q-1)p}{q}\right]+1\right)q, \left[\frac{(q-1)p}{q}\right]+1\right),\ \cdots,\ (qp-(p-1)q, p-1)\,.\end{aligned}$$

Hence we get

$$\begin{aligned}(11)\qquad \sum_{\substack{h,k=1\\ p\mid h+kq}}^{p-1} hk &= p\Bigg\{1\cdot\left(1+2+\cdots+\left[\frac{p}{q}\right]\right)\\ &\quad +2\cdot\left(\left(\left[\frac{p}{q}\right]+1\right)+\left(\left[\frac{p}{q}\right]+2\right)+\cdots+\left[\frac{2p}{q}\right]\right)\\ &\quad +\cdots\cdots\cdots\cdots\cdots\cdots\\ &\quad +q\cdot\left(\left(\left[\frac{(q-1)p}{q}\right]+1\right)+\cdots+(p-1)\right)\Bigg\}-q\sum_{k=1}^{p-1}k^2\\ &= p\Bigg\{\frac{1}{2}p(p-1)+\frac{1}{2}\left(p(p-1)-\left[\frac{p}{q}\right]\left(\left[\frac{p}{q}\right]+1\right)\right)\\ &\quad +\frac{1}{2}\left(p(p-1)-\left[\frac{2p}{q}\right]\left(\left[\frac{2p}{q}\right]+1\right)\right)+\cdots\\ &\quad +\frac{1}{2}\left(p(p-1)-\left[\frac{(q-1)p}{q}\right]\left(\left[\frac{(q-1)p}{q}\right]+1\right)\right)\Bigg\}-\frac{1}{6}pq(p-1)(2p-1)\\ &= \frac{1}{2}p^2q(p-1)-\frac{1}{6}pq(p-1)(2p-1)-\frac{p}{2}\sum_{k=1}^{q-1}\left(\left[\frac{kp}{q}\right]^2+\left[\frac{kp}{q}\right]\right)\,.\end{aligned}$$

Substituting the equations (9), (10) and (11), we obtain the following

THEOREM 4. *The eta-invariant for the* 3*-dimensional lens space* $L(p; q)$ *is represented as follows*:

$$\eta(p;q)=\frac{1}{3p}(p-1)(2pq-3p-q+3)-\frac{2}{p}\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2 .$$

§4. Complete invariantness of eta-invariant.

We now prove the complete invariantness of the eta-invariant for the 3-dimensional lens space with a certain type of p, that is,

THEOREM 5. *Let p be a prime number or $p=kp'$ where p' is a prime number and $k=2$ or 3. Then two lens spaces $L(p;q)$ and $L(p;q')$ are isometric to each other if and only if $\eta(p;q)=\pm\eta(p;q')$* (cf. H. Donnelly [6] Prop. 4.2).

PROOF. Necessity is well known and to prove sufficiency we use the following

THEOREM 6. *Two lens spaces $L(p;q)$ and $L(p;q')$ are isometric to each other if and only if*

$$q\equiv\pm q' \text{ modulo } p \quad \text{or} \quad qq'\equiv\pm 1 \text{ modulo } p .$$

(See, for example, M. M. Cohen [5] or A. Ikeda and Y. Yamamoto [7].)

We obtain

$$6q\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2\equiv(q-1)(2q-1) \qquad \text{modulo } p \tag{12}$$

from the reciprocity formula (1) and, on the other hand, we obtain

$$6\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2\equiv 2q-3+(-1)^n a_{n-1} \qquad \text{modulo } p$$

from the equation (8). Hence we get

$$(q-1)(2q-1)\equiv q\{2q-3+(-1)^n a_{n-1}\} \qquad \text{modulo } p$$

so that

$$qa_{n-1}\equiv(-1)^n \qquad \text{modulo } p .$$

Note that, since p and q are relatively prime, the positive integer a_{n-1} $\left(<\frac{p}{2}\right)$ is uniquely determined by this equation for given p, q and n satisfying $q_n=1$.

Now, as

(13) $$3p\eta(p;q)=(p-1)(2pq-3p-q+3)-6\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2$$

is an integer, we get

$$3p\eta(p;q)\equiv -q-(-1)^n a_{n-1} \qquad \text{modulo } p\,.$$

Define q_i' $(q_{n'}'=1)$ and $a_{n'-1}'$ in the similar way corresponding to q' and assume that $\eta(p;q)=\eta(p;q')$. Then we get

$$q+(-1)^n a_{n-1}\equiv q'+(-1)^{n'}a_{n'-1}' \qquad \text{modulo } p\,.$$

Multiplying qq' to the both sides of this equation, we get

$$q^2q'+q'\equiv qq'^2+q \qquad \text{modulo } p$$

which reduces to

(14) $$(q-q')(qq'-1)\equiv 0 \qquad \text{modulo } p\,.$$

Hence we get

(15) $$q=q' \quad \text{or} \quad qq'\equiv 1 \qquad \text{modulo } p\,.$$

In fact, if $p=3p'$, for example, we may obtain

$$q-q'=3\alpha \qquad \text{and} \qquad qq'-1=p'\beta$$

or

$$q-q'=p'\beta \qquad \text{and} \qquad qq'-1=3\alpha$$

for some integers α and β. However, in either case, we obtain from the equation containing 3α that

$$q\equiv q'\equiv 1 \text{ modulo } 3 \quad \text{or} \quad q\equiv q'\equiv -1 \text{ modulo } 3$$

and substituting these to the equation containing $p'\beta$ we get that $p'\beta$ is a multiple of p if p' is not equal to 3. When $p'=3$, i.e., $p=9$, we can verify Theorem 5 by computation.

On the other hand, if we assume $\eta(p;q)=-\eta(p;q')$, then we get

$$q+q'=p \qquad \text{or} \qquad qq'\equiv -1 \quad \text{modulo } p\,.$$

Thus we have proved Theorem 5 via Theorem 6.

REMARK 1. As I. Iwasaki [8] pointed out, we can obtain the equation (14) by the reciprocity formula of $\eta(p;q)$ which is equivalent to the reciprocity formula (1) without knowing the exact value of the sum

$\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2$. However, we do not need even the reciprocity formula. In fact, we obtain the equation (12) from the equation

$$\sum_{k=1}^{q-1}\left(kp-\left[\frac{kp}{q}\right]q\right)^2=\sum_{k=1}^{q-1}\gamma_k^2=\frac{1}{6}q(q-1)(2q-1)$$

and we obtain

$$3p\eta(p;q)\equiv q-3-6\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2 \qquad \text{modulo } p$$

from the equation (13). Thus, if $\eta(p;q)=\eta(p;q')$ then we get

$$q-q'\equiv 6\left\{\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2-\sum_{k=1}^{q'-1}\left[\frac{kp}{q'}\right]^2\right\} \qquad \text{modulo } p$$

and hence we get

$$qq'(q-q')\equiv q'(q-1)(2q-1)-q(q'-1)(2q'-1) \qquad \text{modulo } p$$

so that

$$(q-q')(qq'-1)\equiv 0 \qquad \text{modulo } p\ .$$

In the similar way, we get

$$(q+q')(qq'+1)\equiv 0 \qquad \text{modulo } p$$

if we assume $\eta(p;q)=-\eta(p;q')$.

REMARK 2. Theorem 5 does not hold when $p=5p'$ where p' is a prime number greater than 5. In fact, we know that $L(65;8)$ and $L(65;18)$ have the same eta-invariant but are not isometric to each other. However, we can prove that Theorem 5 holds if $p=5p'$ and

$$q\equiv q'\equiv \pm 1 \text{ modulo } 5 \quad (\text{or } q\equiv -q'\equiv \pm 1 \text{ modulo } 5)$$

when $\eta(p;q)=\eta(p;q')$ (or $\eta(p;q)=-\eta(p;q')$ respectively).

But, in this case, we must exclude the case $p'=5$ since we know that $L(25;4)$ and $L(25;9)$ have the same eta-invariant but are not isometric to each other.

§ 5. Further arguments.

Let p be a composite number, q and q' be positive integers less than or equal to $\left[\frac{p}{2}\right]$ and relatively prime to p satisfying $q_n=q'_{n'}=1$, and assume that $\eta(p;q)=\pm\eta(p;q')$. Then we get from Theorem 3 and Theorem 4 that

$$\begin{aligned}3p\eta(p;q)-\{\pm 3p\eta(p;q')\}\\ &=\pm q'-q\pm(-1)^{n'}a'_{n'-1}-(-1)^n a_{n-1}\\ &\quad+p\Big\{\frac{3}{2}(\pm((-1)^{n'-1}-1)+1-(-1)^{n-1})\\ &\quad\pm\sum_{i=0}^{n'}(-1)^i\left[\frac{q'_{i-1}}{q'_i}\right]-\sum_{j=0}^{n}(-1)^j\left[\frac{q_{j-1}}{q_j}\right]\Big\}=0\ ,\end{aligned}$$

and hence we get the following simultaneous equation:

$$\pm q'-q\pm(-1)^{n'}a'_{n'-1}-(-1)^n a_{n-1}=sp\ , \tag{16}$$

$$\begin{aligned}&\frac{3}{2}\{\pm((-1)^{n'-1}-1)+1-(-1)^{n-1}\}\\ &\pm\sum_{i=0}^{n'}(-1)^i\left[\frac{q'_{i-1}}{q'_i}\right]-\sum_{j=0}^{n}(-1)^j\left[\frac{q_{j-1}}{q_j}\right]=-s\end{aligned} \tag{17}$$

where $s=0$ or ± 1.

Note that we had used the equation (16) to prove Theorem 5 and obtained the equation (14). When p is a composite number ab, we may have from the equation (14) that

$$q'-q=\alpha a \qquad \text{and} \qquad qq'-1=\beta b+\left[\frac{qq'-1}{p}\right]p$$

for some integers α $(0<|\alpha|<b)$ and β $(0<\beta<a)$. However, taking the equation (17) into account we obtain the following result.

THEOREM 7. *Let p, q, and q' $(q\neq q')$ be as above.*
(i) *If $n=n'=1$ and $\eta(p;q)=-\eta(p;q')$ then we get $qq'+1=p$.*
(ii) *If $n=2$, $n'=1$ then $\eta(p;q)\neq\pm\eta(p;q')$ for any q and q'.*
(iii) *If $n=n'=2$ and $\eta(p;q)=\eta(p;q')$ then we get $qq'\equiv 1$ modulo p.*

PROOF. Case (i). Since $s=0$ in this case, the equations (16) and (17) are the following simple equation:

$$q+q'-\left[\frac{p}{q}\right]-\left[\frac{p}{q'}\right]=0\ .$$

Multiplying qq' to both sides of this equation, we obtain

$$(q+q')(qq'+1-p)=0$$

and hence

$$qq'+1=p\ .$$

Case (ii). Suppose $\eta(p;q)=\eta(p;q')$, then the equations (16) and (17) are rewritten as follows:

$$(16')\qquad q'-q-\left[\frac{p}{q'}\right]-\left(1+\left[\frac{p}{q}\right]\left[\frac{q}{q_1}\right]\right)=sp\ ,$$

$$(17')\qquad 3+\left[\frac{p}{q'}\right]-q'-\left[\frac{p}{q}\right]+\left[\frac{q}{q_1}\right]-q_1=-s\ .$$

If $s=0$ then substituting $q=\left[\frac{q}{q_1}\right]q_1+1$ and the equation (17′) we get from the equation (16′) that

$$\left(1-\left[\frac{p}{q}\right]-q_1\right)\left(1+\left[\frac{q}{q_1}\right]\right)=0\ .$$

However, this equation does not hold. So we have only to consider the case: $s=-1$ and hence $q'<q$. In this case we get from the equations (16′) and (17′) that

$$-p=\left(1-\left[\frac{p}{q}\right]-q_1\right)\left(1+\left[\frac{q}{q_1}\right]\right)-1$$

so that

$$(p-q-q_1)(q-1)(q_1-1)=0\ .$$

However, this equation never holds since $n>1$ and $q+q_1<p$.

Similarly, we can conclude that the equation $\eta(p;q)=-\eta(p;q)$ never holds.

Case (iii). Since $s=0$ in this case, the equations (16) and (17) are rewritten as follows:

$$(16'')\qquad q'-q+\left[\frac{p}{q'}\right]\left[\frac{q'}{q'_1}\right]-\left[\frac{p}{q}\right]\left[\frac{q}{q_1}\right]=0\ ,$$

$$(17'')\qquad \left[\frac{p}{q'}\right]-\left[\frac{q'}{q'_1}\right]+q'_1-\left[\frac{p}{q}\right]+\left[\frac{q}{q_1}\right]-q_1=0\ .$$

By multiplying qq' to both sides of the equation (16″), we get

$$(q'-q)(qq'-1)=\left(q'\left[\frac{q}{q_1}\right]-q\left[\frac{q'}{q'_1}\right]\right)p$$

so that

$$\alpha\left(\beta+\left[\frac{qq'-1}{p}\right]a\right)=q'\left[\frac{q}{q_1}\right]-q\left[\frac{q'}{q'_1}\right]. \tag{18}$$

On the other hand, we get from the equation (16″) that

$$\left(\left[\frac{p}{q'}\right]+q'_1\right)\left[\frac{q'}{q'_1}\right]-\left(\left[\frac{p}{q}\right]+q_1\right)\left[\frac{q}{q_1}\right]=0\,. \tag{19}$$

Substituting the equation

$$\left[\frac{p}{q'}\right]+q'_1=\left[\frac{p}{q}\right]+q_1+\left[\frac{q'}{q'_1}\right]-\left[\frac{q}{q_1}\right]$$

to the equation (19), we get

$$\begin{aligned}\left(\left[\frac{p}{q}\right]+q_1\right)\left(\left[\frac{q'}{q'_1}\right]-\left[\frac{q}{q_1}\right]\right)&+\left(\left[\frac{q'}{q'_1}\right]-\left[\frac{q}{q_1}\right]\right)\left[\frac{q'}{q'_1}\right]\\ &=\left(\left[\frac{p}{q}\right]+q_1+\left[\frac{q'}{q'_1}\right]\right)\left(\left[\frac{q'}{q'_1}\right]-\left[\frac{q}{q_1}\right]\right)\\ &=0\end{aligned}$$

so that we get

$$\left[\frac{q'}{q'_1}\right]=\left[\frac{q}{q_1}\right].$$

Thus the equation (18) reduces to

$$\begin{aligned}\alpha\beta&=(q'-q)\left[\frac{q}{q_1}\right]-\left[\frac{qq'-1}{p}\right]\alpha a\\ &=\left(\left[\frac{q}{q_1}\right]-\left[\frac{qq'-1}{p}\right]\right)\alpha a\,.\end{aligned}$$

Hence, $\beta\equiv 0$ modulo a and we get $qq'-1\equiv 0$ modulo p.

REMARK 3. As for the cases $n'=1$ and $n\geqq 3$, we cannot deduce the equation (15) from the equation (14). Lens spaces in Remark 2 are the counter-examples to these cases.

On the other hand, we obtain the following theorem concerning the equivalence condition of lens spaces and the length of the sequence of the remainders in the Euclidean algorithm for calculating the gcd(p, q).

THEOREM 8. *Let q and q' be positive integers relatively prime to p satisfying $q_n=q'_{n'}=1$.*

(i) *If* $q+q'=p$ *and* $q<\frac{p}{2}$, *then*

$$q_i=q'_{i+1} \quad \textit{for} \quad i=0, 1, \cdots, n-1 .$$

Hence, in particular, $n'=n+1$.

(ii) *If* q *and* q' *are smaller than or equal to* $\frac{p}{2}$ *and satisfy the equation* $qq'\equiv(-1)^n$ *modulo* p, *then*

$$a'_{j-1}=q_{n-j} \quad \textit{for} \quad j=0, 1, \cdots, n , \tag{20}$$

and

$$a_{i-1}=q'_{n-i} \quad \textit{for} \quad i=0, 1, \cdots, n . \tag{21}$$

Hence, in particular, $n'=n$.

PROOF. (i) is trivial and so we prove (ii).

Since a_{n-1} $\left(1\leqq a_{n-1}<\frac{p}{2}\right)$ is defined uniquely by the equation $qa_{n-1}\equiv(-1)^n$ modulo p and q' satisfies the equation $qq'\equiv(-1)^n$ modulo p, we get $q'=a_{n-1}$.

It follows from the equation (6) that $p=a'_0q'_0+a'_{-1}q'_1=a_{n-1}q_{n-1}+a_{n-2}q_n$, i.e.,

$$\left[\frac{p}{q'}\right]q'+q'_1=q'q_{n-1}+a_{n-2} .$$

Hence we get from this equation and the inequalities $0<a_{n-2}<a_{n-1}=q'$ that $q'_1=a_{n-2}$ and $q_{n-1}=\left[\frac{p}{q'}\right]=a'_0$.

In general, it follows from the equation (6) that

$$a_{k+1}=a_{k-1}+a_k\left[\frac{q_k}{q_{k+1}}\right] \quad \text{for} \quad k=0, 1, \cdots, n-1 ,$$

and hence we get

$$\left[\frac{q_k}{q_{k+1}}\right]=\left[\frac{a_{k+1}}{a_k}\right] \quad \text{for} \quad k=-1, 0, \cdots, n-1 .$$

Using this equation, we prove (ii) by induction. Suppose that

$$q'_j=a_{n-j-1}, \quad q'_{j+1}=a_{n-j-2}, \quad a'_{j-1}=q_{n-j}, \quad \text{and} \quad a'_j=q_{n-j-1} .$$

Then

$$\begin{aligned} a'_{j+1} &= a'_{j-1} + a'_j\left[\frac{q'_j}{q'_{j+1}}\right] \\ &= \left[\frac{a_{n-j-1}}{a_{n-j-2}}\right]a'_j + a'_{j-1} \\ &= \left[\frac{q_{n-j-2}}{q_{n-j-1}}\right]q_{n-j-1} + q_{n-j} \\ &= q_{n-j-2}\ , \end{aligned}$$

and hence we get the equation (20). We get the equation (21) similarly. In particular, we get $q'_n = a_{-1} = 1$ so that $n' = n$.

According to Theorem 8 together with Theorem 7 and Remark 3, one may conjecture that $\eta(p; q) = (-1)^n \eta(p; q')$ and $n = n'$ for q and q' smaller than $\frac{p}{2}$ imply the equation $qq' \equiv (-1)^n$ modulo p. This conjecture is true for almost all cases. In fact, we obtain the following theorem.

Theorem 9. *Let $p = ab$ $(a, b > 1)$, a composite number, and q and q' $\left(q \neq q' \leqq \left[\frac{p}{2}\right]\right)$ be positive integers relatively prime to p. Moreover, assume that $n = n'$ and $\eta(p; q') = (-1)^n \eta(p; q)$. If $\eta(p; q) \neq 0$ and q' is a unique integer satisfying the above assumption for given q, then the equation*

$$qq' \equiv (-1)^n \qquad \textit{modulo } p$$

holds.

Theorem 9 is easily shown by the fact that $q' = a_{n-1}$ satisfies the equation $qq' \equiv (-1)^n$ modulo p and this equation implies $n = n'$ and $\eta(p; q') = (-1)^n \eta(p; q)$. As for the exceptional cases of the assumption in Theorem 9, we obtain the following lemma.

Lemma 10. *Let p, q, and q' be as in Theorem 9 and let $n = n'$.*

(i) *If $\eta(p; q) = \eta(p; q') = 0$ for $q' \neq q$, then q and q' satisfy the following equations:*

$$(22)\qquad \begin{aligned} & q' - (-1)^n q = \alpha a && \textit{for some } \alpha\ (0 < |\alpha| < b)\ , \\ & qq' - (-1)^n = \beta b + \left[\frac{qq' - (-1)^n}{p}\right]p && \textit{for some } \beta\ (0 < \beta < a)\ . \end{aligned}$$

(ii) *If $\eta(p; q'') = (-1)^n \eta(p; q)$ for some q'' $\left(q' \neq q'' \leqq \left[\frac{p}{2}\right] \text{ and } n = n''\right)$, then q and q' or q and q'' satisfy the equations (22).*

Proof. At first, note that from the assumption $\eta(p; q') = (-1)^n \eta(p; q)$

we obtain the equation

(23) $$\{q'-(-1)^n q\}\{qq'-(-1)^n\}\equiv 0 \quad \text{modulo } p .$$

Case (i). It follows from the assumption $\eta(p;q)=0$ that

$$\begin{aligned} q &= -(-1)^n a_{n-1}+\frac{3}{2}\{1+(-1)^n\}p-p\sum_{i=0}^{n}(-1)^i\left[\frac{q_{i-1}}{q_i}\right] \\ &\equiv (-1)^{n-1}a_{n-1} \quad \text{modulo } p . \end{aligned}$$

Since $0<a_{n-1}<\frac{p}{2}$, we conclude that n is odd and $q=a_{n-1}$. Hence we get

$$qa_{n-1}=q^2\equiv -1 \quad \text{modulo } p .$$

In the same way, assuming $\eta(p;q')=0$ we get

$$q'^2\equiv -1 \quad \text{modulo } p .$$

It follows from the equation

$$q'^2-q^2=(q+q')(q'-q)\equiv 0 \quad \text{modulo } p$$

that

$$q+q'=\alpha_+ a \quad \text{and} \quad q'-q=\alpha_- b$$

for some integers α_+ $(0<\alpha_+<b)$ and α_- $(0<|\alpha_-|<a)$. Hence we have

$$q(q'-q)=q\alpha_- b .$$

On the other hand, the left hand side of this equation is equal to $qq'-q^2\equiv qq'+1$ modulo p. Hence we have

$$qq'-(-1)^n=qq'+1=\beta b+\left[\frac{qq'+1}{p}\right]p$$

for some integer β $(0<\beta<a)$.

Case (ii). If we assume that $qq''\equiv(-1)^n$ modulo p, then, by the uniqueness of q'' satisfying this equation, q' should satisfy the inequality: $qq'\not\equiv(-1)^n$ modulo p. However, since q' satisfies the equation (23), q' should satisfy the equation (22).

In contrast with Theorem 9, we obtain the following theorem which, combined with Theorem 9 and Lemma 10, shows the existence of some relation between the orientation of a lens space $L(p;q)$ and the length of the Euclidean algorithm for a pair (p, q).

THEOREM 11. *Let p, q, and q' be as in Theorem 9 and let $n=n'$. If $\eta(p;q')=-(-1)^n\eta(p;q)$ then q and q' satisfy the equations*

$$(24)\qquad \begin{aligned} &q'+(-1)^n q=\alpha a && \text{for some } \alpha\ (0<|\alpha|<b)\ , \\ &qq'+(-1)^n=\beta b+\left[\frac{qq'+(-1)^n}{p}\right]p && \text{for some } \beta\ (0<\beta<a)\ . \end{aligned}$$

PROOF. It follows from the assumption that

$$\{q'+(-1)^n q\}\{qq'+(-1)^n\}\equiv 0 \qquad \text{modulo } p\ .$$

Multiplying a_{n-1} to this equation, we have

$$\begin{aligned} \{q'+(-1)^n q\}\{qa_{n-1}q'+(-1)^n a_{n-1}\} &\equiv (-1)^n\{q'+(-1)^n q\}(q'+a_{n-1}) && \text{modulo } p \\ &\equiv 0 && \text{modulo } p\ . \end{aligned}$$

Since $0\neq q'+(-1)^n q<p$ and $0<q'+a_{n-1}<p$, we obtain

$$q'+(-1)^n q=\alpha a \qquad \text{for some } \alpha\ (0<|\alpha|<b)\ ,$$

and

$$q'+a_{n-1}=\gamma b \qquad \text{for some } \gamma\ (0<\gamma<a)\ .$$

Thus we obtain Theorem 11.

Note that the entirely same proof shows that Theorem 11 remains true even if $n'\neq n$. Also note that the equation: $qq'-(-1)^n\equiv 0$ modulo p never follows from $\eta(p;q')=(-1)^n\eta(p;q)$ when $n'\neq n$. From Theorems 7, 9, and 11 and the above facts, we obtain the following main theorem.

THEOREM 12. *Let $p=ab$ $(1<a, b)$, a composite number not of the type $2p'$ or $3p'$ where p' is a prime number, and let q and q' $\left(q, q'\leqq\left[\frac{p}{2}\right]\right)$ be positive integers relatively prime to p satisfying $q_n=q'_{n'}=1$.*

(I) *If $\eta(p;q')=(-1)^n\eta(p;q)$ and $n=n'$, then $qq'-(-1)^n\equiv 0$ modulo p except the following cases.*

(i) $\eta(p;q)\ (=\eta(p;q'))=0$.

(ii) *There exists a positive integer q'' $\left(\leqq\left[\frac{p}{2}\right]\right)$ which is different from q and q' and satisfies the equations:*

$$\eta(p;q'')=(-1)^n\eta(p;q) \qquad \text{and} \qquad n''=n\ .$$

(II) *If $\eta(p;q')=(-1)^n\eta(p;q)$ and $n'\neq n$, then $qq'-(-1)^n\not\equiv 0$ modulo p.*

(III) *If $\eta(p;q')=-(-1)^n\eta(p;q)$ then $qq'+(-1)^n\not\equiv 0$ modulo p.*

Note that in these exceptional cases q and q' satisfy the equations:

$$q' \pm (-1)^n q = \alpha a \qquad \text{for some integer } \alpha \ (0<|\alpha|<b)$$

and

$$qq' \pm (-1)^n = \beta b + \left[\frac{qq' \pm (-1)^n}{p}\right] p \qquad \text{for some integer } \beta \ (0<\beta<a) ,$$

according as $qq' \pm (-1)^n \not\equiv 0$ modulo p.

Also note that there exists no exceptional case for $n=n'=1$ or 2 in (I) and there exist no q $(n=2)$ and q' $(n'=1)$ satisfying $\eta(p; q') = \pm\eta(p; q)$.

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Present Address:
DEPARTMENT OF MATHEMATICS
GAKUSHUIN UNIVERSITY
MEJIRO, TOSHIMA-KU, TOKYO 171