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An Extension of the Method of Iwahori Algebra

Dedicated to Professor Nagayoshi Iwahori on his 60th birthday

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Introduction

This paper is a study of three different types of induced representations of algebraic groups over an algebraically closed field K. As an application of it we can extend the method of Hecke algebras or Iwahori algebras of finite Chevalley groups introduced by N. Iwahori over $\boldsymbol{Z}$, the ring of integers (see [4]) to the case of Chevalley groups G over K.

In section 1 we shall define the induced modules, but in case of Chevalley group G these three induced modules are given as follows. Let B be a certain Borel subgroup of G as in [10, §3] and $K^{\times}=K-\{0\}$. Let $\lambda: B\to K^{\times}$ be a rational linear character of B into $K^{\times}$. We shall write λ_B^G, $KG*\bar{\lambda}$ and $\mathrm{ind}_B^G\lambda$ respectively for the three induced modules induced from λ, where KG is the algebra of G over K.

DEFINITIONS.

$$\lambda_B^G=\{f: G\to K \mid f(bg)=\lambda(b)f(g) \text{ for any } b\in B \text{ and } g\in G\}$$

(see [5]). We define $g*f$, where $g\in G$ and $f\in\lambda_B^G$, to be the map of G into K which takes $x\in G$ to $f(xg)$, i.e.,

$$g*f(x)=f(xg) \qquad (x,\ g\in G)\ .$$

$$\mathrm{ind}_B^G\lambda=\{f\in K[G] \mid f(bg)=\lambda(b)f(g) \text{ for any } b\in B \text{ and } g\in G\}\ ,$$

i.e.,

$$\mathrm{ind}_B^G\lambda=K[G]\cap\lambda_B^G, \text{ where } K[G] \text{ is the coordinate ring of } G\ .$$

We define $\bar{\lambda}$ to be the map of G into K which takes $x\in G-B$ to 0 and $x\in B$ to $\lambda(x)$, then $\bar{\lambda}\in\lambda_B^G$ and

$$KG\otimes_{KB}L\cong KG*\bar{\lambda} \qquad \text{(see Proposition 3.1)}$$

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where L is a one-dimensional KB-module which affords the character λ.

$\mathrm{ind}_B^G \lambda$ and $KG*\bar{\lambda}$ are KG-submodules of λ_B^G and these three induced modules coincide with each other if G is finite (see Corollary 1.2 and Remarks to (1.7)).

In section 2 we examine the basic properties of the induced modules such as λ_B^G in case of abstract groups and show Frobenius Reciprocity and Transitivity of Induction etc.

In section 3 we review the structure of certain spaces of KG-homomorphisms such as $\mathrm{Hom}_{KG}(KG*\bar{\lambda}, \lambda_B^G)$, which will turn out to be a generalization of Iwahori algebra. However, those KG-homomorphism spaces were already studied in [8] in slightly different way.

In section 4 we show the extension as follows. It is well known that if $\mathrm{ind}_B^G \lambda \neq 0$, $\mathrm{ind}_B^G \lambda$ contains a unique B-stable line generated by, say, f, and $KG*f$ is an irreducible finite dimensional rational KG-module (see [10, §12, Theorem 40]). Similar to the fact that the Iwahori algebra deeply relates with the ordinary representation theory of finite Chevalley groups (see [2] and [3]), we can describe the weight element f or $KG*f$ as the image of certain KG-homomorphism of $KG*\overline{\lambda^{w_0}}$ into λ_B^G, where w_0 is the element of maximal length of the Weyl group W of G. More precisely $\mathrm{Hom}_{KG}(KG*\overline{\lambda^{w_0}}, \lambda_B^G)$ has certain K-basis $\{a_w \mid w \in W_\lambda w_0\}$ (see Proposition 4.2) and we have

$$f=\Big(\sum_{w \in W_\lambda w_0} f(\omega_w)a_w\Big)(\overline{\lambda^{w_0}})$$

where ω_w is a fixed representative of w in N (see Theorem 4.7).

We have also got a similar result to the theorems on modular representations of finite Chevalley groups as in [6]. Let U be a certain maximal connected unipotent subgroup of G contained in B (see [10, §3]) and 1_U be the trivial one-dimensional linear character of U into $K^\times$, then 1_U^G contains $\mathrm{ind}_B^G \lambda$ as KG-submodule. Thus f is also contained in 1_U^G and we can describe it as the image of certain KG-homomorphism of $KG*\overline{1_U}$ into 1_U^G, that is, $\mathrm{Hom}_{KG}(KG*\overline{1_U}, 1_U^G)$ contains a certain linearly independent subset $\{A_{h\omega_w} \mid h \in H,\ w \in W\}$ (see Proposition 4.3), and

$$f=\Big(\sum_{w \in W_\lambda w_0,\, h \in H} \lambda(h)f(\omega_w)A_{h\omega_w}\Big)(\overline{1_U})$$

(see Theorem 4.9). One can also find a similar formula in the case of modular representations of finite Chevalley groups in [6, Proposition (3.1)].

§ 1. Definitions.

We first explain three different concepts of induced modules in group representation theory.

LEMMA 1.1 (see [5]). *Let G be a group and H be a subgroup of G. Let kG and kH be the group algebras of G and H over a field k respectively. Let L be a left kH-module. We write L_H^G for the set of all mappings*

$$f: G \to L \quad \text{such that } f(hg) = hf(g) \text{ for any } h \in H \text{ and } g \in G .$$

Then

(i) *L_H^G becomes a left kG-module by the following operation*

$$(f_1+f_2)(g) = f_1(g)+f_2(g) \quad (f_1, f_2 \in L_H^G,\ g \in G) ,$$
$$(cf)(g) = cf(g) \quad (f \in L_H^G,\ g \in G,\ c \in k) ,$$
$$(g * f)(x) = f(xg) \quad (f \in L_H^G,\ g, x \in G) .$$

(ii) *Let $G = \cup_{m \in \mathscr{M}} Hx_m$ (disjoint union) and $\{l_i \mid i \in \mathscr{I}\}$ be a k-basis of L, i.e., $L = \bigoplus_{i \in \mathscr{I}} kl_i$ (direct sum), then $\{x_m^{-1} \otimes l_i \mid (i, m) \in \mathscr{I} \times \mathscr{M}\}$ forms a k-basis of $kG \otimes_{kH} L$.*

(iii) *Let f_{im} be the mapping of G into L such that $f_{im}(hx_j) = \delta_{mj} hl_i$ ($h \in H$ and $j \in \mathscr{M}$), then $f_{im} \in L_H^G$ for any $(i, m) \in \mathscr{I} \times \mathscr{M}$, where $\delta_{mm} = 1$ and $\delta_{mj} = 0$ if $m \neq j$.*

(iv) *Let ι be a mapping of $kG \otimes_{kH} L$ into L_H^G which takes each $x_m^{-1} \otimes l_i$ to f_{im}, where $(i, m) \in \mathscr{I} \times \mathscr{M}$, then ι is an injective kG-homomorphism.*

Proof is straightforward.

COROLLARY 1.2. *Let G, H, L, k and ι be as in Lemma 1.1. If $\mathscr{M}$ is a finite set, i.e., $[G : H] < \infty$, then ι is bijective.*

PROOF. Let $f \in L_H^G$ and $f(x_m) = \sum_{i \in \mathscr{I}} c_{im} l_i$ where $m \in \mathscr{M}$ and $c_{im} \in k$, then $\{c_{im} \mid (i, m) \in \mathscr{I} \times \mathscr{M}\}$ is a finite set. Since

$$\sum_{(i,m) \in \mathscr{I} \times \mathscr{M}} c_{im} f_{im} \in L_H^G \quad \text{and} \quad \Big(\sum_{(i,m) \in \mathscr{I} \times \mathscr{M}} c_{im} f_{im}\Big)(x_j) = \sum_{i \in \mathscr{I}} c_{ij} l_i = f(x_j)$$

where $j \in \mathscr{M}$, we have shown that $f = \sum_{(i,m) \in \mathscr{I} \times \mathscr{M}} c_{im} f_{im} \in \iota(kG \otimes_{kH} L)$. Hence ι is bijective. Q.E.D.

It can be easily shown that ι is not bijective in general.

Now let K be an algebraically closed field and $(G, \mathscr{S}_G)$ be an algebraic grouo over K, i.e., $(G, \mathscr{S}_G)$ is a variety with a sheaf of K-valued functions

$\mathcal{S}_G$ on G and has the group operations which are morphisms of varieties (see e.g. [9]). Since any finite group $\mathcal{G}$ can be embedded into a symmetric group S_n ($\subset GL_n(K)$), where $n=|\mathcal{G}|$, by the regular representation, from now on we assume that a given finite group $\mathcal{G}$ is contained in $(G, \mathcal{S}_G)$. Since $\mathcal{G}$ is a closed subgroup of G, it has the induced sheaf of K-valued functions $\mathcal{S}_{\mathcal{G}}$ such that

$$\mathcal{S}_{\mathcal{G}}(O)=\mathcal{O}_{U_i\cap\mathcal{G}}(O) \qquad \text{for any open set } O \text{ in } \mathcal{G}$$

which is contained in $U_i\cap\mathcal{G}$ ($1\leq i\leq l$), where U_i's are affine open covering of G and $\mathcal{O}_{U_i\cap\mathcal{G}}$ is a canonical sheaf of functions on the affine algebraic variety $U_i\cap\mathcal{G}$.

LEMMA 1.3. *Let (V, A) be an affine algebraic variety over K where A is the coordinate ring of V. Let S be a finite subset of V and $M(S, K)$ be the set of all mappings of S into K. Then*

$$\{f|S \mid f\in A\}=M(S, K)\,.$$

PROOF. Clearly $f|S\in M(S, K)$ for any $f\in A$. Assume that S has t different elements $\{s_1, s_2, \cdots, s_t\}$. Let $V_i=V-(S-\{s_i\})$. Since finite sets are closed in V, V_i is a union of finite principal open sets V_f of V. $V_i=\cup_f V_f$. Since $s_i\in V_i$ and $s_j\notin V_i$ if $j\neq i$, there exists $f_i\in A$ such that

$$f_i(s_i)\neq 0 \quad\text{and}\quad f_i(s_j)=0 \text{ if } j\neq i, \qquad \text{for any } 1\leq i\leq t\,.$$

Hence $\{f|S \mid f\in A\}=M(S, K)$. Q.E.D.

COROLLARY 1.4. *Let $(X, \mathcal{S}_X)$ be a variety over K with a finite affine open covering $\{U_i|i=1, 2, \cdots, l\}$. Let F be a finite subset of X with the induced sheaf of K-valued functions $\mathcal{S}_F$ such that*

$$\mathcal{S}_F(O)=\mathcal{O}_{U_i\cap F}(O) \qquad \textit{for any open set } O \textit{ in } F$$

which is contained in $U_i\cap F$ ($1\leq i\leq l$). Then

$$\mathcal{S}_F(S)=M(S, K) \qquad \textit{for any subset } S \textit{ in } F\,.$$

PROOF. Since $\mathcal{S}_F(S\cap U_i)=\mathcal{O}_{U_i\cap F}(S\cap U_i)=M(S\cap U_i, K)$ and $S=\cup_i(S\cap U_i)$, $f\in\mathcal{S}_F(S)$ if and only if $f|S\cap U_i\in M(S\cap U_i, K)$ for any $1\leq i\leq l$. Hence $\mathcal{S}_F(S)=M(S, K)$. Q.E.D.

DEFINITION 1.5. Let $(G, \mathcal{S}_G)$ be an algebraic group over K and M be a vector space over K. We define $\mathrm{Map}(G, M)$ to be the K-space of all mappings f of G into M such that

$f(G)$ spans a finite dimensional K-subspace N of M

(we write $K\langle f(G)\rangle$ for N) and

$f: G \to N$ is a morphism of varieties .

PROPOSITION 1.6 (see [1]). *Let* $(G, \mathscr{S}_G)$ *be an algebraic group over* K *and* M *be a vector space over* K. *Then*

(i) Map(G, M) *is a left* KG*-module by the following operation*:

$$\begin{array}{ccc} G\times \mathrm{Map}(G, M) & \longrightarrow & \mathrm{Map}(G, M)\ , \\ \cup\!\!\!\!\!\!\;\;\; & & \cup\!\!\!\!\!\!\;\;\; \\ (g, f) & \longmapsto & g*f \end{array}$$

where $(g*f)(x)=f(xg)$ *for an* $x\in G$.

(ii) *If* $M=K$, *then* $\mathrm{Map}(G, K)=\mathscr{S}_G(G)$.

(iii) $\mathscr{S}_G(G)\otimes_K M$ *is a left* KG*-module by the following operation*

$$\begin{array}{ccc} G\times (\mathscr{S}_G(G)\otimes_K M) & \longrightarrow & \mathscr{S}_G(G)\otimes_K M\ , \\ \cup\!\!\!\!\!\!\;\;\; & & \cup\!\!\!\!\!\!\;\;\; \\ (g, f\otimes m) & \longmapsto & (g*f)\otimes m \end{array}$$

and the map

$$\begin{array}{ccc} \rho : \mathscr{S}_G(G)\otimes_K M & \longrightarrow & \mathrm{Map}(G, M)\ , \\ \cup\!\!\!\!\!\!\;\;\; & & \cup\!\!\!\!\!\!\;\;\; \\ f\otimes m & \longmapsto & \rho(f\otimes m) \end{array}$$

where $\rho(f\otimes m)(g)=f(g)m$ $(g\in G)$, *is a* KG*-isomorphism*.

(iv) Map(G, M) *is a locally finite rational* KG*-module, i.e.*, $KG*f$ *is a finite dimensional rational* KG*-module for any* $f\in \mathrm{Map}(G, M)$, *if* $\mathscr{S}_G(G)$ *is locally finite and rational as left module.*

DEFINITION 1.7. Let $(G, \mathscr{S}_G)$ be an algebraic group over K and H be a closed subgroup of G. Let V be a left KH-module, then we define the induced KG-module $\mathrm{ind}_H^G V$ induced from V to be the KG-submodule

$$\mathrm{ind}_H^G V=\{f\in \mathrm{Map}(G,V)\mid f(hg)=hf(g) \text{ for all } h\in H \text{ and } g\in G\}$$

of $\mathrm{Map}(G,V)$.

REMARKS TO (1.7). Let G, H and V be as in Definition 1.7. Then

(i) $\mathrm{ind}_{\{1\}}^G V=\mathrm{Map}(G,V)$, where $\{1\}$ is the trivial subgroup of G.

(ii) $\mathrm{ind}_{\{1\}}^G K=\mathscr{S}_G(G)$, where K is considered as the one-dimensional trivial left module of $\{1\}$.

(iii) If $\mathrm{Map}(G,V)$ is locally finite and rational, e.g., G is affine, then $\mathrm{ind}_H^G V$ is also locally finite and rational.

(iv) Let $(\mathscr{G}, \mathscr{S}_{\mathscr{G}})$ be a finite subgroup of G and $\mathscr{H}$ be a subgroup of $\mathscr{G}$ and W be a $K\mathscr{H}$-module. Then

(a) $\mathrm{ind}_{\mathscr{H}}^{\mathscr{G}} W$ is the set of all mappings $f\colon \mathscr{G} \to W$ such that $f(hg)=hf(g)$ for all $h \in \mathscr{H}$ and $g \in \mathscr{G}$, i.e., $\mathrm{ind}_{\mathscr{H}}^{\mathscr{G}} W = W_{\mathscr{H}}^{\mathscr{G}}$.

(b) There exists a $K\mathscr{G}$-isomorphism ι of $K\mathscr{G} \otimes_{K\mathscr{H}} W$ into $\mathrm{ind}_{\mathscr{H}}^{\mathscr{G}} W$ such that

$$\iota(x_m^{-1} \otimes w_i) = f_{im}\colon \begin{array}{ccc} \mathscr{G} & \longrightarrow & W \\ \cup & & \cup \\ hx_j & \longmapsto & \delta_{mj} h w_i \end{array}$$

where $\mathscr{G} = \bigcup_{m \in \mathscr{M}} \mathscr{H} x_m$ (disjoint union), $W = \bigoplus_{i \in \mathscr{I}} K w_i$ (direct sum) and $h \in \mathscr{H}$.

So far we have defined three types of induced modules of an algebraic group $(G, \mathscr{S}_G)$ over K, V_H^G, $\mathrm{ind}_H^G V$ and $KG \otimes_{KH} V$ where $(H, \mathscr{S}_H)$ is a closed subgroup of $(G, \mathscr{S}_G)$ and V is a KH-module. Though V_H^G contains $\mathrm{ind}_H^G V$ and $KG \otimes_{KH} V$ and all these three modules coincide in case of finite groups, they are not equal in general.

§2. Basic properties of V_H^G.

Let G be a group and H be a subgroup of G. Let kG and kH be the group algebras of G and H over a field k respectively. We show some basic properties of V_H^G such as Frobenius Reciprocity and Transitivity of Induction etc., where V is a left kH-module.

PROPOSITION 2.1. *Let G be a group, H be a subgroup of G and V be a kH-module.*

(i) *Let*

$$\varepsilon_V\colon \begin{array}{ccc} V_H^G & \longrightarrow & V\,, \\ \cup & & \cup \\ f & \longmapsto & f(1) \end{array}$$

then ε_V is a kH-homomorphism.

(ii) *For any kG-module M and kH-homomorphism φ of M into V, there exists a unique kG-homomorphism $\tilde{\varphi}\colon M \to V_H^G$ which makes the following diagram commutative.*

$$\begin{array}{ccc} M & \xrightarrow{\ \varphi\ } & V \\ & {\scriptstyle \exists 1\tilde{\varphi}} \searrow \quad \circlearrowleft \quad \nearrow {\scriptstyle \varepsilon_V} & \\ & V_H^G & \end{array}$$

PROOF. (i) $\varepsilon_V(h*f)=f(h)=hf(1)=h\varepsilon_V(f)$ for any $h\in H$ and $f\in V_H^G$.

(ii) Let $m\in M$, we define $\tilde{\varphi}(m)$ to be the map of G into V such that $G\ni g\mapsto\varphi(gm)\in V$. Since φ is a kH-homomorphism, $\tilde{\varphi}(m)\in V_H^G$. Let $x,\ g\in G$, then

$$x*(\tilde{\varphi}(m))(g)=\tilde{\varphi}(m)(gx)=\varphi(gxm)=\tilde{\varphi}(xm)(g)\ .$$

Hence $x*\tilde{\varphi}(m)=\tilde{\varphi}(xm)$ for any $x\in G$ and $m\in M$, that is, $\tilde{\varphi}$ is a kG-homomorphism. Clearly $\varepsilon_V\circ\tilde{\varphi}(m)=\tilde{\varphi}(m)(1)=\varphi(m)$ for any $m\in M$. Let $f\colon M\to V_H^G$ be a kG-homomorphism such that $\varepsilon_V\circ f=\varphi$. Since $\{f(m)\}(1)=\varphi(m)$ for any $m\in M$ and $\{f(m)\}(g)=g*\{f(m)\}(1)=f(gm)(1)=\varphi(gm)$, we have $f=\tilde{\varphi}$. Q.E.D.

COROLLARY TO (2.1) (Frobenius Reciprocity). *Let M be a kG-module, then*

$$\begin{array}{ccc}\mathrm{Hom}_{kH}(M,V) & \cong & \mathrm{Hom}_{kG}(M,V_H^G)\\ \uplus & & \uplus\\ \varphi & \longmapsto & \tilde{\varphi}\end{array}$$

as k-spaces where V and $\tilde{\varphi}$ are as in (ii).

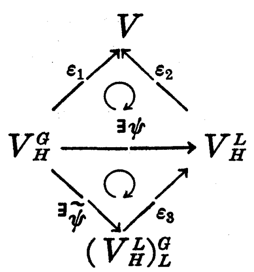
PROPOSITION 2.2 (Transitivity of Induction). *Let G be a group and $L\supset H$ be subgroups of G. Let V be a kH-module, then*

$$(V_H^L)_L^G\cong V_H^G \qquad \textit{as } kG\textit{-modules}\ .$$

PROOF. Let

$$\begin{array}{ccc}\varepsilon_1: V_H^G & \longrightarrow & V\ ,\\ \uplus & & \uplus\\ f & \longmapsto & f(1)\end{array}\qquad \begin{array}{ccc}\varepsilon_2: V_H^L & \longrightarrow & V\\ \uplus & & \uplus\\ f & \longmapsto & f(1)\end{array}\quad\text{and}\quad \begin{array}{ccc}\varepsilon_3: (V_H^L)_L^G & \longrightarrow & V_H^L\ .\\ \uplus & & \uplus\\ f & \longmapsto & f(1)\end{array}$$

From Proposition 2.1 there exist a kL-homomorphism $\psi\colon V_H^G\to V_H^L$ and a kG-homomorphism $\tilde{\psi}\colon V_H^G\to(V_H^L)_L^G$ which makes the following diagram commutative.



Notice that

$$\begin{array}{rcl}(\tilde{\Psi}(f))(g): L & \longrightarrow & V \\ \cup\!\!\!\!\!\!\cdot & & \cup\!\!\!\!\!\!\cdot \\ l & \longmapsto & f(lg)\end{array} \qquad \text{for any } f \in V_H^G \text{ and } g \in G .$$

Now let $\mu \in (V_H^L)_L^G$. We define $\bar{\mu}$ to be the map of G into V such that $\bar{\mu}(g) = (\mu(g))(1)$ where $g \in G$. Since

$$\begin{array}{rcccl}\bar{\mu}: G & \longrightarrow & V_H^L & \xrightarrow{\varepsilon_2} & V \\ \cup\!\!\!\!\!\!\cdot & & \cup\!\!\!\!\!\!\cdot & & \cup\!\!\!\!\!\!\cdot \\ g & \longmapsto & \mu(g) & \longmapsto & (\mu(g))(1)\end{array}$$

and $\bar{\mu}(hg) = (\mu(hg))(1) = (h * \mu(g))(1) = (\mu(g))(h) = h(\mu(g)(1)) = h(\bar{\mu}(g))$ for any $h \in H$ and $g \in G$, we have $\bar{\mu} \in V_H^G$. Since

$$\begin{array}{rcl}\tilde{\Psi}(\bar{\mu})(g): L & \longrightarrow & V \\ \cup\!\!\!\!\!\!\cdot & & \cup\!\!\!\!\!\!\cdot \\ l & \longmapsto & \bar{\mu}(lg)\end{array}$$

and $\bar{\mu}(lg) = (\mu(lg))(1) = (l * \mu(g))(1) = (\mu(g))(l)$ for any $l \in L$ and $g \in G$, we have $\tilde{\Psi}(\bar{\mu}) = \mu$. Hence $\tilde{\Psi}$ is surjective. Injectivity of $\tilde{\Psi}$ is clear. Q.E.D.

DEFINITION 2.3. Let G be a group and M and N be kG-modules, then $M \otimes_k N$ becomes a kG-module by the following operation.

$$\begin{array}{rcl}G \times M \otimes_k N & \longrightarrow & M \otimes_k N \\ \cup\!\!\!\!\!\!\cdot & & \cup\!\!\!\!\!\!\cdot \\ (g, \sum_i m_i \otimes n_i) & \longmapsto & \sum_i g m_i \otimes g n_i\end{array}$$

DEFINITION 2.4. Let G be a group and H a subgroup of G. Let V be a left kH-module where k is a field. We define a kG-submodule $\hat{V}_H^G$ of V_H^G to be the set of all mappings $f: G \to V$ such that $f(G)$ generates a finite dimensional k-subspace $k\langle f(G) \rangle$ of V and

$$f(hg) = hf(g) \qquad \text{for any } h \in H \text{ and } g \in G .$$

PROPOSITION 2.5 (Tensor Identity). *Let G be a group and H be a subgroup of G. Let V be a kH-module, and W be a kG-module such that kGw is finite dimensional for any $w \in W$. Then*

$$\begin{array}{rcl}\hat{V}_H^G \otimes_k W & \overset{\rho}{\cong} & (\widehat{V \otimes_k W})_H^G \\ \cup\!\!\!\!\!\!\cdot & & \cup\!\!\!\!\!\!\cdot \\ f \otimes w & \longmapsto & [\rho(f \otimes w): g \mapsto f(g) \otimes gw]\end{array}$$

as kG-modules.

PROOF (see [1]). Let $\hat{M}(G, V)$ be the set of all mappings f of G into

V such that $f(G)$ generates a finite dimensional k-subspace of V. Then $\hat{M}(G,V)$ becomes a left kG-module by the following operation:

$$\begin{aligned}
&(f_1+f_2)(g)=f_1(g)+f_2(g) && (f_1, f_2\in\hat{M}(G,V),\ g\in G)\ ,\\
&(cf)(g)=cf(g) && (f\in\hat{M}(G,V),\ g\in G,\ c\in k)\ ,\\
&(g*f)(x)=f(xg) && (f\in\hat{M}(G,V),\ g,\ x\in G)\ .
\end{aligned}$$

We first show that the map

$$\begin{array}{rcl}
\Phi:\hat{M}(G,V)\otimes_k W & \longrightarrow & \hat{M}(G,V\otimes_k W)\\
\cup\!\!\!\!\!\;\shortmid & & \cup\!\!\!\!\!\;\shortmid\\
f\otimes w & \longmapsto & [\Phi(f\otimes w)\colon g\mapsto f(g)\otimes gw]
\end{array}$$

is a kG-isomorphism. Let $\{v_i\mid i\in I\}$ be a k-basis of V. Since $V=\bigoplus_{i\in I}kv_i$ (direct sum) and $\hat{M}(G,kv_i)\subset\hat{M}(G,V)$,

$$\hat{M}(G,V)=\hat{M}(G,\bigoplus_{i\in I}kv_i)=\bigoplus_{i\in I}\hat{M}(G,kv_i)\ .$$

Hence $\hat{M}(G,V)\otimes_k W=(\bigoplus_{i\in I}\hat{M}(G,kv_i))\otimes_k W=\bigoplus_{i\in I}(\pi_i\otimes 1_W)(\hat{M}(G,V)\otimes_k W)$ where $\pi_i\colon\hat{M}(G,V)\to\hat{M}(G,kv_i)$ is the projection and $1_W\colon W\to W$ is the identity map. Similarly since $V\otimes_k W=\bigoplus_{i\in I}kv_i\otimes_k W$, we have $\hat{M}(G,V\otimes_k W)=\bigoplus_{i\in I}\hat{M}(G,kv_i\otimes_k W)$. Thus we first check that

$$\begin{array}{rcl}
\Phi\mid\hat{M}(G,kv_i)\otimes_k W:\hat{M}(G,kv_i)\otimes_k W & \longrightarrow & \hat{M}(G,kv_i\otimes W)\\
\cup\!\!\!\!\!\;\shortmid & & \cup\!\!\!\!\!\;\shortmid\\
f\otimes w & \longmapsto & [\Phi(f\otimes w)\colon g\mapsto f(g)\otimes gw]
\end{array}$$

is a k-isomorphism. Let

$$\begin{array}{rcl}
\varphi:\hat{M}(G,k)\otimes_k W & \longrightarrow & \hat{M}(G,W)\ ,\\
\cup\!\!\!\!\!\;\shortmid & & \cup\!\!\!\!\!\;\shortmid\\
f\otimes w & \longmapsto & [\varphi(f\otimes w)\colon g\mapsto f(g)gw]\\
\varphi_1\colon M(G,k)\otimes_k W & \longrightarrow & \hat{M}(G,W)\qquad\text{and}\\
\cup\!\!\!\!\!\;\shortmid & & \cup\!\!\!\!\!\;\shortmid\\
f\otimes w & \longmapsto & [\varphi_1(f\otimes w)\colon g\mapsto f(g)w]\\
\varphi_2\colon\hat{M}(G,W) & \longrightarrow & \hat{M}(G,W)\ ,\\
\cup\!\!\!\!\!\;\shortmid & & \cup\!\!\!\!\!\;\shortmid\\
\tau & \longmapsto & [\varphi_2(\tau)\colon g\mapsto g\tau(g)]
\end{array}$$

then φ_1 and φ_2 are well-defined k-linear maps and $\varphi=\varphi_2\circ\varphi_1$. Let $\{f_t\mid t\in T\}$ and $\{w_j\mid j\in J\}$ be k-basis of $\hat{M}(G,k)$ and W respectively, then $\{f_t\otimes w_j\mid t\in T, j\in J\}$ forms a k-basis of $\hat{M}(G,k)\otimes_k W$. Assume that $\varphi_1(\sum_{t,j}c_{tj}f_t\otimes w_j)(x)=0$ for any $x\in G$ where $\{c_{tj}\}\subset k$ and almost all c_{tj}'s are zero. Since

$$\varphi_1(\sum_{t,j}c_{tj}f_t\otimes w_j)(x)=\sum_{t,j}c_{tj}f_t(x)w_j=\sum_j(\sum_t c_{tj}f_t(x))w_j=0$$

for any $x \in G$, $\sum_t c_{tj} f_t(x)=0$ for any $x \in G$ and $j \in J$. Thus all c_{tj}'s are zero and φ_1 is injective. Now let $h \in \hat{M}(G, W)$ and $\{w_1, w_2, \cdots, w_n\}$ be a k-basis of $N=k\langle h(G)\rangle$ and

$$\begin{array}{ccl} h: G & \longrightarrow & N=kw_1 \oplus \cdots \oplus kw_n \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ g & \longmapsto & h(g)=f_1(g)w_1+\cdots+f_n(g)w_n \end{array}$$

where $f_1(g), \cdots, f_n(g) \in k$, then $h=\varphi_1(\sum_{t=1}^{n} f_t \otimes w_t)$. Hence φ_1 is bijective. Since the map

$$\begin{array}{ccl} \hat{M}(G, W) & \longrightarrow & \hat{M}(G, W) \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ \tau & \longmapsto & [g \mapsto g^{-1}\tau(g)] \end{array}$$

is φ_2^{-1}, φ_2 is also bijective. Hence $\varphi=\varphi_2 \circ \varphi_1$ is a k-isomorphism. Thus we have shown that Φ is a k-isomorphism. Let $f \otimes w \in \hat{M}(G, V) \otimes_k W$, then

$$\begin{aligned} \Phi(g(f \otimes w))(x) &= \Phi(g * f \otimes gw)(x) = g * f(x) \otimes xgw = f(xg) \otimes xgw \\ &= (g * \Phi(f \otimes w))(x) \qquad \text{for any } g, x \in G, \end{aligned}$$

which implies that Φ is a kG-isomorphism. Finally we show that $\Phi(\hat{V}_H^G \otimes_k W)=(\widehat{V \otimes_k W})_H^G$. Let $f \otimes w \in \hat{V}_H^G \otimes_k W$, then

$$\Phi(f \otimes w)(hg)=f(hg) \otimes hgw=hf(g) \otimes hgw=h\Phi(f \otimes w)(g)$$

for any $h \in H$ and $g \in G$. Hence $\Phi(\hat{V}_H^G \otimes_k W) \subset (\widehat{V \otimes_k W})_H^G$. Conversely let $\tau \in (\widehat{V \otimes_k W})_H^G$ and $\{w_j | j \in J\}$ be a k-basis of W, then we have $\tau(g)= \sum_{j \in J} \tau_j(g) \otimes gw_j$ where $\tau_j(g) \in V$ $(j \in J)$. Suppose that there exists $\sum_{j \in J} f_j \otimes w_j \in \hat{M}(G, V) \otimes_k W$ such that $\Phi(\sum_{j \in J} f_j \otimes w_j)=\tau$, then $\tau(g)= \sum_{j \in J} \tau_j(g) \otimes gw_j=\sum_{j \in J} f_j(g) \otimes gw_j$ and $f_j(g)=\tau_j(g)$ for any $j \in J$ and $g \in G$. Hence it is enough to show that each $f_j \in \hat{V}_H^G$. Since

$$\tau(hg)=\sum_{j \in J} \tau_j(hg) \otimes hgw_j=h\tau(g)=\sum_{j \in J} h\tau_j(g) \otimes hgw_j ,$$

we have $f_j(hg)=hf_j(g)$ for any $h \in H$ and $g \in G$. Therefore $f_j \in \hat{V}_H^G$ and $\rho=\Phi | \hat{V}_H^G \otimes_k W$ is a kG-isomorphism. Q.E.D.

§3. Modules induced from linear representations.

Let G be a group and H a subgroup of G. Let λ be a linear character of H into $k^{\times}=k-\{0\}$, where k is a field. We consider k to be a one-dimensional kH-module such that

$$\begin{array}{ccc} H\times k & \longrightarrow & k\,. \\ \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid \\ (h,\,x) & \longmapsto & \lambda(h)x \end{array}$$

Thus we have $\lambda_H^G=\{f\colon G\to k \mid f(hg)=\lambda(h)f(g)$ for any $h\in H$ and $g\in G\}$.

Now let $\bar{\lambda}\colon G\to k$ be an extension of $\lambda\colon H\to k$ such that $\bar{\lambda}(h)=\lambda(h)$ for any $h\in H$ and $\bar{\lambda}(g)=0$ for any $g\in G-H$, then $\bar{\lambda}\in\lambda_H^G$ and $h*\bar{\lambda}=\lambda(h)\bar{\lambda}$ for any $h\in H$.

PROPOSITION 3.1 (see [8, Proposition (1.2) and (1.3)]). *Let G be a group and H a subgroup of G. Let λ be a linear character of H into $k^\times$, where k is a field. Let $\bar{\lambda}\colon G\to k$ be the extension of λ. Let $G=\bigcup_{m\in\mathscr{M}} Hx_m$ (disjoint union) and f_m be the mapping of G into k such that*

$$f_m(hx_j)=\delta_{mj}\lambda(h) \qquad (h\in H,\ j\in\mathscr{M})\,.$$

We assume that one of $\{x_m\}$ is 1. Then

(i) *$f_m\in\lambda_H^G$ for any $m\in\mathscr{M}$ and*

$$\begin{array}{rcl} \iota : kG\otimes_{kH} k & \longrightarrow & \lambda_H^G \\ \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid \\ x_m^{-1}\otimes 1 & \longmapsto & f_m \end{array}$$

is an injective kG-homomorphism and $\iota(kG\otimes_{kH} k)=kG*\bar{\lambda}$;

(ii) *$f_m=x_m^{-1}*\bar{\lambda}$ for any $m\in\mathscr{M}$;*

(iii) *since $x_m^{-1}*\bar{\lambda}$ takes the value zero outside of the coset Hx_m, for any scalar $c_m\in k$, we can define an element $\sum_{m\in\mathscr{M}} c_m x_m^{-1}*\bar{\lambda}$ of λ_H^G to be $\sum_{m\in\mathscr{M}} c_m x_m^{-1}*\bar{\lambda}(hx_{m'})=c_{m'}\lambda(h)$ where $h\in H$ and $m'\in\mathscr{M}$;*

(iv) *$\lambda_H^G=\{\sum_{m\in\mathscr{M}} c_m x_m^{-1}*\bar{\lambda} \mid \forall c_m\in k\}$.*

PROOF. (i) From Lemma 1.1, $f_m\in\lambda_H^G$ and ι is an injective kG-homomorphism. Let $x_{m_*}=1$, then $\iota(x_{m_*}^{-1}\otimes 1)=f_{m_*}=\bar{\lambda}$. Hence $\iota(kG\otimes_{kH} k)=kG*\bar{\lambda}$.

(ii) is clear from the proof of (i).

(iii) It is clear that $\sum_{m\in\mathscr{M}} c_m x_m^{-1}*\bar{\lambda} : G\to k$ is a well-defined map. Let $h\in H$ and $g\in G$. We can assume that $hg\in Hx_{m'}$ for some $m'\in\mathscr{M}$. Hence $(\sum_{m\in\mathscr{M}} c_m x_m^{-1}*\bar{\lambda})(hg)=c_{m'}\lambda(h')$ where $hg=h'x_{m'}$ for some $h'\in H$. Since $(\sum_{m\in\mathscr{M}} c_m x_m^{-1}*\bar{\lambda})(g)=c_{m'}\lambda(h^{-1}h')$, we have

$$\Big(\sum_{m\in\mathscr{M}} c_m x_m^{-1}*\bar{\lambda}\Big)(hg)=c_{m'}\lambda(h')=\lambda(h)c_{m'}\lambda(h^{-1}h')=\lambda(h)\Big(\sum_{m\in\mathscr{M}} c_m x_m^{-1}*\bar{\lambda}\Big)(g)\,.$$

Thus $\sum_{m\in\mathscr{M}} c_m x_m^{-1}*\bar{\lambda}\in\lambda_H^G$.

(iv) Let f be an element of λ_H^G, then we have

$$f=\sum_{m\in\mathscr{M}} f(x_m)x_m^{-1}*\bar{\lambda}\,.$$

Q.E.D.

Now let H_i be a subgroup of G and $\lambda_i: H_i \to k^\times$ be a linear character of H_i into $k^\times$, where $i=1, 2$. Let $\bar{\lambda}_i$ be the extension of λ_i and $Y_i = kG * \bar{\lambda}_i$ $(i=1, 2)$. Let $G = \cup_{m \in \mathcal{M}} H_2 x_m$ (disjoint) and $G = \cup_{n \in \mathcal{N}} H_1 y_n$ (disjoint). We assume that one of $\{x_m\}$ and one of $\{y_n\}$ are 1. Y_2 has a k-basis $\{x_m^{-1} * \bar{\lambda}_2\}$ and Y_1 has a k-basis $\{y_n^{-1} * \bar{\lambda}_1\}$ and

$$(\lambda_1)_{H_1}^G = \{ \sum_{n \in \mathcal{N}} c_n y_n^{-1} * \bar{\lambda}_1 \mid \forall c_n \in k\} .$$

We shall use the following notation, for $x, y \in G$:

$$y^x = x^{-1} y x , \quad H_1^x = x^{-1} H_1 x , \quad H_1^{(x)} = H_1^x \cap H_2 \qquad \text{and}$$

$$\begin{array}{ccc} \lambda_1^x : H_1^{(x)} & \longrightarrow & k^\times . \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ h & \longmapsto & \lambda_1(xhx^{-1}) \end{array}$$

We also write $H_2^{(x)}$ for $H_1 \cap H_2^x$, where $H_2^x = x^{-1} H_2 x$. Let $G = \cup_{i \in I} D_i$, where the D_i's are distinct (H_1, H_2)-double cosets $H_1 x H_2$ in G, and $J \subset I$ be the set of indices j such that for some $x \in D_j$, $\lambda_1^x = \lambda_2$ on $H_1^{(x)}$. It can be easily checked that if $\lambda_1^x = \lambda_2$ on $H_1^{(x)}$ for some $x \in D_j$, then $\lambda_1^x = \lambda_2$ on $H_1^{(x)}$ for all $x \in D_j$.

Assume $x \in G$ and $H_2 = \cup_s H_1^{(x)} h_s$ (disjoint), then we have $H_1 x H_2 = \cup_s H_1 x h_s$ (disjoint) and also $H_2 x^{-1} H_1 = \cup_s h_s^{-1} x^{-1} H_1$ (disjoint) (see [7, Lemma (2.1)]).

PROPOSITION 3.2 (see [8, Proposition (2.1)]). *Let G, H_i, λ_i, Y_i and J etc. be as above. Let $g_j \in D_j^{-1}$ be a fixed representative of each double coset D_j^{-1} $(j \in J)$. Let*

$$H_2 = \bigcup_s H_1^{(g_j^{-1})} h_s \qquad (disjoint) .$$

We always assume that one of $\{h_s\}$, that is, h_{s_*} is 1. Then

(i) *Since $H_2 g_j H_1 = \cup_s h_s^{-1} g_j H_1$ (disjoint) and $G \supset \cup_j H_2 g_j H_1$ (disjoint), we can assume $\{y_n^{-1}\} \supset \{h_s^{-1} g_j\}$ and define an element $\sum_s \lambda_2(h_s) h_s^{-1} g_j * \bar{\lambda}_1$ of $(\lambda_1)_{H_1}^G$, as in Proposition (3.1), to be $(\sum_s \lambda_2(h_s) h_s^{-1} g_j * \bar{\lambda}_1)(h g_j^{-1} h_{s'}) = \lambda_2(h_{s'}) \lambda_1(h)$ where $h \in H_1$ and $(\sum_s \lambda_2(h_s) h_s^{-1} g_j * \bar{\lambda}_1)(x) = 0$ if $x \notin D_j$.*

(ii) *Let $A_j(\bar{\lambda}_2) = \sum_s \lambda_2(h_s) h_s^{-1} g_j * \bar{\lambda}_1$, then*

$$\begin{array}{ccc} A_j : Y_2 & \longrightarrow & (\lambda_1)_{H_1}^G \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ x_m^{-1} * \bar{\lambda}_2 & \longmapsto & x_m^{-1} A_j(\bar{\lambda}_2) \end{array}$$

is a well defined kG-homomorphism for each $j \in J$.

(iii) *$\{A_j\}_{j \in J}$ are linearly independent in* $\mathrm{Hom}_{kG}(Y_2, (\lambda_1)_{H_1}^G)$.

PROOF. (i) From Proposition 3.1 (iv), certainly $\sum_s \lambda_2(h_s)h_s^{-1}g_j * \bar{\lambda}_1$ belongs to $(\lambda_1)_{H_1}^G$.

(ii) Since $\{x_m^{-1} * \bar{\lambda}_2\}$ is a k-basis of Y_2, A_j is a well-defined k-linear map. Let $g \in G$ and assume

$$g x_m^{-1} * \bar{\lambda}_2 = x_{m_0}^{-1} h * \bar{\lambda}_2 = \lambda_2(h) x_{m_0}^{-1} * \bar{\lambda}_2$$

where $g x_m^{-1} = x_{m_0}^{-1} h$ for some $m_0 \in \mathscr{M}$ and $h \in H_2$. Then we have

$$A_j(g x_m^{-1} * \bar{\lambda}_2) = \lambda_2(h) x_{m_0}^{-1} A_j(\bar{\lambda}_2) .$$

We shall show that $g A_j(x_m^{-1} * \bar{\lambda}_2) = A_j(g x_m^{-1} * \bar{\lambda}_2)$. Since $h^{-1}H_2 = H_2 = \bigcup_s h^{-1} h_s^{-1} H_1^{(g_j^{-1})}$ (disjoint), there exists $r_s \in H_1^{(g_j^{-1})}$ such that $h^{-1}h_s^{-1} = h_{s'}^{-1} r_s$ for each s. Notice $g A_j(x_m^{-1} * \bar{\lambda}_2) = g x_m^{-1} A_j(\bar{\lambda}_2) = x_{m_0}^{-1} h A_j(\bar{\lambda}_2)$. Let $x \notin D_j$, then $xh \notin D_j$ and we have $\lambda_2(h) A_j(\bar{\lambda}_2)(x) = 0 = h A_j(\bar{\lambda}_2)(x)$. Assume $h_0 g_j^{-1} h_s \in H_1 g_j^{-1} H_2 = D_j = \bigcup_s H_1 g_j^{-1} h_s$ where $h_0 \in H_1$, then we have

$$\lambda_2(h) A_j(\bar{\lambda}_2)(h_0 g_j^{-1} h_s) = \lambda_2(h)\lambda_2(h_s)\lambda_1(h_0)$$

and

$$\begin{aligned} h A_j(\bar{\lambda}_2)(h_0 g_j^{-1} h_s) &= A_j(\bar{\lambda}_2)(h_0 g_j^{-1} h_s h) = A_j(\bar{\lambda}_2)(h_0 g_j^{-1} r_s^{-1} h_{s'}) \\ &= A_j(\bar{\lambda}_2)(h_0 g_j^{-1} r_s^{-1} g_j g_j^{-1} h_{s'}) = \lambda_2(h_{s'})\lambda_1(h_0 g_j^{-1} r_s^{-1} g_j) \\ &= \lambda_2(h_{s'})\lambda_1(h_0)\lambda_1^{g_j^{-1}}(r_s^{-1}) = \lambda_2(h_{s'})\lambda_1(h_0)\lambda_2(r_s^{-1}) \\ &= \lambda_2(r_s^{-1} h_{s'})\lambda_1(h_0) = \lambda_2(h_s h)\lambda_1(h_0) . \end{aligned}$$

Hence $h A_j(\bar{\lambda}_2) = \lambda_2(h) A_j(\bar{\lambda}_2)$ for any $h \in H_2$. Thus $A_j(g x_m^{-1} * \bar{\lambda}_2) = \lambda_2(h) x_{m_0}^{-1} A_j(\bar{\lambda}_2) = x_{m_0}^{-1} h A_j(\bar{\lambda}_2) = g x_m^{-1} A_j(\bar{\lambda}_2) = g A_j(x_m^{-1} * \bar{\lambda}_2)$ for all $g \in G$ and $m \in \mathscr{M}$.

(iii) Suppose $\sum_{j \in J} t_j A_j = 0$, where $t_j \in k$ and almost all t_j's are zero, then $\sum_{j \in J} t_j A_j(\bar{\lambda}_2)(h g_{j_0}^{-1} h_s) = t_{j_0} \lambda_2(h_s)\lambda_1(h) = 0$ for any $j_0 \in J$ and $h g_{j_0}^{-1} h_s \in D_{j_0}$. Hence $t_j = 0$ for all $j \in J$ and $\{A_j\}_{j \in J}$ are linearly independent. Q.E.D.

Since $A_j(\bar{\lambda}_2) = \sum_s \lambda_2(h_s) h_s^{-1} g_j * \bar{\lambda}_1 \in (\lambda_1)_H^G$ and $A_j(\bar{\lambda}_2)$ vanishes outside of the coset D_j, for any scalar $c_j \in k$ we can define an element $(\sum_{j \in J} c_j A_j)(\bar{\lambda}_2)$ of $(\lambda_1)_{H_1}^G$ to be

$$\Big(\sum_{j \in J} c_j A_j\Big)(\bar{\lambda}_2) = \sum_{j \in J} \Big(\sum_s c_j \lambda_2(h_s) h_s^{-1} g_j * \bar{\lambda}_1\Big) .$$

Since $(\lambda_1)_{H_1}^G$ is a kG-module, we can define

$$\Big(\sum_{j \in J} c_j A_j\Big)(x_m^{-1} * \bar{\lambda}_2) \quad \text{to be} \quad x_m^{-1}\Big(\sum_{j \in J} c_j A_j\Big)(\bar{\lambda}_2) \in (\lambda_1)_{H_1}^G$$

for each x_m^{-1}. Thus we have

$$\Big(\sum_{j \in J} c_j A_j\Big)(x_m^{-1} * \bar{\lambda}_2) = x_m^{-1}\Big(\sum_{j \in J} c_j A_j\Big)(\bar{\lambda}_2) = x_m^{-1} \sum_{j \in J} \Big(\sum_s c_j \lambda_2(h_s) h_s^{-1} g_j * \bar{\lambda}_1\Big) .$$

We also define $(\sum_{j\in J} c_jA_j)(\sum_{m\in\mathscr{M}} t_m x_m^{-1}*\bar{\lambda}_2)$ to be $\sum_{m\in\mathscr{M}} t_m(\sum_{j\in J} c_jA_j)(x_m^{-1}*\bar{\lambda}_2)$ where almost all t_m's $\in k$ are zero.

THEOREM 3.3 (see [8, Theorem (2.2)]). *Let G, H_i, λ_i, Y_i and A_j $(j\in J)$ etc. be as before.*

(i) *Let $\mathscr{J}=\{j\in J \mid |D_j^{-1}/H_1|<\infty\}$ and $E=\mathrm{Hom}_{kG}(Y_2, Y_1)$, then E is a k-subspace of $\mathrm{Hom}_{kG}(Y_2, (\lambda_1)_{H_1}^G)$ and $\{A_j \mid j\in\mathscr{J}\}$ forms a k-basis of E.*

(ii) *For any scalars $\{c_j\in k \mid j\in J\}$, $\sum_{j\in J} c_jA_j$ is a well-defined kG-homomorphism of Y_2 into $(\lambda_1)_{H_1}^G$.*

(iii) *Let f be an arbitrary element of $\mathrm{Hom}_{kG}(Y_2, (\lambda_1)_{H_1}^G)$, then there exists a unique scalar $c_j\in k$ for each $j\in J$ such that*

$$f=\sum_{j\in J} c_jA_j\ .$$

PROOF. (i) Since $Y_1\subset(\lambda_1)_{H_1}^G$, E is a k-subspace of $\mathrm{Hom}_{kG}(Y_2, (\lambda_1)_{H_1}^G)$. From [7, Theorem (1.3)] it is clear that $\{A_j \mid j\in\mathscr{J}\}$ forms a k-basis of E.

(ii) Since $\{x_m^{-1}*\bar{\lambda}_2\}$ is a k-basis of Y_2, $\sum_{j\in J} c_jA_j$ is a well-defined k-linear map of Y_2 into $(\lambda_1)_{H_1}^G$. Let $g\in G$ and $m\in\mathscr{M}$, and assume $gx_m^{-1}=x_{m_0}^{-1}h$ for some $m_0\in\mathscr{M}$ and $h\in H_2$. Then

$$(\sum_{j\in J} c_jA_j)(gx_m^{-1}*\bar{\lambda}_2)=(\sum_{j\in J} c_jA_j)(x_{m_0}^{-1}h*\bar{\lambda}_2)=\lambda_2(h)x_{m_0}^{-1}(\sum_{j\in J} c_jA_j)(\bar{\lambda}_2)\ .$$

Since $g(\sum_{j\in J} c_jA_j)(x_m^{-1}*\bar{\lambda}_2)=gx_m^{-1}(\sum_{j\in J} c_jA_j)(\bar{\lambda}_2)=x_{m_0}^{-1}h(\sum_{j\in J} c_jA_j)(\bar{\lambda}_2)$, we only have to show that

$$h(\sum_{j\in J} c_jA_j)(\bar{\lambda}_2)=\lambda_2(h)(\sum_{j\in J} c_jA_j)(\bar{\lambda}_2)\ .$$

Let $x\notin D_j$ for any $j\in J$, then since $xh\notin D_j$ for any $j\in J$, we have

$$\begin{aligned} h\{(\sum_{j\in J} c_jA_j)(\bar{\lambda}_2)\}(x) &= \{\sum_{j\in J}(\sum_{s} c_j\lambda_2(h_s)h_s^{-1}g_j*\bar{\lambda}_2)\}(xh)=0 \\ &= \lambda_2(h)\{(\sum_{j\in J} c_jA_j)(\bar{\lambda}_2)\}(x)\ . \end{aligned}$$

Assume $x\in D_j$ for some $j\in J$, then since xh belongs to $D_j=H_1g_j^{-1}H_2$, we have

$$\begin{aligned} h\{(\sum_{j\in J} c_jA_j)(\bar{\lambda}_2)\}(x) &= c_jA_j(\bar{\lambda}_2)(xh)=c_j\{hA_j(\bar{\lambda}_2)\}(x) \\ &= c_jA_j(h*\bar{\lambda}_2)(x)=c_j\lambda_2(h)A_j(\bar{\lambda}_2)(x)=\lambda_2(h)(\sum_{j\in J} c_jA_j)(\bar{\lambda}_2)(x)\ . \end{aligned}$$

Hence $\sum_{j\in J} c_jA_j$ is a well-defined kG-homomorphism.

(iii) Let take a fixed representative g_i from each (H_2, H_1)-double coset D_i^{-1} $(i\in I)$ such that $\{g_i\}_{i\in I}\supset\{g_j\}_{j\in J}$. Let $H_2=\cup_q H_1^{(g_i^{-1})}r_q$ (disjoint) for each $i\in I$, then we have $H_2g_iH_1=\cup_q r_q^{-1}g_iH_1$ (disjoint) and we can take a k-basis of Y_1 to be $\cup_{i\in I}\{r_q^{-1}g_i*\bar{\lambda}_1 \mid H_2=\cup_q H_1^{(g_i^{-1})}r_q \text{ (disjoint)}\}$. We always assume that one of $\{r_q\}$, that is, r_{q*} is 1.

Let f be an arbitrary element of $\mathrm{Hom}_{kG}(Y_2, (\lambda_1)_{H_1}^G)$, then we have $f(\bar{\lambda}_2)=\sum_{q,i} c_{q,i}r_q^{-1}g_i*\bar{\lambda}_1$, where $c_{q,i}\in k$ and almost all $\{c_{q,i}\}$ are not necessarily zero. Since $h*f(\bar{\lambda}_2)=f(h*\bar{\lambda}_2)=\lambda_2(h)f(\bar{\lambda}_2)$ for any $h\in H_2$, we have

$$h\sum_{q,i} c_{q,i}r_q^{-1}g_i*\bar{\lambda}_1=\lambda_2(h)\sum_{q,i} c_{q,i}r_q^{-1}g_i*\bar{\lambda}_1=\sum_{q,i} c_{q,i}\lambda_2(h)r_q^{-1}g_i*\bar{\lambda}_1 .$$

Hence $(\sum_q c_{q,i}r_q^{-1}g_i*\bar{\lambda}_1)(xh)=(\sum_q c_{q,i}\lambda_2(h)r_q^{-1}g_i*\bar{\lambda}_1)(x)$ for any $x\in D_i$, because $H_1g_i^{-1}H_2h=H_1g_i^{-1}H_2=\cup_q H_1g_i^{-1}r_q$. Let $r_{q_0}\in\{r_q\}$, then we have

$$(\sum_q c_{q,i}r_q^{-1}g_i*\bar{\lambda}_1)(g_i^{-1}r_{q_0})=c_{q_0,i}=(\sum_q c_{q,i}\lambda_2(r_{q_0})r_q^{-1}g_i*\bar{\lambda}_1)(g_i^{-1})=c_{q*,i}\lambda_2(r_{q_0}) .$$

Thus we have $c_{q_0,i}=\lambda_2(r_{q_0})c_{q*,i}$ for any $r_{q_0}\in\{r_q\}$. Hence

$$\begin{aligned} f(\bar{\lambda}_2)&=\sum_i\sum_q c_{q*,i}\lambda_2(r_q)r_q^{-1}g_i*\bar{\lambda}_1 \\ &=\sum_i c_{q*,i}(\sum_q \lambda_2(r_q)r_q^{-1}g_i*\bar{\lambda}_1) . \end{aligned}$$

Let $h\in H_1^{(g_i^{-1})}$, then

$$\begin{aligned} c_{q*,i}(\sum_q \lambda_2(r_q)r_q^{-1}g_i*\bar{\lambda}_1)(g_i^{-1}h)&=c_{q*,i}\lambda_2(h) \\ =c_{q*,i}(\sum_q \lambda_2(r_q)r_q^{-1}g_i*\bar{\lambda}_1)(h^{g_i}g_i^{-1})&=c_{q*,i}\lambda_1(h^{g_i}) . \end{aligned}$$

Hence $c_{q*,i}\neq 0$ only when $\lambda_1^{g_i^{-1}}=\lambda_2$ on $H_1^{(g_i^{-1})}$. Thus we have

$$f(\bar{\lambda}_2)=\sum_{j\in J} c_{q*,j}(\sum_q \lambda_2(r_q)r_q^{-1}g_j*\bar{\lambda}_1)=(\sum_{j\in J} c_{q*,j}A_j)(\bar{\lambda}_2) .$$

Since $\sum_{j\in J} c_{q*,j}A_j$ is a kG-homomorphism from (ii), we have $f=\sum_{j\in J} c_{q*,j}A_j$. It is clear that the scalars $\{c_{q*,j}\mid j\in J\}$ are uniquely determined by f, because $f(\bar{\lambda}_2)(g_j^{-1})=c_{q*,j}$ for each $j\in J$. Q.E.D.

REMARK. Let $(G, \mathscr{S}_G)$ be an algebraic group over K and $(H, \mathscr{S}_H)$ be a closed subgroup of G. Let $\lambda\colon H\to K^\times$ be a homomorphism of algebraic groups of H into $(K^\times, K[X, X^{-1}])$, i.e., one-dimensional rational representation of H over K. Clearly $\lambda\colon H\to K^\times$ is a one-dimensional rational representation of H if and only if $\lambda\colon H\to K^\times$ is a group homomorphism and $\lambda\in\mathscr{S}_H(H)$. Since a map $f\colon G\to K$ is a morphism of varieties if and only if $f\in\mathscr{S}_G(G)$, we have

$$\mathrm{ind}_H^G\lambda=\lambda_H^G\cap\mathscr{S}_G(G) \qquad \text{(see Proposition 1.6)} .$$

However $KG*\bar{\lambda}=\lambda_H^G\cap KG$ does not hold in general, where KG is embedded into $K_{\{1\}}^G=\{f: G\to K \mid f$ is a mapping$\}$ as follows:

$$\begin{array}{ccc} KG & \hookrightarrow & K_{\{1\}}^G \\ \cup\!\!\!| & & \cup\!\!\!| \\ g^{-1} & \longmapsto & g^* \end{array}$$

(where g^* is a mapping of G into K which takes g to 1 and g' to 0 if $g\neq g'$), because $\bar{\lambda}$ does not belong to KG if H is infinite.

§4. An application to Chevalley groups.

In this section we follow the same notation as in [10, §3], that is, G is a Chevalley group defined over an algebraically closed field K with respect to a given finite dimensional semisimple Lie algebra $\mathscr{L}$ over the complex number field and a finite dimensional faithful module of $\mathscr{L}$. Let Σ be the root system of $\mathscr{L}$, then G has a set of generators $\{x_\alpha(t) \mid \alpha\in\Sigma$ and $t\in K\}$ and the following subgroups.

$U=\langle x_\alpha(t) \mid \alpha>0$ and $t\in K\rangle$.
$H=\langle h_\alpha(t) \mid \alpha\in\Sigma$ and $t\in K^\times\rangle$,
where $h_\alpha(t)=w_\alpha(t)w_\alpha(1)^{-1}$ and $w_\alpha(t)=x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)$.
$B=UH$.
$N=\langle w_\alpha(t) \mid \alpha\in\Sigma$ and $t\in K^\times\rangle$.

Let W be the Weyl group of Σ, then there exists an isomorphism φ of W onto N/H which takes each reflection $w_\alpha\in W$ of $\alpha\in\Sigma$ to $w_\alpha(1)H$. G has a Bruhat decomposition

$$G=\bigcup_{w\in W} BwB \qquad (BwB=Bw'B \Rightarrow w=w')$$

and every element of BwB is uniquely expressed as a product of an element from B, a fixed representative ω_w of w in N and an element from U_w, where

$$U_w=\langle x_\alpha(t) \mid \alpha\in P\cap w^{-1}(-P) \text{ and } t\in K\rangle$$

and P is the set of positive roots of Σ (see [10, §3, Theorem 4 and 4']). Hence G has a (U, U)-double coset decomposition

$$G=\bigcup_{n\in N} UnU \quad (UnU=Un'U \Rightarrow n=n')$$

and $Uh\omega_w U=Uh\omega_w U_w$ for any $h\in H$ and $w\in W$.

Now let $\lambda: B\to K^\times$ be a linear character of B into $K^\times$, then $\mathrm{Ker}\,\lambda\supset U$, because U is the commutator subgroup of B. Hence we can identify the set of all linear characters of B into $K^\times$ with the set of all linear characters of H into $K^\times$, because $B \rhd U$ and $B/U \cong H$.

Let $\bar{\lambda}$ be the extension of λ as in § 3 and $Y_\lambda = KG * \bar{\lambda}$, then

PROPOSITION 4.1 (see [7, Theorem (2.1)]). *Let G be a Chevalley group over an algebraically closed field K, and U, H and B etc. be as before. Let $\lambda: B \to K^\times$ be a linear character of B into $K^\times$. Then*

(i) *the module Y_λ is indecomposable; and*

(ii) *for any pair of linear characters λ and λ' of B into $K^\times$, it holds $\lambda = \lambda'$ if and only if there exists a non-trivial KG-homomorphism of Y_λ into $Y_{\lambda'}$;*

(iii) $\dim_K \mathrm{End}_{KG}(Y_\lambda) = 1$.

Since $B^{h\omega_w} = B^{\omega_w}$ and $\lambda^{h\omega_w} = \lambda^{\omega_w}$ for any $h \in H$ (for the definition of λ^{ω_w}, see § 3), we shall write λ^w for λ^{ω_w}, where $w \in W$. Similarly let X be the set of all linear character of H into $K^\times$, then W operates on X as follows:

$$\begin{array}{ccc} W \times X & \longrightarrow & X \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ (w, \lambda) & \longmapsto & \lambda^w \end{array} \qquad \text{where} \quad \begin{array}{ccc} \lambda^w: H & \longrightarrow & K^\times . \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ h & \longmapsto & \lambda(whw^{-1}) \end{array}$$

We define W_λ to be the isotropy group $\{w \in W \mid \lambda^w = \lambda \text{ on } H\}$ of λ in W ($\lambda \in X$). It can be easily verified that

$$W_{(\lambda|H)} = \{w \in W \mid \lambda^w = \lambda \text{ on } B^{\omega_w} \cap B\}$$

for any linear character λ of B into $K^\times$. Hence we also write W_λ for $W_{(\lambda|H)}$ when λ is a linear character of B into $K^\times$.

PROPOSITION 4.2 (see Proposition 3.2 and Theorem 3.3). *Let G be a Chevalley group over an algebraically closed field K, and U, H and B etc. be as before. Let $\lambda_i: B \to K^\times$ be a linear character of B into $K^\times$ where $i = 1, 2$ and*

$$\mathscr{W} = \{w \in W \mid \lambda_1^w = \lambda_2\} .$$

(i) *Since $B\omega_w^{-1}B = U_w\omega_w^{-1}B$ and $B\omega_w^{-1}B = \cup_{u \in U_w} u\omega_w^{-1}B$ (disjoint), we can define an element $\sum_{u \in U_w} u\omega_w^{-1} * \bar{\lambda}_1$ of $(\lambda_1)_B^G$ as in Proposition 3.2, where $w \in \mathscr{W}$, to be*

$$\Big(\sum_{u \in U_w} u\omega_w^{-1} * \bar{\lambda}_1\Big)(b\omega_w u'^{-1}) = \lambda_1(b) \qquad \textit{where } u' \in U_w \textit{ and } b \in B$$

and

$$(\sum_{u\in U_w} u\omega_w^{-1}*\overline{\lambda_1})(x)=0 \qquad \text{if } x\notin B\omega_w B\,.$$

(ii) *Let* $a_w(\overline{\lambda_2})=\sum_{u\in U_w} u\omega_w^{-1}*\overline{\lambda_1}$, *then*

$$\begin{array}{rcl} a_w\colon\ Y_{\lambda_2} & \longrightarrow & (\lambda_1)_B^G \\ \cup\!\!\!\!\!\shortmid & & \cup\!\!\!\!\!\shortmid \\ x_m^{-1}*\overline{\lambda_2} & \longmapsto & x_m^{-1}a_w(\overline{\lambda_2}) \end{array}$$

is a well-defined KG-homomorphism for each $w\in\mathscr{W}^{\sim}$, *where* $G=\bigcup_{m\in\mathscr{M}} Bx_m$ *(disjoint union) and* $\{x_m\}=\bigcup_{w\in W}\omega_w U_w$.

(iii) $\{a_w\mid w\in\mathscr{W}^{\sim}\}$ *forms a K-basis of* $\mathrm{Hom}_{KG}(Y_{\lambda_2}, (\lambda_1)_B^G)$.

PROPOSITION 4.3 (see Proposition 3.2 and Theorem 3.3). *Let G be a Chevalley group over an algebraically closed field K, and U, H and B etc. be as before. Let* $1_U\colon U\to K^\times$ *be the trivial linear character of U into* $K^\times$, *i.e.,* $1_U(u)=1$ *for any* $u\in U$.

(i) *Since* $U\omega_w^{-1}h^{-1}U=U_w\omega_w^{-1}h^{-1}U$ *for any* $h\in H$ *and* $w\in W$ *and*

$$U\omega_w^{-1}h^{-1}U=\bigcup_{u\in U_w} u\omega_w^{-1}h^{-1}U \qquad (\textit{disjoint})\,,$$

we can define an element $\sum_{u\in U_w} u\omega_w^{-1}h^{-1}*\overline{1_U}$ *of* 1_U^G *as in Proposition* 3.2 *to be*

$$(\sum_{u\in U_w} u\omega_w^{-1}h^{-1}*\overline{1_U})(u_0h\omega_w u'^{-1})=1 \qquad \text{where } u'\in U_w \text{ and } u_0\in U$$

and

$$(\sum_{u\in U_w} u\omega_w^{-1}h^{-1}*\overline{1_U})(x)=0 \qquad \text{if } x\notin Uh\omega_w U\,.$$

(ii) *Let* $A_{h\omega_w}(\overline{1_U})=\sum_{u\in U_w} u\omega_w^{-1}h^{-1}*\overline{1_U}$, *then*

$$\begin{array}{rcl} A_{h\omega_w}\colon\ KG*\overline{1_U} & \longrightarrow & 1_U^G \\ \cup\!\!\!\!\!\shortmid & & \cup\!\!\!\!\!\shortmid \\ x_\mu^{-1}*\overline{1_U} & \longmapsto & x_\mu^{-1}A_{h\omega_w}(\overline{1_U}) \end{array}$$

is a well-defined KG-homomorphism for each $n=h\omega_w\in N$, *where* $G=\bigcup_{\mu\in L} Ux_\mu$ *(disjoint) and* $\{x_\mu\}=\bigcup_{h\in H, w\in W} h\omega_w U_w$.

(iii) *Let* φ *be an arbitrary element of* $\mathrm{Hom}_{KG}(KG*\overline{1_U}, 1_U^G)$, *then there exists a unique scalar* $c_{h\omega_w}\in K$ *for each* $n=hw_w\in N$ *such that*

$$\varphi=\sum_{h\in H, w\in W} c_{h\omega_w}A_{h\omega_w}\,.$$

Now we shall review the representation theory of G over K.

DEFINITION 4.4. Let V be a locally finite rational KG-module. Then we call $v \in V$ a weight vector of the module if there exists a linear character λ of H over K such that

$$hv = \lambda(h)v \qquad \text{for all } h \in H .$$

When $v \neq 0$, we call λ a weight of the module. It is clear that the weight λ belongs to $K[H]$, the coordinate ring of H.

THEOREM 4.5 (see [10, §12, Theorem 40]). *Let λ be a rational linear character of B into $K^\times$. Then if $\mathrm{ind}_B^G \lambda \neq 0$ (see Definition 1.7), $\mathrm{ind}_B^G \lambda$ contains a unique B-stable line Λ. The weight of Λ is λ^{w_0}, where w_0 is the element of maximal length of the Weyl group W of G.*

PROOF. From the Lie-Kolchin theorem it is clear that $\mathrm{ind}_B^G \lambda$ contains a B-stable line, because $\mathrm{ind}_B^G \lambda$ is locally finite rational and B is connected and solvable.

Since G is connected and $U^{w_0}HU$ is open in G (see [10, §5, proof of Theorem 6; Theorem 7]), $Bw_0U = w_0U^{w_0}HU$ is open and dense in G. Let f be an element of

$$\mathrm{ind}_B^G \lambda = \{f \in \mathscr{S}_G(G) \mid f(bg) = \lambda(b)f(g) \text{ for any } b \in B \text{ and } g \in G\} ,$$

then we have $f(bw_0u) = \lambda(b)f(w_0u)$ for any $bw_0u \in Bw_0U$. Notice $U_{w_0} = U$. Let Λ be a B-stable line in $\mathrm{ind}_B^G \lambda$. Assume f be a non-zero element of Λ, then we have

$$f(bw_0u) = \lambda(b)f(w_0u) = \lambda(b)f(w_0)$$

for any $bw_0u \in Bw_0U$, because U is the commutator subgroup of B. Suppose that Λ' is another B-stable line in $\mathrm{ind}_B^G \lambda$ generated by f', then

$$f'(w_0)^{-1}f'(bw_0u) = \lambda(b) = f(w_0)^{-1}f(bw_0u) .$$

Hence $f' = f'(w_0)f(w_0)^{-1}f$ on the dense open subset Bw_0U, which shows $f' = f'(w_0)f(w_0)^{-1}f$ on G. Thus we have shown that $\Lambda = \Lambda'$.

Let $f \in \Lambda$, then $h * f(bw_0u) = f(bw_0uh) = f(bw_0hw_0^{-1}w_0u^h) = \lambda(b)\lambda(w_0hw_0^{-1})f(w_0)$ for any $bw_0u \in Bw_0U$. Since $h * f = \lambda^{w_0}(h)f$ on Bw_0U, the weight of f is λ^{w_0}. Q.E.D.

COROLLARY 4.6 (cf. [6] and Proposition 4.1). *Let λ be a rational linear character of B into $K^\times$ and $\mathrm{soc}(\mathrm{ind}_B^G \lambda)$ be the socle of $\mathrm{ind}_B^G \lambda$, that is, the sum of all irreducible KG-submodules of $\mathrm{ind}_B^G \lambda$. Assume that $\mathrm{ind}_B^G \lambda \neq 0$, then*

(i) *$\mathrm{soc}(\mathrm{ind}_B^G \lambda)$ is irreducible;*

(ii) *$\mathrm{ind}_B^G \lambda$ is an indecomposable KG-module;*

(iii) *for any rational linear character* λ' *of* B *into* $K^{\times}$ *the following three statements are equivalent*

a) $\mathrm{ind}_B^G \lambda \cong \mathrm{ind}_B^G \lambda'$,

b) $\mathrm{soc}(\mathrm{ind}_B^G \lambda) \cong \mathrm{soc}(\mathrm{ind}_B^G \lambda')$,

c) $\lambda = \lambda'$;

(iv) *for any finite dimensional irreducible rational* KG*-module* S, *there exists a unique rational linear character* χ *of* B *into* $K^{\times}$ *such that* $S \cong \mathrm{soc}(\mathrm{ind}_B^G \chi)$.

PROOF. (i) Since $\mathrm{ind}_B^G \lambda$ is locally finite and rational (see Proposition 1.6 and Definition 1.7), it contains a non-trivial rational irreducible KG-submodule and any irreducible KG-submodule is of finite dimension and rational. Let M be an irreducible submodule of $\mathrm{ind}_B^G \lambda$. Then M cotains a B-stable line from the Lie-Kolchin theorem. Hence M contains the unique B-stable line Λ. Thus $M = \mathrm{soc}(\mathrm{ind}_B^G \lambda)$.

(ii) is clear from (i).

(iii) Assume that $\mathrm{ind}_B^G \lambda \cong \mathrm{ind}_B^G \lambda'$, then the unique B-stable lines of $\mathrm{ind}_B^G \lambda$ and $\mathrm{ind}_B^G \lambda'$ have the same weight λ^{w_0}. Hence $\lambda = \lambda'$. Similarly $\mathrm{soc}(\mathrm{ind}_B^G \lambda) \cong \mathrm{soc}(\mathrm{ind}_B^G \lambda')$ implies $\lambda = \lambda'$.

(iv) From the Lie-Kolchin theorem, S contains a B-stable line Λ. Since $\dim_K S/\Lambda = \dim_K S - 1$, from the induction S contains a B-submodule T of dimension $\dim_K S - 1$. Let χ be the weight of S/T. Since

$$\mathrm{Hom}_{KB}(S, S/T) \cong \mathrm{Hom}_{KG}(S, \mathrm{ind}_B^G \chi)$$

as K-spaces from the Frobenius Reciprocity, there exists a non-trivial KG-homomorphism of S into $\mathrm{ind}_B^G \chi$. Hence $S \cong \mathrm{soc}(\mathrm{ind}_B^G \chi)$. Q.E.D.

THEOREM 4.7. *Let* G *be a Chevalley group over an algebraically closed field* K *and* U, H *and* B *etc. be as before. Let* λ *be a rational linear character of* B *into* $K^{\times}$ *such that* $\mathrm{ind}_B^G \lambda \neq 0$ *and* f *be a non-zero weight vector in* $\mathrm{ind}_B^G \lambda$ *of weight* λ^{w_0} *where* w_0 *is the element of maximal length of the Weyl group* W *of* G. *Then*

(i) $\{w \in W \mid \lambda^{w} = \lambda^{w_0}\} = W_{\lambda} w_0$;

(ii) *for any* $w \in W$, $f \mid BwB \neq 0$ *if and only if* $f(\omega_w) \neq 0$;

(iii) $w \in W_{\lambda} w_0$ *if* $f(\omega_w) \neq 0$;

(iv) $f = (\sum_{w \in W_{\lambda} w_0} f(\omega_w) a_w)(\overline{\lambda^{w_0}})$ *where* $\{a_w \mid w \in W_{\lambda} w_0\}$ *is the* K*-basis of* $\mathrm{Hom}_{KG}(KG * \overline{\lambda^{w_0}}, \lambda_B^G)$ *given as in Proposition* 4.2.

PROOF. (i) is clear from the fact that $w_0^2 = 1$.

(ii) Let $b\omega_w u \in B\omega_w U_w$, then $f(b\omega_w u) = \lambda(b) f(\omega_w)$. Hence $f \mid BwB \neq 0$ if and only if $f(\omega_w) \neq 0$.

(iii) Assume that $f(\omega_w) \neq 0$. Since $f(\omega_w h) = \lambda^{w_0}(h) f(\omega_w) = f(\omega_w h \omega_w^{-1} \omega_w) = \lambda^w(h) f(\omega_w)$ for any $h \in H$, we have $\lambda^w = \lambda^{w_0}$, i.e., $w \in W_\lambda w_0$.

(iv) Since $(\sum_{w \in W_\lambda w_0} f(\omega_w) a_w)(\overline{\lambda^{w_0}})(b\omega_w u'^{-1}) = f(\omega_w)\lambda(b)$ for any $b\omega_w u'^{-1} \in B\omega_w U_w$ where $w \in W_\lambda w_0$ and $(\sum_{w \in W_\lambda w_0} f(\omega_w) a_w)(\overline{\lambda^{w_0}}) | Bw'B = 0$ if $w' \in W_\lambda w_0$, we have $f = \sum_{w \in W_\lambda w_0} f(\omega_w) a_w (\overline{\lambda^{w_0}})$. Q.E.D.

REMARKS 4.8. Let G, λ and f etc. be as in Theorem 4.7, then

(i) $KG * \bar{\lambda}$ contains no finite dimensional rational KG-submodule;

(ii) $\lambda_B^G \supset KG * \bar{\lambda} \oplus \mathrm{ind}_B^G \lambda$.

PROOF. (i) Assume that $KG * \bar{\lambda}$ contains a finite dimensional rational irreducible KG-submodule V and $V \cong \mathrm{soc}(\mathrm{ind}_B^G \chi)$ for some rational linear character χ of B into $K^\times$. Since

$$\mathrm{Hom}_{KG}(KG * \overline{\chi^{w_0}}, KG * \bar{\lambda}) = \begin{cases} 0 & \text{if } \chi^{w_0} \neq \lambda \\ K & \text{if } \chi^{w_0} = \lambda \end{cases}$$

from Proposition 4.1, the existence of a non-trivial homomorphism of $KG * \overline{\chi^{w_0}}$ onto V ($\subset KG * \bar{\lambda}$) is a contradiction.

(ii) is clear from (i). Q.E.D.

THEOREM 4.9 (cf. [6, Proposition (3.1)]). *Let G be a Chevalley group over an algebraically closed field K and U, H and B etc. be as before. Let $1_U: U \to K^\times$ be the trivial non-zero linear character of U into $K^\times$ and λ be a rational linear character of B into $K^\times$ such that $\mathrm{ind}_B^G \lambda \neq 0$ and f be a non-zero weight vector in $\mathrm{ind}_B^G \lambda$ of weight λ^{w_0} where w_0 is the element of the maximal length of the Weyl group W of G. Then*

(i) *$\mathrm{ind}_B^G \lambda$ is a KG-submodule of 1_U^G and*

$$1_U^G \supset \mathrm{ind}_B^G \lambda \supset \mathrm{soc}(\mathrm{ind}_B^G \lambda) \ni f;$$

(ii)
$$f = \Big(\sum_{w \in W_\lambda w_0, h \in H} \lambda(h) f(\omega_w) A_{h\omega_w}\Big)(\overline{1_U})$$

where $\{A_{h\omega_w} | w \in W, h \in H\}$ ($\subset \mathrm{Hom}_{KG}(KG * \overline{1_U}, 1_U^G)$) is as in Proposition 4.3.

PROOF. (i) Since $\lambda | U = 1_U$ and

$1_U^G = \{f: G \to K \mid f(ug) = f(g) \text{ for any } u \in U \text{ and } g \in G\}$ and

$\mathrm{ind}_B^G \lambda = \{f \in K[G] \mid f(bg) = \lambda(b) f(g) \text{ for any } b \in B \text{ and } g \in G\}$

where $K[G]$ is the coordinate ring of G, 1_U^G contains $\mathrm{ind}_B^G \lambda$ as submodule.

(ii) Notice $f | BwB = 0$ if $w \notin W_\lambda w_0$. Since

$$\Big\{\Big(\sum_{w \in W_\lambda w_0, h \in H} \lambda(h) f(\omega_w) A_{h\omega_w}\Big)(\overline{1_U})\Big\}(u_0 h \omega_w u'^{-1})$$
$$= \lambda(h) f(\omega_w) = f(u_0 h \omega_w u'^{-1})$$

where $w \in W_\lambda w_0$ and $u_0 h\omega_w u'^{-1} \in Uh\omega_w U_w$ and

$$\{(\sum_{w \in W_\lambda w_0, h \in H} \lambda(h) f(\omega_w) A_{h\omega_w})(\overline{1_U})\}|BwB=0 \qquad \text{if } w \notin W_\lambda w_0$$

from Proposition 4.3, we have

$$f=(\sum_{w \in W_\lambda w_0, h \in H} \lambda(h) f(\omega_w) A_{h\omega_w})(\overline{1_U}) . \qquad \text{Q.E.D.}$$

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