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Bloch Constants and Bloch Minimal Surfaces

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§ 1. Introduction.

The n-th Bloch constant b_n ($n \geqq 2$) will be defined in terms of radii of certain disks on minimal surfaces in the Euclidean space $\boldsymbol{R}^n$. As will be seen, b_2 is the familiar one in the complex analysis. We shall prove that

$$(1.1) \qquad b_n \geqq n^{-1/2} b_2 , \qquad n \geqq 3 .$$

Let each component x_j of a nonconstant map $x=(x_1, \cdots, x_n)$ from the disk $D=\{|w|<1\}$ in the complex plane $|w|<\infty$, $w=u+iv$, into the Euclidean space $\boldsymbol{R}^n$ ($n \geqq 2$) be harmonic in D. Then, the set S of all pairs $(w, x(w))$, $w \in D$, or simply, the map x itself, is called a minimal surface if

$$(1.2) \qquad x_u x_v = 0 , \quad x_u x_u = x_v x_v \qquad \text{in} \quad D ,$$

where

$$x_u = (x_{1u}, \cdots, x_{nu}) , \qquad x_v = (x_{1v}, \cdots, x_{nv})$$

are partial derivatives and the products are inner; S is the one-to-one image of D by x.

Henceforward, $x: D \to \boldsymbol{R}^n$ always means a minimal surface, and somewhat informally, we regard S as a subset of $\boldsymbol{R}^n$.

The surface S is endowed with the metric

$$d(x(w_1), x(w_2)) = \inf_{\gamma} \int_{\gamma} |x_u(w)| \, |dw| ,$$

where $x(w_j) \in S$, $j=1, 2$, $|x_u| = (x_u x_u)^{1/2}$ and γ ranges over all (rectifiable) curves connecting w_1 and w_2 in D. One can also consider this a new metric in D other than the Euclidean metric. Obviously, $|x(w_1) - x(w_2)| \leqq d(x(w_1), x(w_2))$; the left-hand side is the Euclidean metric in $\boldsymbol{R}^n$.

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The open disk of center $x(w_0)$ and radius $r>0$ in S is

$$\Gamma_x(w_0, r)=\{x(w);\ d(x(w), x(w_0))<r\}\ .$$

We shall later observe that the closure of $\Gamma_x(w_0, r)$ is

$$\Gamma_x^*(w_0, r)=\{x(w);\ d(x(w), x(w_0))\leqq r\}\ ;$$

this is not necessarily compact. A point $x(w)$ is called regular or nonbranched if $x_u(w)\neq 0$, nonzero vector, and a subset $S_1\subset S$ is called regular if each point of S_1 is regular. If $\Gamma_x(w_0, r)$ is regular and further if $\Gamma_x^*(w_0, r)$ is compact, then we call $\Gamma_x(w_0, r)$ admissible. Let $b(x)$ be the supremum of $r>0$ such that there exists an admissible $\Gamma_x(w_0, r)$ for some point $x(w_0)$. Let b_n be the infimum of $b(x)$ for all $x: D\to \boldsymbol{R}^n$ subject to the "pinning" condition $|x_u(0)|=1$. We then call b_n the n-th Bloch constant. Since $x: D\to\boldsymbol{R}^n$ can be regarded in the obvious way as $x: D\to\boldsymbol{R}^{n+1}$, it follows that $b_{n+1}\leqq b_n$, $n\geqq 2$.

A minimal surface $x: D\to\boldsymbol{R}^2$ can be regarded as a nonconstant holomorphic or antiholomorphic function f in D and vice versa. For the holomorphic case, an admissible $\Gamma_x(w_0, r)$ is the one-sheeted whole disk $\{|w-f(w_0)|<r\}$ on the Riemannian image, an elementary but never trivial fact. Therefore, b_2 is just the Bloch constant [A2, p. 14] in the complex analysis.

Our first aim is to prove

Theorem 1. *The inequality* (1.1) *holds.*

It is familiar that [A1, p. 364], [AG, p. 672], [H1], [H2, p. 60],

$$0.433\cdots=\frac{\sqrt{3}}{4}<b_2\leqq\frac{\Gamma(1/3)\Gamma(11/12)}{(1+3^{1/2})^{1/2}\Gamma(1/4)}=0.471\cdots\ .$$

The determination of b_2 still remains an outstanding problem. Since $b_3>1/4=0.25$, we have an improvement of E. F. Beckenbach's [B, p. 456] earlier one: $b_3\geqq(16\sqrt{3})^{-1}=0.036\cdots$. His paper contains no definition of the Bloch constant b_3; the result is implicit.

A holomorphic function f in D is called Bloch if

$$\mu(f)\equiv\sup_{w\in D}(1-|w|^2)|f'(w)|$$

is finite. The notion arises from the principal idea of proving the Bloch theorem due to E. Landau [L, pp. 617–618]. The term "Bloch function" in the present meaning now prevails, ignoring R. M. Robinson's earlier paper [Rb].

We call $x: D \to \boldsymbol{R}^n$ Bloch if

$$\mu(x) \equiv \sup_{w \in D}(1-|w|^2)|x_u(w)|$$

is finite. Note that $\mu(f)=\mu(x)$ for nonconstant f. A typical example is a bounded minimal surface $x: D \to \boldsymbol{R}^n$, namely, $|x|$ is bounded in D. Another one is $x: D \to \boldsymbol{R}^n$ whose Gauss curvature $\kappa(w)$ is bounded: $\kappa(w) \leqq -A$ $(A>0)$, $w \in D$. Then, $\mu(x) \leqq 2A^{-1/2}$.

In Section 2 we prove Theorem 1. Section 2a is appended to Section 2, where the notion of n-th strong Landau constant is introduced. In Section 3 we propose some basic facts on Bloch minimal surfaces. In Section 4 we prove some results on Bloch minimal surfaces in connection with the present topics on disks on S. In Section 5 we prove that $x: D \to \boldsymbol{R}^n$ is Bloch if and only if x is of bounded mean oscillation in some sense; the result is analytic rather than geometric.

I wish to express my gratitude to my colleague Norio Ejiri for his criticism at an early stage of preparation.

§ 2. Proof of Theorem 1.

We begin with some basic properties of minimal surfaces; see [N], [O].

There exist holomorphic functions f_j such that $x_j = \operatorname{Re} f_j$, $1 \leqq j \leqq n$, in D, so that the formulae in (1.2) can be unified as

$$\sum_{j=1}^{n} (f_j')^2 \equiv 0 \; ; \tag{2.1}$$

however, the correspondence $x \to (f_1, \cdots, f_n)$ is not necessarily one-to-one. We note that

$$|x_u|^2 \equiv 2^{-1} \sum_{j=1}^{n} |f_j'|^2 \,. \tag{2.2}$$

We call $(f_1, \cdots, f_n)$ an admissible system for $x: D \to \boldsymbol{R}^n$.

LEMMA 2.1. *Let $(f_1, \cdots, f_n)$ be an admissible system for $x: D \to \boldsymbol{R}^n$. Then, for each curve γ in D,*

$$\left\{2^{-1} \sum_{j=1}^{n} \left(\int_\gamma |f_j'(w)||dw|\right)^2\right\}^{1/2} \leqq \int_\gamma |x_u(w)||dw| \,. \tag{2.3}$$

PROOF. The Minkowski inequality for the integrals with the power $p=2^{-1}$ [BB, Section 18, p. 20 ff.] reads

$$2\left(\int |x_u|\right)^2 = \left\{\int (\sum |f_j'|^2)^{1/2}\right\}^2 \geqq \sum \left(\int |f_j'|\right)^2 ,$$

which yields (2.3).

For $x: D\to \boldsymbol{R}^n$ and $w\in D$ we let $\Delta(w, x)$ be the supremum of $r>0$ such that an admissible $\Gamma_x(w, r)$ exists; if $x(w)$ is not regular, then we set $\Delta(w, x)=0$. Then, $b(x)=\sup \Delta(w, x)$, where w ranges over D. The quantity $\Delta(w, x)$ measures the distance from $x(w)$ either to the nearest branch point (possibly, $x(w)$ itself) or to the "boundary" of S. We denote $\Delta(w, f)=\Delta(w, x)$ for f holomorphic in D with $x=(\operatorname{Re} f, \operatorname{Im} f)$. Thus, $b(f)=b(x)$. If f is constant, then $\Delta(w, f)\equiv 0$, $b(f)=0$. The Liouville theorem applied to the inverse of f yields that $\Delta(w, f)<\infty$ at each $w\in D$.

THEOREM 2. *Let $(f_1, \cdots, f_n)$ be an admissible system for $x: D\to \boldsymbol{R}^n$. Then, at each $a\in D$, we have*

$$\left\{2^{-1}\sum_{j=1}^{n}\Delta(a, f_j)^2\right\}^{1/2}\leqq \Delta(a, x)\ . \tag{2.4}$$

PROOF. We may assume that $x(a)$ is regular. We pick up all j with $f_j'(a)\neq 0$; for simplicity we assume that they are

$$f_j'(a)\neq 0\ ,\qquad j=1, 2, \cdots, m\ (\leqq n)\ .$$

Then, for $1\leqq j\leqq m$, the Riemannian image of D by f_j contains the one-sheeted open disk of center $f_j(a)$ and radius $\Delta_j\equiv\Delta(a, f_j)>0$. Let $0<\varepsilon<\min_{1\leqq j\leqq m}\Delta_j$. Then, for $1\leqq j\leqq m$, there exists a compact set δ_j in D, which contains a and which is mapped by f_j one-to-one onto the closed disk $\{|w-f_j(a)|\leqq \Delta_j-\varepsilon/2\}$. The union $\delta=\delta_1\cup\cdots\cup\delta_m$ is, therefore, compact, so that $x(\delta)$ is compact.

We shall show that

$$\Gamma_x^*(a, A(\varepsilon))\subset x(\delta)\ , \tag{2.5}$$

where

$$A(\varepsilon)=\left\{2^{-1}\sum_{j=1}^{m}(\Delta_j-\varepsilon)^2\right\}^{1/2}\ .$$

Then, $\Gamma_x^*(a, A(\varepsilon))$ is compact and $|x_u|$ never vanishes in δ. Thus, $\Delta(a, x)\geqq A(\varepsilon)$. On letting $\varepsilon\to 0$ we have (2.4).

Suppose that (2.5) is false. Then, there exists $w\in D\setminus\delta$ such that $d(x(w), x(a))\leqq A(\varepsilon)<A((3/4)\varepsilon)$. We may find a curve γ connecting a and w in D such that

$$\int_\gamma |x_u(\zeta)||d\zeta|<A\left(\frac{3}{4}\varepsilon\right)\ .$$

For each j, $1 \leqq j \leqq m$, the curve γ contains a subcurve γ_j connecting a and a boundary point of δ_j such that $\gamma_j \subset \delta_j$. Consequently,

$$\int_\gamma |f_j'(\zeta)||d\zeta| \geqq \int_{\gamma_j} |f_j'(\zeta)||d\zeta| \geqq \Delta_j - \frac{\varepsilon}{2}, \qquad 1 \leqq j \leqq m .$$

With the aid of (2.3) we now have

$$A\left(\frac{\varepsilon}{2}\right) \leqq \int_\gamma |x_u(\zeta)||d\zeta| < A\left(\frac{3}{4}\varepsilon\right).$$

This contradiction completes the proof.

PROOF OF THEOREM 1. Suppose that $x: D \to \boldsymbol{R}^n$ satisfy $|x_u(0)| = 1$, and let $(f_1, \cdots, f_n)$ be an admissible system for x. We may suppose that $b(x) < \infty$. Since $2^{-1/2} b(f_j) \leqq b(x)$, $1 \leqq j \leqq n$, by (2.4), it follows that $b(f_j) < \infty$, $1 \leqq j \leqq n$. We then choose all f_j with $f_j'(0) \neq 0$; again, for simplicity,

$$p_j \equiv |f_j'(0)| \neq 0, \qquad 1 \leqq j \leqq m\ (\leqq n).$$

Applying the Bloch theorem: $b_2 > 0$, to each $f_j/f_j'(0)$, we have $b(f_j) \geqq b_2 p_j$, $1 \leqq j \leqq m$. Hence,

$$b_2 p_j \leqq b(f_j) \leqq 2^{1/2} b(x), \qquad 1 \leqq j \leqq m .$$

Since $2 = 2|x_u(0)|^2 = p_1^2 + \cdots + p_m^2$, it now follows that

$$2b_2^2 \leqq 2nb(x)^2,$$

whence $b_2 \leqq n^{1/2} b_n$. This completes the proof.

Although we shall not make use of, the fact that the closure of $\Gamma_x(w_0, r)$ is $\Gamma_x^*(w_0, r)$, described in Section 1, is of independent interest. We first note that, among the axioms of the distance, the one: $w_1 \neq w_2 \Rightarrow \text{dis}(x(w_1), x(w_2)) > 0$, follows from (2.3); actually, at least one f_j is non-constant.

We shall show that for each $x(w)$ with $d(x(w_0), x(w)) = r$, and each $\varepsilon > 0$, there exists $x(w_1) \in \Gamma_x(w_0, r) \cap \Gamma_x(w, \varepsilon)$. First, there exists a compact disk δ of center w contained in D such that $x(\delta) \subset \Gamma_x(w, \varepsilon)$; we may assume $w_0 \notin \delta$. Next, for each $k \geqq 2$, there exists a curve γ_k connecting w_0 and w in D such that

$$\int_{\gamma_k} |x_u(\zeta)||d\zeta| < r + k^{-1} .$$

Then, by a point w_k on the boundary circle $\partial\delta$ of δ, γ_k is divided into subcurves γ_{1k} and γ_{2k}; γ_{1k} connects w_0 and w_k, and γ_{2k} connects w_k and w. Therefore,

$$(2.6)\qquad d(x(w_0), x(w_k)) \leqq \int_{\gamma_{1k}} = \int_{\gamma_k} - \int_{\gamma_{2k}} < r + k^{-1} - d(x(w_k), x(w))\ .$$

Choose a converging subsequence of $\{w_k\}$, which we denote again by $\{w_k\}$, such that $|w_k - w_1| \to 0$, $w_1 \in \partial\delta$. It is easy to observe that

$$d(x(w_k), x(w_1)) \to 0\ .$$

Letting $k \to \infty$ in (2.6) we finally have

$$d(x(w_0), x(w_1)) \leqq r - d(x(w_1), x(w)) < r\ .$$

REMARK. Beckenbach's proof of $b_3 > 0$ makes use of a method of cubic equations quite peculiar to the dimension $n=3$, which we call B method; see the proof of [B, Lemma 1]. One should surely feel, in the first reading of [B], that a particular property of the roots of a cubic equation appears to play an important role, so that, one would suspect the possibility of extending the result to higher dimensions by B method. It should be noted that B method (in spirit, we let, after rotation and translation and in our notation, $p_1 = p_2 = p_3$ in the proof of Theorem 1) yields no improvement of $b_3 \geqq 3^{-1/2} b_2$. As another remark, we point out that Beckenbach assumed that $|x_u(0)| \geqq 1$ for $x: D \to \boldsymbol{R}^n$. On dividing: $|x_u(0)|^{-1} x$, one finds that $b(x) \geqq |x_u(0)| b_n \geqq b_n$.

§ 2a. The strong Landau constants.

Let $x: D \to \boldsymbol{R}^n$ $(n \geqq 2)$, let $w \in D$, and let $\Delta_L(w, x)$ be the supremum of $r > 0$ such that $\Gamma_x^*(w, r)$ is compact; no regularity restriction on $\Gamma_x(w, r)$ is posed. Let $L(x)$ be the supremum of $\Delta_L(w, x)$ for $w \in D$. For $n \geqq 2$, the infimum L_n of $L(x)$ for all $x: D \to \boldsymbol{R}^n$ subject to the condition $|x_u(0)| = 1$ is called the n-th strong Landau constant by the reason mentioned soon. Since $\Delta(w, x) \leqq \Delta_L(w, x)$, it follows that $b(x) \leqq L(x)$, and hence $b_n \leqq L_n$, $n \geqq 2$. It is easy to see that $L_n \leqq L_{n-1}$, $n \geqq 3$.

Without saying the details, we denote $\Delta_L(w, f)$ and $L(f)$ for f holomorphic, and possibly constant, in D. The constant L_2 is then not greater than the Landau constant L; L is the supremum of the radii of all open Euclidean disks contained in the (set-theoretic) image of D by f holomorphic in D and normalized by $|f'(0)| = 1$. The constant L is introduced by E. Landau [L]. It is known that [A1, p. 364], [Rm, p. 389], [P, p. 690], (2a.1) $$2^{-1}<L\leqq\frac{\Gamma(1/3)\Gamma(5/6)}{\Gamma(1/6)}=0.543\cdots;$$

the right-hand side is said to be an unpublished result of Robinson [A1, p. 364]. Since $b_2\leqq L_2\leqq L$, it would be an interesting problem to prove $b_2<L_2<L$. We shall not be concerned with this but with

THEOREM 1a. *$L_n\geqq n^{-1/2}L_2$, $n\geqq 3$.*

The proof of Theorem 1a is similar to that of Theorem 1, once the following analogue of Theorem 2 is established.

THEOREM 2a. *Let $(f_1,\cdots,f_n)$ be an admissible system for $x\colon D\rightarrow \boldsymbol{R}^n$. Then, at each $a\in D$, we have*

(2a.2) $$\left\{2^{-1}\sum_{j=1}^{n}\Delta_L(a,f_j)^2\right\}^{1/2}\leqq\Delta_L(a,x).$$

PROOF. There is a nuisance of the possibility of $\Delta_L(a,f_j)=\infty$. We pick up all nonconstant f_j; for simplicity, we assume that they are f_j, $1\leqq j\leqq m$. Then,

$$\Delta_j\equiv\Delta_L(a,f_j)>0\,,\qquad 1\leqq j\leqq m\ (\leqq n)\,.$$

We first consider the case where all $\Delta_j<\infty$, $1\leqq j\leqq m$, and we let $0<\varepsilon<\min_{1\leqq j\leqq m}\Delta_j$ again. Let δ_j be the component, containing a, of the inverse image of

$$\{|w-f_j(a)|\leqq\Delta_j-\varepsilon/2\}\quad\text{by}\quad f_j\,,\qquad 1\leqq j\leqq m\,.$$

Since f_j is an open map, it follows that each δ_j is compact, so that $\delta=\delta_1\cup\cdots\cup\delta_m$ is compact. Following the same lines as in the proof of Theorem 2 up to proving (2.5) for our present Δ_j, we now have (2a.2) by the limiting process of $\varepsilon\rightarrow 0$.

To prove that $\Delta_L(a,x)=\infty$ in case there exists one $\Delta_L(a,f_j)=\infty$, we let $\delta(k)$ be the component, containing a, of the inverse image of $\{|w-f_j(a)|\leqq k+2\}$ by f_j, $k=1,2,\cdots$. Then, $x(\delta(k))$ is compact, and we can show that

$$\Gamma_x^*(a,2^{-1/2}k)\subset x(\delta(k))$$

with the aid of

$$2^{-1/2}\int_\gamma|f_j'(w)||dw|\leqq\int_\gamma|x_u(w)||dw|$$

resulting from (2.3). Thus, $\Delta_L(a, x) \geqq 2^{-1/2}k$, and hence the limiting process yields the requested.

REMARK. There would be no "precise" extension of the Landau constant to $n \geqq 3$ because there is no fixed "surface" covered by S.

§ 3. Bloch minimal surfaces.

Let $(f_1, \cdots, f_n)$ be an admissible system for $x: D \to \boldsymbol{R}^n$. Then, it follows from (2.2) that

$$2^{-1/2}\mu(f_j) \leqq \mu(x) \leqq 2^{-1/2} \sum_{k=1}^{n} \mu(f_k), \qquad 1 \leqq j \leqq n. \tag{3.1}$$

Therefore, x is Bloch if and only if all f_j are Bloch. It follows from (2.1) that if $n-1$ members of $f_1, \cdots, f_n$ are Bloch, then the rest is Bloch.

If $x: D \to \boldsymbol{R}^n$ is bounded, then $\mathrm{Re}\, f_j$ is bounded, so that f_j is Bloch, $1 \leqq j \leqq n$; see the remark after Lemma 3.1 below. Therefore, x is Bloch by (3.1).

The Gauss curvature [O, p. 76] $\kappa(w)$ of $x: D \to \boldsymbol{R}^n$ at a regular point $x(w)$ is defined by

$$\kappa(w) = -|x_u(w)|^{-2} \Delta \log |x_u(w)| .$$

Suppose that $\kappa(w) \leqq -A$ $(A>0)$ at each regular point. We shall show that $\mu(x) \leqq 2A^{-1/2}$. The proof follows the same lines as in the proof of [A1, Theorem A]; we include a sketch of it for completeness. For each r, $0<r<1$, we set $\lambda(w) = r(r^2 - |w|^2)^{-1}$. Let $y = 2^{-1}A^{1/2}x$. Our aim is to show that

$$|y_u(w)| \leqq \lambda(w) \qquad \text{for} \quad |w| < r .$$

Then, letting $r \to 1$ we have the requested. We suppose that the open set $E = \{w; |w|<r, |y_u(w)| > \lambda(w)\}$ is nonempty. Since $|y_u|$ never vanishes in E, we have

$$\Delta \log(|y_u|/\lambda) \geqq 4(|y_u|^2 - \lambda^2) > 0 \qquad \text{in} \quad E ,$$

so that the nonconstant and positive subharmonic function $s = \log(|y_u|/\lambda)$ has no maximum in E. We choose a sequence $\{w_k\}_{k \geqq 1}$ of points in E such that $s(w_k)$ converges to the supremum $Q>0$ of s in E, and further $w_k \to w_0$, $|w_0| \leqq r$. The two possibilities, $|w_0| = r$ and $|w_0| < r$, then lead us to a contradiction: $Q = -\infty$ and $Q = 0$, respectively.

Many criteria for a holomorphic function in D to be Bloch are known

[P]; see some recent works [Y1], [Y2], [Y4], for example. It is not difficult, with the aid of (3.1) partially, to obtain analogous criteria for x to be Bloch. Among them we pick up three which might be noteworthy from a geometrical viewpoint.

The disk D is endowed with the Poincaré metric, or the non-Euclidean hyperbolic metric; the distance is

$$\sigma(w_1, w_2)=\tanh^{-1}\frac{|w_1-w_2|}{|1-\bar{w}_1 w_2|} .$$

Let $U(a, r)$ be the disk of center $a\in D$ and the radius $\tanh^{-1} r$, that is,

$$U(a, r)=\left\{w; \frac{|w-a|}{|1-\bar{a}w|}<r\right\} , \qquad 0<r<1 .$$

The area of the image $x(U(a, r))$ counting the multiplicities is then given by

$$\text{Area } x(U(a, r))=\iint_{U(a,r)}|x_u(w)|^2 du dv .$$

THEOREM 3. *A minimal surface $x: D\to \boldsymbol{R}^n$ is Bloch if and only if there exists r, $0<r<1$, such that*

$$\sup_{a\in D} \text{Area } x(U(a, r))<\infty .$$

For the proof of the corresponding result for the holomorphic functions, see [Y1].

The next theorem is never obvious.

THEOREM 4. *A minimal surface $x: D\to \boldsymbol{R}^n$ is Bloch if and only if x is uniformly continuous as a map from D endowed with σ into the Euclidean space $\boldsymbol{R}^n$.*

For the proof we shall make use of

LEMMA 3.1. *Let f be holomorphic and $|\operatorname{Re} f|<K$ in a disk $\{|w|<M\}$, $M>0$. Then, $|f'(0)|\leqq M^{-1}e^{2K}$.*

PROOF. We may assume that f is nonconstant. Then, $g(w)=\exp\{f(Mw)-K\}$, $w\in D$, is bounded, $|g|<1$. The Schwarz-Pick lemma now reads

$$(1-|w|^2)|g'(w)|\leqq 1-|g(w)|^2<1 ,$$

which, together with $|g|^{-1}\leqq e^{2K}$, yields

$$(1-|w|^2)|f'(Mw)| \leqq M^{-1}e^{2K}, \qquad w \in D.$$

The lemma follows on setting $w=0$.

REMARK. If $|\mathrm{Re}\, f|<K$ in D, then f is Bloch by

$$(1-|w|^2)|f'(w)| \leqq e^{2K}, \qquad w \in D.$$

Let Φ be the family of all one-to-one conformal mappings from D onto D. Then,

$$(1-|w|^2)|\phi'(w)| = 1-|\phi(w)|^2, \qquad w \in D,$$

for each $\phi \in \Phi$. Therefore, $x: D \to \boldsymbol{R}^n$ is Bloch if and only if the composed minimal surface $x \circ \phi: D \to \boldsymbol{R}^n$ for some (and hence each) $\phi \in \Phi$, is Bloch by $\mu(x)=\mu(x\circ\phi)$.

PROOF OF THEOREM 4. Suppose first that x is Bloch, $\mu=\mu(x)<\infty$. On integrating both sides of

$$|x_u(w)||dw| \leqq \mu(1-|w|^2)^{-1}|dw|$$

along the Poincaré geodesic γ connecting w_1 and w_2 in D we have

$$(3.2) \qquad |x(w_1)-x(w_2)| \leqq d(x(w_1), x(w_2)) \leqq \int_\gamma |x_u(w)||dw| \leqq \mu\sigma(w_1, w_2),$$

so that x is uniformly continuous.

Conversely, suppose that x is uniformly continuous. Then, there exists M, $0<M<1$, such that

$$\sigma(w_1, w_2) < \tanh^{-1} M \Longrightarrow |x(w_1)-x(w_2)| < 1.$$

For each $a \in D$, we consider a particular member

$$\phi_a(w) = \frac{w+a}{1+\bar{a}w}$$

of Φ. Let $(f_1, \cdots, f_n)$ be an admissible system. Then,

$$|w|<M \Longrightarrow \sigma(\phi_a(w), \phi_a(0)) = \tanh^{-1}|w| < \tanh^{-1} M,$$

so that, the real part $x_j\circ\phi_a - x_j(a)$ of $f_j\circ\phi_a - f_j(a)$ is bounded by 1 for $|w|<M$ because

$$|x_j\circ\phi_a(w) - x_j(a)| \leqq |x\circ\phi_a(w) - x\circ\phi_a(0)| < 1.$$

It follows from Lemma 3.1 that $(1-|a|^2)|f_j'(a)|\leqq M^{-1}e^2$. Since a is arbitrary, $\mu(f_j)\leqq M^{-1}e^2$, $1\leqq j\leqq n$, whence, (3.1) shows that x is Bloch. This completes the proof.

We fix n. A family $\mathscr{M}$ of minimal surfaces $x: D\to \boldsymbol{R}^n$ is called normal (in the sense of P. Montel) if each sequence $\{x^{(k)}\}_{k\geqq 1}$ extracted from $\mathscr{M}$ contains a subsequence $\{y^{(k)}\}_{k\geqq 1}$ such that, for each $\varepsilon>0$ and for each compact set $\delta\subset D$, there exists a number $J=J(\delta, \varepsilon)$ with the property:

$$\sup_{w\in\delta}|y^{(j)}(w)-y^{(k)}(w)|<\varepsilon$$

for all $j, k>J$. Given $x: D\to \boldsymbol{R}^n$ we consider the family of minimal surfaces

$$\mathscr{M}(x)=\{x\circ\phi-x\circ\phi(0);\ \phi\in\Phi\}\ .$$

THEOREM 5. *For $x: D\to \boldsymbol{R}^n$ to be Bloch it is necessary and sufficient that $\mathscr{M}(x)$ is normal.*

PROOF. Suppose that x is Bloch with $\mu=\mu(x)$. We shall show that
(i) for each $a\in D$ and each $y\in\mathscr{M}(x)$,

$$|y(a)|\leqq\mu\sigma(a, 0)\ ;$$

(ii) for each $a\in D$ and each $\varepsilon>0$, there exists $r>0$ such that

$$|w-a|<r\Longrightarrow|y(w)-y(a)|<\varepsilon \text{ for all } y\in\mathscr{M}(x)\ .$$

Then, $\mathscr{M}(x)$ is uniformly bounded at each point $a\in D$ by (i) and equicontinuous by (ii). The Ascoli-Arzelà's diagonal process theorem (see [Ry, p. 155]) then shows that $\mathscr{M}(x)$ is normal.

Since $\sigma(a, 0)=\sigma(\phi(a), \phi(0))$, (i) is a consequence of (3.2). Choose $r>0$ so that

$$|w-a|<r\Longrightarrow\sigma(w, a)<\varepsilon/\mu\ .$$

Since $\sigma(w, a)=\sigma(\phi(w), \phi(a))$, (ii) is again a consequence of (3.2).

To prove the sufficiency we assume that $\mathscr{M}(x)$ is normal, yet x is not Bloch. Then, there exists a sequence $\{a_k\}_{k\geqq 1}$ with $(1-|a_k|^2)|x_u(a_k)|\to\infty$. Let

$$\phi_k(w)=\frac{w+a_k}{1+\bar{a}_k w}\ , \qquad w\in D\ .$$

Then,

$$y^{(k)} \equiv x \circ \phi_k - x(a_k) \in \mathscr{M}(x) ,$$

so that there exists a subsequence of $\{y^{(k)}\}$, which we denote again by $\{y^{(k)}\}$ such that $\{y^{(k)}\}$ converges to a map y from D into $\boldsymbol{R}^n$ uniformly on each compact set in D; each component of y is harmonic in D. After the componentwise observation, it follows that $(y^{(k)})_u$ also converges to y_u locally and uniformly. In particular, $(y^{(k)})_u(0) \to y_u(0)$, so that $|(y^{(k)})_u(0)| = (1-|a_k|^2)|x_u(a_k)| \not\to \infty$; this is a contradiction.

§ 4. Disks on Bloch minimal surfaces.

We begin with a characterization of b_n in terms of $\mu(x)$ and $b(x)$.

PROPOSITION 1. *Fix* $n \geqq 2$. *Then,*

$$\mu(x) \leqq b_n^{-1} b(x) \tag{4.1}$$

for each $x: D \to \boldsymbol{R}^n$. *This is sharp in the sense that if* $c>0$ *satisfies* $\mu(x) \leqq cb(x)$ *for each* $x: D \to \boldsymbol{R}^n$, *then* $b_n^{-1} \leqq c$.

In the case $n=2$ we obtain

$$\mu(f) \leqq b_2^{-1} b(f) \leqq 4 \cdot 3^{-1/2} b(f) \tag{4.2}$$

for each f holomorphic in D. Proposition 1 has

COROLLARY 4.1. *If* $b(x) < \infty$ *for* $x: D \to \boldsymbol{R}^n$, *then* x *is Bloch.*

We note that for f holomorphic in D, we have

$$b(f) \leqq \mu(f) \tag{4.3}$$

with the aid of W. Seidel and J. L. Walsh's theorem [SW, Theorem 2, p. 133]. This, combined with (4.2), yields the well-known criterion: *f is Bloch if and only if $b(f) < \infty$.* It is open whether or not the converse of Corollary 4.1 is true in case $n \geqq 3$.

PROOF OF PROPOSITION 1. The sharpness is immediate. For x with $|x_u(0)|=1$, we have $1 \leqq cb(x)$, so that the definition of b_n shows that $c^{-1} \leqq b_n$. Now, we must prove that

$$(1-|a|^2)|x_u(a)| \leqq b_n^{-1} b(x) , \qquad a \in D . \tag{4.4}$$

Assuming that $x(a)$ is regular, we set

$$y = (1-|a|^2)^{-1} |x_u(a)|^{-1} x \circ \phi_a$$

in D. Then, $|y_u(0)|=1$, so that, by the definition of b_n, there exists w in D such that $\Delta(w, y)\geqq b_n$. Therefore,

$$b(x)\geqq \Delta(\phi_a(w), x)=(1-|a|^2)|x_u(a)|\Delta(w, y) \geqq b_n(1-|a|^2)|x_u(a)| ,$$

whence (4.4).

Let q stand for b or μ and let c_q be the infimum of $c>0$ such that

$$(1-|w|^2)|f'(w)|\leqq cq(f)^{1/2}\Delta(w, f)^{1/2} , \qquad w\in D ,$$

for each f holomorphic and Bloch in D. Actually, c_q is the minimum in the sense that

$$(1-|w|^2)|f'(w)|\leqq c_q q(f)^{1/2}\Delta(w, f)^{1/2} , \qquad w\in D , \tag{4.5}$$

for each f Bloch in D.

The first result, perhaps, of estimating c_b explicitly, would be [SW, Theorem 10, p. 208], where

$$c_b\leqq 2\cdot 5^{1/2}b_2^{-1/2} ;$$

the right term is at least, $6.51\cdots$. L. V. Ahlfors implicitly proved that

$$(1-|w|^2)|f'(w)|\leqq 2\cdot 3^{-1/2}\{\Delta(w, f)/b(f)\}^{1/2}\{3b(f)-\Delta(w, f)\} \tag{4.6}$$

if $b(f)<\infty$ (see [A1, pp. 363–364], [A2, pp. 12–15]). It now follows that

$$(1-|w|^2)|f'(w)|\leqq 2\cdot 3^{1/2}b(f)^{1/2}\Delta(w, f)^{1/2} , \qquad w\in D ,$$

so that $c_b\leqq 2\cdot 3^{1/2}=3.46\cdots$. C. Pommerenke [P, Theorem 1, (i)] improved the Ahlfors estimate (4.6); he proved that the right-hand side of (4.6) can be multiplied by an absolute constant P, $0<P<1$, so that $c_b<2\cdot 3^{1/2}$. However, it appears to be difficult to find more explicit estimate of P than $0<P<1$ by his method. It is easy to prove that

$$1\leqq c_\mu\leqq c_b\leqq b_2^{-1/2}c_\mu .$$

For the c_μ part, we observed in [Y3, Theorem 1] that

$$c_\mu\leqq \min_{r>0} 2r^{1/2}(\tanh r)^{-1}=2.62\cdots .$$

Our next task is to extend (4.5) to $\boldsymbol{R}^n$.

PROPOSITION 2. *If $x\colon D\to \boldsymbol{R}^n$ is Bloch, then at each $w\in D$,*

$$(1-|w|^2)|x_u(w)|\leqq c_q n^{1/4} q(x)^{1/2} \Delta(w, x)^{1/2},$$

where $q=b$ *or* μ.

PROOF. It follows from (2.4) and (3.1) that

$$q(f_j)\leqq 2^{1/2}q(x), \qquad 1\leqq j\leqq n.$$

Squaring both sides of (4.5) for f_j, summing up with respect to j, and considering the Schwarz inequality, we have

$$\begin{aligned}2(1-|w|^2)^2|x_u(w)|^2&=(1-|w|^2)^2\sum|f_j'(w)|^2\\ &\leqq c_q^2\sum q(f_j)\Delta(w, f_j)\leqq c_q^2 2^{1/2}q(x)n^{1/2}(\sum \Delta(w, f_j)^2)^{1/2}\\ &\leqq 2c_q^2 q(x)n^{1/2}\Delta(w, x),\end{aligned}$$

where (2.4) is considered.

REMARK 1. The L_n version of Proposition 1 is valid:

$$\mu(x)\leqq L_n^{-1}L(x)$$

holds for each $x: D\rightarrow \boldsymbol{R}^n$ $(n\geqq 2)$. If $c>0$ satisfies $\mu(x)\leqq cL(x)$ for each $x: D\rightarrow \boldsymbol{R}^n$, then $L_n^{-1}\leqq c$.

REMARK 2. With the aid of the inequality $2^{-1}b(f_j)^2\leqq b(x)^2$, $1\leqq j\leqq n$, resulting from (2.4), Proposition 1 teaches us another proof of Theorem 1. We have

$$\begin{aligned}(1-|w|^2)^2|x_u(w)|^2&=2^{-1}\sum(1-|w|^2)^2|f_j'(w)|^2\\ &\leqq 2^{-1}\sum b_2^{-2}b(f_j)^2\leqq nb_2^{-2}b(x)^2,\end{aligned}$$

whence, $\mu(x)\leqq n^{1/2}b_2^{-1}b(x)$. Therefore, $b_n^{-1}\leqq n^{1/2}b_2^{-1}$, or, $b_n\geqq n^{-1/2}b_2$. Similarly, we have another proof of Theorem 1a, which we leave as an exercise.

REMARK 3. We may show that $c_b\geqq b_2^{-1}=2.11\cdots$. Actually, $\mu(f)\leqq c_b b(f)$, together with Proposition 1, shows that $c_b\geqq b_2^{-1}$. As a consequence, we further obtain

$$c_\mu\geqq b_2^{1/2}c_b\geqq b_2^{-1/2}\geqq 1.45\cdots.$$

In conclusion,

$$\begin{aligned}2.11\cdots\leqq c_b<3.46\cdots,\\ 1.45\cdots\leqq c_\mu\leqq 2.62\cdots.\end{aligned}$$

§ 5. Integral criteria.

We shall show that $x: D\rightarrow \boldsymbol{R}^n$ is Bloch if and only if x is of bounded mean oscillation in D, that is,

(A) $$\sup_{\substack{a\in D\\ 0<\rho\leq 1}} mD(a,\rho)^{-1}\iint_{D(a,\rho)}|x(w)-x(a)|\,dudv<\infty\ ,$$

where

$$D(a,\rho)=\{|w-a|<\rho(1-|a|)\}\ ,$$
$$mD(a,\rho)=\pi\rho^2(1-|a|)^2\ ,\quad \text{the area of } D(a,\rho)\ .$$

We shall actually prove much more.

THEOREM 6. *For* $x\colon D\to \boldsymbol{R}^n$ *the following are mutually equivalent.*

(B) x *is Bloch.*

(C) *There exists* $c>0$ *such that*

$$\sup_{a\in D} mD(a,1)^{-1}\iint_{D(a,1)}\exp(c|x(w)-x(a)|)\,dudv<\infty\ .$$

(D) *There exists* ρ, $0<\rho<1$, *such that*

$$\sup_{a\in D} mD(a,\rho)^{-1}\iint_{D(a,\rho)}\log|x(w)-x(a)|\,dudv<\infty\ .$$

As will be apparent, we may say that (C) is the strongest and (D) is the weakest condition in integrals; the case $n=2$ in Theorem 6 yields criteria for a holomorphic function in D to be Bloch.

Postponing the proof of the theorem we show that (C)$\Rightarrow$(A)$\Rightarrow$(D). As will be proved later in Lemma 5.1, $\log|x-x(a)|$ is subharmonic in D. Therefore,

$$|x-x(a)|^p=\exp(p\log|x-x(a)|)\ ,\qquad p>0\ ,$$
$$\exp(c|x-x(a)|)\ ,\qquad c>0\ ,$$

are subharmonic in D. With the aid of

$$\log X\leqq X\leqq c^{-1}e^{cX}\ ,\qquad X\geqq 0\ ,$$

and the fact that, for a fixed $a\in D$, the area mean in $D(a,\rho)$ of a subharmonic function in D is a nondecreasing function of $\rho(1-|a|)$, hence of ρ, [Rd, p. 8], we have (C)$\Rightarrow$(A)$\Rightarrow$(D).

LEMMA 5.1. *For* $x\colon D\to\boldsymbol{R}^n$, *and for each fixed* $x_0\in\boldsymbol{R}^n$, $\log|x-x_0|$ *is a subharmonic function in* D.

PROOF. See [BR, p. 653] for the case $n=3$. We may suppose that $n\geqq 3$ and $x_0=0$. Since $\log|x(w)|=-\infty$ if $x(w)=0$, we consider w with $x(w)\neq 0$. At this point, we have

(5.1) $$2^{-1}|x|^4\Delta\log|x|=|x_u|^2|x|^2-(xx_u)^2-(xx_v)^2\ .$$

To prove $\Delta \log x \geqq 0$ at w, we may further assume that $x_u(w) \neq 0$. Let $e_1, \cdots, e_n$ be an orthonormal basis in $\boldsymbol{R}^n$ such that

$$e_1 = \frac{x_u(w)}{|x_u(w)|}, \qquad e_2 = \frac{x_v(w)}{|x_v(w)|}.$$

Let

$$x(w) = \sum_{j=1}^{n} c_j e_j .$$

In view of (1.2) we observe that the right-hand side of (5.1) at w is

$$|x_u|^2 \Big(\sum_{j=3}^{n} c_j^2 \Big) \geqq 0 .$$

PROOF OF THEOREM 6. (B)$\Rightarrow$(C). There exists $c>0$ with $c\mu<2$, $\mu = \mu(x)$. For each fixed $a \in D$,

$$w = a + (1-|a|)\zeta \in D(a, 1) \Longleftrightarrow \zeta \in D .$$

Therefore,

$$|\phi_{-a}(w)| = \frac{|w-a|}{|1-\bar{a}w|} \leqq |\zeta| ,$$

and for each $w \in D(a, 1)$,

(5.2) $$|x(w)-x(a)| \leqq \mu \tanh^{-1} |\phi_{-a}(w)| \leqq \mu \tanh^{-1} |\zeta| ,$$

so that

$$\exp(c|x(w)-x(a)|) \leqq \left(\frac{1+|\zeta|}{1-|\zeta|} \right)^{c\mu/2} .$$

Consequently,

$$mD(a, 1)^{-1} \iint_{D(a,1)} \exp(c|x(w)-x(a)|) du dv$$
$$\leqq \pi^{-1} \iint_D \left(\frac{1+|\zeta|}{1-|\zeta|} \right)^{c\mu/2} d\xi d\eta , \qquad \zeta = \xi + i\eta ;$$

the right-hand side is a positive constant independent of a, so that (C) holds.

Since (C)$\Rightarrow$(D) is trivial by Lemma 5.1, it remains to be proved that (D)$\Rightarrow$(B). Let K be the supremum in (D). To estimate $(1-|a|^2)|x_u(a)|$ at a, we note that $-\infty \leqq \log |x_u(a)| < +\infty$. It then follows from (1.2) that

$$\frac{|x(w)-x(a)|}{|w-a|} \to |x_u(a)| \quad \text{as} \quad w \to a . \tag{5.3}$$

Set

$$V(w)=\log|x(w)-x(a)|-\log|w-a|$$

for $w \in D\setminus\{a\}$. Then, V is subharmonic in $D\setminus\{a\}$ and is bounded from above in a small punctured disk $\{0<|w-a|<r\}$. From M. Brelot's removable singularity theorem [Rd, Section 7.15, p. 48] it follows that we may define $V(a)$ so that V is subharmonic in the whole disk D. By the upper semicontinuity of V at a, we then have by (5.3),

$$\log|x_u(a)| \leqq V(a) .$$

Therefore,

$$\begin{aligned}\log|x_u(a)| &\leqq V(a) \leqq mD(a, \rho)^{-1}\iint_{D(a,\rho)} V(w)dudv \\ &= mD(a, \rho)^{-1}\iint_{D(a,\rho)} \log|x(w)-x(a)|dudv \\ &\quad -\log(1-|a|)\rho+2^{-1} ,\end{aligned} \tag{5.4}$$

whence

$$|x_u(a)| \leqq e^{1/2+K}\{(1-|a|)\rho\}^{-1} , \quad \text{or} ,$$
$$(1-|a|^2)|x_u(a)| \leqq 2\rho^{-1}e^{1/2+K} ,$$

which completes the proof of the theorem.

REMARK 1. It would be of interest to compare

$$\|x\| \equiv \sup_{a\in D} mD(a, 1)^{-1}\iint_{D(a,1)}|x(w)-x(a)|dudv$$

with $\mu(x)$; the result is

$$2^{-1}e^{-1/2}\mu(x) \leqq \|x\| \leqq \mu(x) . \tag{5.5}$$

Suppose that $\mu=\mu(x)<\infty$. It then follows from (5.2) that at each $a \in D$,

$$\begin{aligned}mD(a, 1)^{-1}&\iint_{D(a,1)}|x(w)-x(a)|dudv \\ &\leqq \mu\pi^{-1}\iint_{D}\tanh^{-1}|\zeta|d\xi d\eta=\mu ,\end{aligned}$$

whence the right-hand side of (5.5) follows.

Suppose next that $\|x\|<\infty$. At each point a, and for $0<\rho<1$, it follows from (5.4) that

$$\begin{aligned}|x_u(a)|&=\exp\log|x_u(a)|\\&\leqq(1-|a|)^{-1}\rho^{-1}e^{1/2}mD(a,\rho)^{-1}\iint_{D(a,\rho)}|x(w)-x(a)|\,dudv\\&\leqq(1-|a|)^{-1}\rho^{-1}e^{1/2}\|x\|\,,\end{aligned}$$

whence

$$(1-|a|^2)|x_u(a)|\leqq 2\rho^{-1}e^{1/2}\|x\|\,.$$

Letting $\rho\to 1$ we obtain the left-hand side of (5.5).

REMARK 2. We have the following Schwarz lemma:

For $x: D\to \boldsymbol{R}^n$, bounded, $|x|<1$, in D, with $x(0)=0$,

(5.6) $$|x(w)|\leqq|w|$$

holds for each $w\in D$ and $|x_u(0)|\leqq 1$. The equality in (5.6) holds for a w_0, $0<|w_0|<1$, or $|x_u(0)|=1$ if and only if x maps D one-to-one onto a unit disk lying in a plane.

See [BR, pp. 656–657] for the case $n=3$. The subharmonic function V in D for the present x with $a=0$ (hence, $-\infty\leqq\log|x_u(0)|\leqq V(0)$) considered in the proof of Theorem 6 has the nonpositive supremum in D. Hence, $\log|x(w)|\leqq\log|w|$ for $0<|w|<1$, and $\log|x_u(0)|\leqq 0$. The "if" part in the second half is obvious because x can be considered as a holomorphic or antiholomorphic function in D after a rotation of the plane about the origin. To prove the "only if" part, we first note that $V(w)\equiv 0$ by the maximum principle, whence $|x(w)|^2\equiv|w|^2$. Then, for an admissible system $(f_1,\cdots,f_n)$ for x we have

$$2\sum|f_j'(w)|^2=4|x_u(w)|^2=\Delta(|x(w)|^2)\equiv 4\,.$$

Therefore, $f_j(w)\equiv c_jw+d_j$, where $c_j=\alpha_j+i\beta_j$ and d_j are constants with $\operatorname{Re} d_j=0$ by $x(0)=0$ $(1\leqq j\leqq n)$, and

$$\sum(\alpha_j^2-\beta_j^2+2i\alpha_j\beta_j)\equiv\sum(f_j'(w))^2\equiv 0\,.$$

Thus, $x_j(w)=\alpha_ju-\beta_jv$, $1\leqq j\leqq n$, so that, $x(D)$ is contained in the plane generated by the (real) orthonormal vectors

$$(\alpha_1,\cdots,\alpha_n)\qquad\text{and}\qquad(\beta_1,\cdots,\beta_n)\,.$$

After a suitable rotation, we can regard x as a holomorphic or anti-holomorphic function, and the Schwarz lemma in the complex analysis now shows the conclusion.

Added in proof.

On the basis of pp. 184-185 in D. Gnuschke-Hauschild and C. Pommerenke's paper: "On Bloch functions and gap series", J. reine angew. Math. **367** (1986), 172-186, the sentence citing Pommerenke's paper just after the display (4.6) should be deleted.

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