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Domains Integral over Each Underring

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§1. Introduction and notation.

Let R be a (commutative integral) domain with quotient field K. One has often studied R by analyzing properties of its overrings (that is, the rings contained between R and K). This article contributes to "dual" analyses via underrings. (We shall say that B is an *underring* of R in case B is a subring of R also having quotient field K.) Some studies of domains via underrings have already been done in [3], [4]. For instance, [4, Theorem 2.2] characterized the R such that each underring of R is seminormal. (It turned out that the same class of rings is characterized if "underring" is replaced by "subring" and/or "seminormal" is replaced by "a Euclidean domain." The rings R in question are the overrings of $\boldsymbol{Z}$ (up to isomorphism) and the fields algebraic over a finite field.) This work completed earlier studies such as [5], [1, Proposition 2.11], and [3].

The domains characterized in [4, Theorem 2.2] may also be viewed as the domains each of whose underrings (resp., subrings) is integrally closed. One would nevertheless expect, rather generally, that requiring prescribed behavior of the underrings of R would be less restrictive than requiring analogous behavior of all the subrings of R. To test this intuition and seek results with different flavor, we move to the "other extreme" from [4, Theorem 2.2]: we now ask to characterize the R such that R is integral over each of its underrings. It should be noted that the answer to *this* question changes if "underrings" is replaced by "subrings." The answers are given in Proposition 2.2, our main results (Theorem 2.3 and Corollary 3.6), and Remark 2.4(a)–(c). It will be seen that consideration of *underrings* leads to certain global fields of positive

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characteristic. Section 3 illuminates the positive characteristic case by developing a suitable "relative" theory.

R, K retain the above meanings throughout, all subrings contain the 1 of the larger ring, ch denotes characteristic, $\boldsymbol{F}_p$ denotes the prime field of characteristic $p>0$, and $\boldsymbol{A}$ denotes the ring of all algebraic integers. Any unexplained material is standard, as in [6], [8].

§ 2. The domains of the title.

In any study of domains via behavior of their underrings, certain domains are catalogued by default. These are the domains having no proper underrings. They are characterized in Proposition 2.1, whose proof follows immediately from [3, Corollary] or [4, Theorem 2.2]. Recall, by way of contrast, that R has just R and K as *overrings* if and only if R is a valuation domain of (Krull) dimension at most 1.

PROPOSITION 2.1. *The following conditions on R are equivalent:*
(1) *R is the only underring of R;*
(2) *Either $R\cong \boldsymbol{Z}$ or $R=K$ is a field algebraic over some $\boldsymbol{F}_p$.*

The next three results serve to vindicate our intuition. One of their morals is that, by focusing on underrings rather than subrings, we produce a catalogue of domains which features at least some global fields of positive characteristic. (Dually, note that each overring of R is integral over R if and only if $R=K$; but *no* domain R has the property that each domain containing R is integral over R.)

PROPOSITION 2.2. *The following conditions on R are equivalent:*
(1) *R is integral over each subring of R;*
(2) *Either R is isomorphic to a subring of $\boldsymbol{A}$ or $R=K$ is a field algebraic over some $\boldsymbol{F}_p$.*

PROOF. Proposition 2.1 easily yields (2)$\Rightarrow$(1), while the converse follows easily from [8, Theorem 48]. We leave the details to the reader.

We next present the main result of this section.

THEOREM 2.3. (i) *The following two conditions on R are equivalent:*
(1) *R is integral over each underring of R;*
(2) *One of the following two conditions holds:*
(a) *R is isomorphic to a subring of $\boldsymbol{A}$;*
(b) $\mathrm{ch}(R)=p>0$, *and R is contained in each valuation ring V of K such that K is the quotient field of $V\cap R$.*
(ii) *The above condition* (b) *holds if K is algebraic over $\boldsymbol{F}_p$ (in*

which case, $R=K$).

(iii) *Suppose that the above condition* (b) *holds and that* K *is not algebraic over* $\boldsymbol{F}_p$. *Then* $\mathrm{t.d.}(K/\boldsymbol{F}_p)=1$. *Indeed, there exists a singleton transcendence basis* $\{X\}$ *of* K *over* $\boldsymbol{F}_p$ *such that* R *is integral over* $\boldsymbol{F}_p[X]$.

PROOF. (i) (2)$\Rightarrow$(1): By Proposition 2.2, it is enough to show that (b)$\Rightarrow$(1). Deny. Then there exists an underring B of R so that R is not integral over B. Hence (cf. [6, Theorem 19.8]), R fails to be contained in some valuation overring V of B. As $B\subset V\cap R$, K is the quotient field of $V\cap R$, contradicting (b), as desired.

(1)$\Rightarrow$(2): Assume $\mathrm{ch}(R)=0$. There are two cases, determined by whether K is algebraic over $\boldsymbol{Q}$.

Suppose that K *is* algebraic over $\boldsymbol{Q}$. We claim that (a) holds. It will suffice to show that R is integral over $\boldsymbol{Z}$. To this end, let T denote the integral closure of $\boldsymbol{Z}$ in K. By clearing denominators, one sees as usual that K is the quotient field of T. By algebraicity of $K/\boldsymbol{Q}$, [7, Corollary 1.7] yields that K is the quotient field of $D=R\cap T$; in other words, D is an underring of R. Thus, by (1), R is integral over D. Since $D\subset T$, D is integral over $\boldsymbol{Z}$, whence R is integral over $\boldsymbol{Z}$, as asserted.

Next, we shall show that the assumption that K is *not* algebraic over $\boldsymbol{Q}$ leads to a contradiction. Choose a transcendence basis $\{X_i\}$ of K over $\boldsymbol{Q}$ such that $\{X_i\}\subset R$; let X denote one of the X_i. Let S denote the intersection of R with the integral closure of $\boldsymbol{Z}[\{X_i\}]$ in K. As $K/\boldsymbol{Q}(\{X_i\})$ is algebraic, the above reasoning yields that K is the quotient field of S. (More specifically: clear denominators and apply [7, Corollary 1.7].) As S is then an underring of R, so too is $E=\boldsymbol{Z}+2S$; hence, by (1), R is integral over E. As $X\in R$, one infers an equation

$$X^n+(m_1+2s_1)X^{n-1}+\cdots+(m_n+2s_n)=0\ ,$$

with $m_i\in\boldsymbol{Z}$ and $s_i\in S$ for all i. Setting

$$f=2^{-1}(X^n+m_1X^{n-1}+\cdots+m_{n-1}X+m_n)\ ,$$

we find $f=-(s_1X^{n-1}+\cdots+s_n)\in\boldsymbol{Q}[X]\cap S\subset\boldsymbol{Q}(\{X_i\})\cap S=\boldsymbol{Z}[\{X_i\}]$. Accordingly, the leading coefficient of f, namely $1/2$, is an integer: this is the desired contradiction.

We next consider the case $\mathrm{ch}(R)=p>0$, and must infer (b) from (1). The contrapositive is easy to see: if V is a valuation ring of K such that $R\not\subset V$ and K is the quotient field of $V\cap R$, then R is not integral over its underring $V\cap R$.

(ii) If K is algebraic over $\boldsymbol{F}_p$, then $R=K$ (cf. [8, Theorem 48]).

In this case, (b) holds trivially. (Alternately, apply Proposition 2.2 or Proposition 2.1).

(iii) Let $\{X_i\}$ denote a transcendence basis of K over $\boldsymbol{F}_p$ such that $\{X_i\}\subset R$. Let X be one of the X_i; and *if possible*, let Y be a *different* one of the X_i. We proceed to adapt the methods used to prove the characteristic-zero case. Let S_1 denote the intersection of R with the integral closure of $\boldsymbol{F}_p[\{X_i\}]$ in K. One shows as above that S_1 is an underring of R; hence, $E_1=\boldsymbol{F}_p+YS_1$ is too. By (1), X is integral over E_1. Manipulating an integrality equation as above, we find a monic polynomial $g\in \boldsymbol{F}_p[X]$ such that

$$Y^{-1}g\in \boldsymbol{F}_p(\{X_i\})\cap S_1=\boldsymbol{F}_p[\{X_i\}]\ .$$

As $g\in Y\boldsymbol{F}_p[\{X_i\}]$ is nonzero, $\deg_Y(g)\geqq 1$, contradicting $g\in \boldsymbol{F}_p[X]$. Thus, no such Y exists; in other words, t.d.$(K/\boldsymbol{F}_p)=1$. It remains only to show that R is integral over $\boldsymbol{F}_p[X]$.

To this end, let B denote the integral closure of $\boldsymbol{F}_p[X]$ in R. By clearing denominators, each $r\in R\setminus\{0\}$ can be expressed as $r=bc^{-1}$ for suitable $b\in B$, $c\in \boldsymbol{F}_p[X]\setminus\{0\}$. In particular, R is an overring of B; that is, B is a underring of R. By (1), R is integral over B. As B is integral over $\boldsymbol{F}_p[X]$, R is thus integral over $\boldsymbol{F}_p[X]$. The proof is complete.

It seems difficult to give a neat list of the domains satisfying (b) in the statement of Theorem 2.3. In particular, not all R of transcendence degree 1 over $\boldsymbol{F}_p$ satisfy (b). The first three parts of the next result elaborate on these points.

REMARK 2.4. (a) The following result will be useful. If $L\subset F$ is an algebraic field extension and S is an underring of $F[X]$ which contains L, then $F[X]$ is integral over S.

For a proof, it is enough to show that X is integral over S, since each element of F is integral over S. Now, since S is an underring of $F[X]$, we can write $X^{-1}=ab^{-1}$, with $a, b\in S\setminus\{0\}$. As $a\in F[X]$, we have $a=\alpha_0X^n+\cdots+\alpha_n$, $\alpha_i\in F$, $\alpha_0\neq 0$. Replace a with bX^{-1}, multiply through by X, and solve for X^n. We see that X is integral over $T=S[(\alpha_0)^{-1}, \alpha_1, \cdots, \alpha_n]$. Since each element of F is integral over S, T is integral over S, and the assertion follows.

As an application of the above result, we have the following: if X is transcendental over $\boldsymbol{F}_p$, then $\boldsymbol{F}_p[X]$ is integral over each of its underrings. Thus, the domains characterized via underring behavior in Theorem 2.3 include rings not catalogued via the analogous subring behavior in Proposition 2.2.

For another application of the above result, this time with K infinitely-generated over $\boldsymbol{F}_p$, let F denote an algebraic closure of $\boldsymbol{F}_p$, and note that $R=F[X]$ is integral over each of its underrings.

(b) Let X be transcendental over $\boldsymbol{F}_p$, and let $K=\overline{\boldsymbol{F}_p(X)}$, an algebraic closure of $\boldsymbol{F}_p(X)$. Suppose that R is integral over $\boldsymbol{F}_p[X]$. Then R fails to be integral over some underring of R. (In particular, the necessary conditions in Theorem 2.3 (iii) do not imply (1).) Indeed, by using Riemann's theorem, Gilmer and Heinzer [7, Remark 3.2] showed that under these conditions, $V\cap R$ has quotient field K for each valuation ring V of K. (In applying [7], one must also note that $\dim(V)\leqq 1$: see [6, Theorem 20.9]; and that $\boldsymbol{F}_p\subset V$.) Since the intersection of all the valuation rings of $\overline{\boldsymbol{F}_p(X)}$ is algebraic over $\boldsymbol{F}_p$, R does not lie in this intersection. Accordingly, Theorem 2.3 (i) yields the assertion about R.

(c) The situation for global fields of positive characteristic is just as varied as for the infinitely-generated cases studied in (a), (b). On the one hand, let n be any positive integer, let $\boldsymbol{F}_{p^n}$ replace F in the final paragraph of (a), and let R $(=\boldsymbol{F}_{p^n}[X])$ be the resulting integral closure of $\boldsymbol{F}_p[X]$ in $K=\boldsymbol{F}_{p^n}(X)$. As above, one shows that R is integral over each of its underrings. By taking $n\geqq 2$, we have examples not subsumed under the first application in (a), since Luroth's theorem shows that K is then not purely transcendental over $\boldsymbol{F}_p$.

On the other hand, not all global fields lead to rings of the type characterized in Theorem 2.3. To see this, let K be a global field algebraic over some $\boldsymbol{F}_p(X)$ such that R, the integral closure of $\boldsymbol{F}_p[X]$ in K, is contained in all but two valuation rings, say $V_1\neq V_2$, of K. (This can be guaranteed by arranging that the (X^{-1})-adic valuation splits in K, where $[K\colon \boldsymbol{F}_p(X)]=2$. For instance, take $p\neq 2$ and $K=\boldsymbol{F}_p(X, y)$, where $y^2+X^{-1}-1=0$. The key points are that if T denotes the integral closure of $\boldsymbol{F}_p[X^{-1}]$ in K, then both $y+1$ and $y-1$ are nonunits of T; and no maximal ideal of T can contain both $y+1$ and $y-1$.) Then, by Riemann's theorem (cf. [7, Remark 3.1], with $D=V_2$), K is the quotient field of $V_1\cap R$. As $V_1\cap R\subsetneqq R$, Theorem 2.3 (i) yields that R is not integral over at least one of its underrings.

(d) This section's theorem suggests numerous related questions about underrings. In closing, we mention one of these. Which domains R have the property that R is flat over each of its underrings? Since a domain cannot have a proper overring which is both flat and integral, this question is at "the other extreme" from that of Theorem 2.3. By the theory of normal pairs [2, Theorem 3] or by [4, Theorem 2.2], if each underring of R is integrally closed, then R is flat over each of its underrings. It would be interesting to find other sufficient conditions.

§ 3. The relative theory.

Corollary 3.6 will augment Theorem 2.3 by giving a new characterization of the domains of positive characteristic which are integral over each of their underrings. It will be convenient to say, given domains $S \subset R$, that R is *S-linear* in case R is integral over each of its underrings that contains S. (The terminology is suggested by the role of the affine line, in the guise of $\boldsymbol{F}_p[X]$, below.) Notice, if S denotes the prime ring of R, that R is S-linear if and only if R is integral over each of its underrings. We begin our study of S-linearity with a result that reworks part of section 2.

LEMMA 3.1. *Let $S \subset R$ be domains. Then:*

(i) *If R is S-linear, then* $\mathrm{t.d.}(R/S) \leqq 1$.

(ii) *Suppose that R is algebraic over S. Then R is S-linear if and only if R is integral over S.*

(iii) *If R is S-linear and $X \in R$ is transcendental over S, then R is integral over $S[X]$ and S is a field.*

PROOF. (i) Replace $\boldsymbol{F}_p$ with S in the first paragraph of the proof of Theorem 2.3 (iii).

(ii) The "if" assertion is trivial. Conversely, replacing $\boldsymbol{Z}$ with S, adapt the second paragraph of the proof that $(1) \Rightarrow (2)$ in Theorem 2.3 (i).

(iii) Replace $\boldsymbol{F}_p$ with S in the last paragraph of the proof of Theorem 2.3, to show R integral over $S[X]$. Moreover, if S is not a field, choose a nonzero prime ideal P of S and find a prime J of R such that $J \cap S[X] = PS[X]$. By S-linearity, R is integral over its underring $S+J$; hence, R/J is integral over $(S+J)/J \cong S/J \cap S = S/P$. However, R/J contains the S/P-algebra $S[X]/PS[X] \cong (S/P)[X]$, which is *not* integral over S/P. This contradiction completes the proof.

As usual, it will be convenient to let A' denote the integral closure of a ring A.

PROPOSITION 3.2. *Let $S \subset R$ be domains. Then R' is S-linear if and only if R is S-linear.*

PROOF. The "only if" assertion holds since integrality is transitive. Conversely, assume that R is S-linear, and let D be any underring of R' that contains S. It will suffice to find $X \in D$ such that R' is integral over $S[X]$, for R' will then be integral over D, as desired. Thus, by

Lemma 3.1 (iii) and the transitivity of integrality, it will suffice to find $X \in D \cap R$ which is transcendental over S. Hence, we need only prove that t.d.$(D \cap R/S)=1$.

As Lemma 3.1 (ii) makes the assertion clear in case R is algebraic over S, Lemma 3.1 (i) reduces us to t.d.$(R/S)=1$. Hence t.d.$(D/S)=1$, and we may pick $Y \in D$ which is transcendental over S. Thus t.d.$(D \cap R[Y]/S)=1$, and so K is algebraic over (the quotient field of) $D \cap R[Y]$. Hence, by [7, Corollary 1.8], K *is* that quotient field. Now, since Y is integral over R, the conductor $(R\colon R[Y])$ is nonzero, and an application of [7, Proposition 2.2] yields that $D \cap R$ has quotient field K. In particular, t.d.$(D \cap R/S)=$t.d.$(R/S)=1$, completing the proof.

PROPOSITION 3.3. *Let $S \subset R$ be domains. Let L be a subfield of K which contains S. Assume that R is S-linear, R is integrally closed, and* t.d.$(R \cap L/S)=1$. *Then $R \cap L$ is S-linear and $R \cap L$ has quotient field L.*

PROOF. Choose $X \in A=R \cap L$, transcendental over S. By Lemma 3.1 (iii), R is integral over $S[X]$, and so R is the integral closure of $S[X]$ in K. Hence, A is the integral closure of $S[X]$ in L. Moreover, Lemma 3.1 (i) and the hypotheses yield t.d.$(R/S)=1$, and it follows easily that L is algebraic over $S[X]$. Then, by clearing denominators, $A_{S[X]\setminus\{0\}}=L$; in particular, L is the quotient field of A.

It remains to show that if D is an underring of A which contains S, then A is integral over D. As t.d.$(D/S)=$t.d.$(L/S)=1$, we may pick $Y \in D$, transcendental over S. By Lemma 3.1 (iii), R is integral over $S[Y]$. As $S[Y] \subset D \subset A \subset R$, the proof is complete.

We come next to this section's main result, motivated in part by Remark 2.4 (b), (c).

THEOREM 3.4. *Let $S \subset R$ be domains whose corresponding extension of quotient fields $F \subset K$ is not algebraic. If R is S-linear, then precisely one valuation ring of K/F does not contain R.*

PROOF. A valuation ring of K contains R if and only if it contains R'. Thus, in view of Proposition 3.2, we may replace R with R'; that is, suppose R integrally closed. Moreover, since the intersection of all valuation rings of K/F is algebraic over F and K/F is not algebraic, at least one valuation ring of K/F does not contain R.

Case 1: K finitely generated over F. If the assertion fails, choose distinct valuation rings V_1, V_2 of K/F, neither of which contains R. By

Lemma 3.1 (i) and the hypothesis, t.d.$(R/S)=1$. Now, let D denote the intersection of all valuation rings of K/F other than V_2. By Riemann's theorem (cf. [7, Remark 3.1]), K is the quotient field of D. We claim that $A=V_1\cap R$ also has quotient field K. Indeed, it is enough to show that $B=A_{S\setminus\{0\}}$ has quotient field K. Note that $B=V_1\cap R_{S\setminus\{0\}}$ contains F. As $R_{S\setminus\{0\}}$ is integrally closed, it is an intersection of a family $\{W_i\}$ of valuation rings of K/F. As $R\not\subset V_2$, no W_i is V_2, and so $D\subset \cap W_i=R_{S\setminus\{0\}}$. As $D\subset V_1$, it follows that $D\subset B$, whence K is the quotient field of B, proving the claim. Since R is S-linear, R is integral over A, although R is not contained in the valuation overring V_1 of A. This (desired) contradiction settles case 1.

General case. Once again, suppose the assertion fails, and choose distinct valuation rings V_1, V_2 of K/F, neither of which contains R. As above, t.d.$(R/S)=1$. Pick $X\in R$, transcendental over F. Since K is algebraic over $F(X)$, $E_i=V_i\cap F(X)$ is a nontrivial valuation ring of $F(X)/F$. Hence, E_i is a DVR. In particular, $\dim(E_i)=1$, and so (cf. [6, Theorem 19.16 (c)]) $\dim(V_i)=1$. Thus V_1 and V_2 are incomparable, yielding an element α in the maximal ideal of V_1 but not in the maximal ideal of V_2. Put $L=F(X,\alpha)$ and $R_1=R\cap L$. By Proposition 3.3, R_1 is S-linear and has quotient field L. As $S[X]\subset R_1$ and Lemma 3.1 (iii) yields that R is integral over $S[X]$, R is integral over R_1. Since $D_i=V_i\cap L$ is a valuation ring of L/F, one has $R_1\not\subset D_i$ (lest $R\subset D_i\subset V_i$, an absurdity). By case 1, $D_1=D_2$, contradicting $\alpha^{-1}\in D_2\setminus D_1$. This completes the proof.

As a partial converse to the above result, we next offer

PROPOSITION 3.5. *Let $S\subset R$ be domains whose corresponding extension of quotient fields $F\subset K$ is not algebraic. Then R is S-linear if and only if at most one valuation ring of K/S does not contain R.*

PROOF. The "if" assertion follows by combining Theorem 3.4 with Lemma 3.1 (iii). As for the converse, by Proposition 3.2, we may suppose that R is integrally closed. If the assertion fails, there exists an underring D of R such that $S\subset D$ and R is not integral over D. The latter condition gives a valuation overring V of D such that $R\not\subset V$. Then, by hypothesis, D is contained in each valuation ring of K/S and, *a fortiori*, in each valuation ring of K/F. Hence D is algebraic over F, although K is not. This contradiction completes the proof.

Our next result is the promised characteristic p result. In conjunction with Theorem 2.3, it gives the final result.

COROLLARY 3.6. *Assume that* $\mathrm{ch}(R)=p>0$ *and that R is not algebraic* *over $\boldsymbol{F}_p$. Then R is integral over each underring of R if and only if precisely one valuation ring of K does not contain R.*

PROOF. Set $S=\boldsymbol{F}_p\ (=F)$. As each domain of characteristic p contains $\boldsymbol{F}_p$, the assertion follows by combining Theorem 3.4 and Proposition 3.5.

COROLLARY 3.7. *R is integral over each underring of R if and only if one of the following three conditions holds:*

(1) *R is isomorphic to a subring of $\boldsymbol{A}$;*

(2) *$R=K$ is an algebraic field extension of some $\boldsymbol{F}_p$;*

(3) $\mathrm{ch}(R)>0$ *and precisely one valuation ring of K does not contain R.*

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