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The Signature of Kähler Surfaces Immersed into CP^m

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Abstract. In this note we give some interesting topological restrictions for the immersion of Kaehler surfaces into the complex projective space $CP^m(1)$.

§1. Introduction.

Let M be a 2-dimensional compact Kaehler submanifold immersed into the complex projective space $CP^m(1)$ endowed with the Fubini-Study metric of constant holomorphic sectional curvature 1. We denote by sign(M) and σ the signature of M and the second fundamental form of the immersion respectively.

In this paper we obtain the following theorems.

THEOREM 1. *For M we have:*

$$32\pi^2 \operatorname{sign}(M) \geqq \int_M (4-|\sigma|^4) *1 \tag{1.1}$$

where $*$ denotes the Hodge star operator and the equality holds if and only if M is an imbedded submanifold congruent to the standard imbedding of $CP^2(1)$ or $CQ^2=CP^1\times CP^1$ into $CP^m(1)$.

THEOREM 2. *If M has scalar curvature $\tau\geqq 3$, then*

$$\operatorname{sign}(M)\leqq \operatorname{sign}(CP^2)$$

where the equality holds if and only if M is congruent to the standard imbedding of $CP^2(1/2)$ or $CP^2(1)$ into $CP^m(1)$.

From Theorem 1 we obtain

COROLLARY 1. A) *If M has positive total scalar curvature, then the*

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second Betti number of M satisfies

$$b_2 \leqq 2 + \frac{1}{32\pi^2} \int_M (|\sigma|^4 - 4) * 1$$

where the equality holds if and only if M is congruent to the standard imbedding of $CP^2(1)$ or CQ^2 into $CP^m(1)$.

B) *If M has* sign$(M) \leqq 0$, *then*

$$\int_M |\sigma|^4 * 1 \geqq 4 \operatorname{vol}(M)$$

where the equality holds if and only if M is congruent to the standard imbedding of CQ^2 into $CP^m(1)$.

Theorem 1 has another interesting consequence. Indeed, Theorem 1 together with Theorem 2 yields

COROLLARY 2. *If M has scalar curvature $\tau \geqq 4$, then M is congruent to the standard imbedding of CQ^2 or $CP^2(1)$ into $CP^m(1)$.*

REMARK. Corollary 2 is one of Ogiue's conjectures [4]. During the preparation of this note it came to my knowledge that this Ogiue's conjecture has been proved in [5] for every $n \geqq 2$.

§2. Preliminaries.

Let M be a 2-dimensional compact Kaehler manifold. Let $\{\vartheta^1, \vartheta^2\}$ be a local field of unitary coframes. Then the Kaehler 2-form ϕ, the Ricci form γ and the scalar curvature τ are given by

$$\phi = \frac{\sqrt{-1}}{8\pi} \sum \vartheta^\alpha \wedge \bar{\vartheta}^\alpha , \quad \gamma = \frac{\sqrt{-1}}{4\pi} \sum \rho_{\alpha\bar{\beta}} \vartheta^\alpha \wedge \bar{\vartheta}^\beta , \quad \tau = 2 \sum \rho_{\alpha\bar{\alpha}}$$

where $\rho_{\alpha\bar{\beta}}$ are the local components of the Ricci tensor ρ of M. It is well-known that the first Chern class c_1 is represented by γ. We denote by $|R|$ and $|\rho|$ the lengths of the curvature and Ricci tensors respectively. We recall that the signature of M can be expressed by the following formulas (cf. for example [1] and [2] p. 125):

$$96\pi^2 \operatorname{sign}(M) = \int_M (4|\rho|^2 - 2|R|^2) * 1 , \tag{2.1}$$

$$\operatorname{sign}(M) = \sum_{p,q=0}^{2} (-1)^q b_{p,q} \tag{2.2}$$

where $b_{p,q}$ denotes the dimension of the space of the harmonic forms of bidegree (p, q) on M.

From a classification theorem of Nakagawa-Takagi [3] (see also Takeuchi [6]), we have the following

LEMMA 2.1. *Let M be a compact Kaehler surface immersed in $CP^m(1)$. Then M has parallel second fundamental form if and only if it is an imbedded submanifold congruent to the standard imbedding of one in the following table*:

surface	p	τ	vol	sign
(a) $CP^2(1)$	0	6	$8\pi^2$	1
(b) $CP^2(1/2)$	3	3	$32\pi^2$	1
(c) CQ^2	1	4	$16\pi^2$	0

where p is the essential complex codimension. The imbeddings (b) *and* (c) *are called respectively the Veronese imbedding and the Segre imbedding.*

§3. Proof of Theorem 1.

Since M is holomorphically isometrically immersed in $CP^m(1)$, the second fundamental form σ of the immersion satisfies the following equations

$$\tau=6-|\sigma|^2 , \tag{3.1}$$

$$|\rho|^2=9-3|\sigma|^2+\mathrm{Tr}(\sum_{\alpha} A_\alpha^2)^2 , \tag{3.2}$$

$$|R|^2=12-4|\sigma|^2+2\sum_{\alpha,\beta} (\mathrm{Tr}\, A_\alpha A_\beta)^2 , \tag{3.3}$$

$$\frac{1}{2}\Delta|\sigma|^2=|\bar{\nabla}\sigma|^2+2|\sigma|^2-2\,\mathrm{Tr}(\sum_{\alpha} A_\alpha^2)^2-\sum_{\alpha,\beta} (\mathrm{Tr}\, A_\alpha A_\beta)^2 , \tag{3.4}$$

where Δ is the Laplacian, $\bar{\nabla}\sigma$ the covariant derivative of σ and A_α the Weingarten maps associated with orthonormal basis $\xi_1, \cdots, \xi_{2(m-2)}$ of the normal space. Equations (3.1) and (3.4) can be found in [4]. Equations (3.2) and (3.3) can be obtained from the equation of Gauss. It is also shown that (cf. [4] p. 87)

$$2\sum_{\alpha,\beta} (\mathrm{Tr}\, A_\alpha A_\beta)^2 \leqq |\sigma|^4 . \tag{3.5}$$

Taking the integral of the both sides of (3.4) and using Green's Theorem, we have

$$\int_M |\bar{\nabla}\sigma|^2 *1 = \int_M \{2\,\mathrm{Tr}(\sum_\alpha A_\alpha^2)^2 + \sum_{\alpha,\beta} (\mathrm{Tr}\, A_\alpha A_\beta)^2 - 2|\sigma|^2\} *1 \,. \tag{3.6}$$

Now combining (2.2) with (3.2), (3.3) and (3.6), we obtain

$$48\pi^2\, \mathrm{sign}(M) = \int_M \{|\bar{\nabla}\sigma|^2 + 6 - 3\sum_{\alpha,\beta} (\mathrm{Tr}\, A_\alpha A_\beta)^2\} *1 \,. \tag{3.7}$$

From (3.7), (3.5) and (3.1) we get

$$48\pi^2\, \mathrm{sign}(M) \geqq \int_M |\bar{\nabla}\sigma|^2 *1 + \frac{3}{2}\int_M (4 - |\sigma|^4) *1 \geqq \frac{3}{2}\int_M (4 - |\sigma|^4) *1 \,,$$

from which (1.1) follows. Suppose that the equality holds in (1.1), that is,

$$48\pi^2\, \mathrm{sign}(M) = \frac{3}{2}\int_M (4 - |\sigma|^4) *1 \,, \tag{3.8}$$

then M has parallel second fundamental form. On the other hand (3.8) is not satisfied for $M = CP^2(1/2)$. Therefore (3.8) and Lemma 2.1 imply that $M = CP^2(1)$ or $M = CQ^2$.

§4. Proof of Theorem 2.

Let g be the Kaehler metric of M induced from the immersion $j: M \to CP^m(1)$ and ϕ the associated Kaehler form. Now since the total scalar curvature $\int_M \tau *1$ is positive, a result of Yau [7] implies that all plurigenera of M vanish. In particular we have $b_{2,0} = 0$. Then, using $b_{2,0} = 0$, $b_{2,2} = b_{0,0} = 1$, $b_{p,q} = b_{q,p}$ and Serre duality, from (2.2) we obtain

$$\mathrm{sign}(M) = 2 - b_{1,1} \leqq 1 \,.$$

If $\mathrm{sign}(M) = 1$, then $b_{1,1} = 1$ and consequently M is cohomologically Einsteinian, i.e., $c_1 = a\omega$ for some constant a, where $\omega = [\phi]$ is the cohomology class represented by ϕ. On the other hand, by a direct computation we find

$$\phi \wedge \gamma = \frac{\tau}{2}\phi^2 \,. \tag{4.1}$$

Thus by taking integration of both sides of equation (4.1) we obtain $a = (1/2)\mathrm{vol}(M)\int_M \tau *1 > 0$. Therefore M has positive first Chern class. Then from a classification theorem of Yau [7] we have that M is biholomorphic to either $CP^1 \times CP^1$ or to a surface obtained from CP^2 by blowing up k points, $0 \leqq k \leqq 8$, in general position. However, since $b_{1,1} = 1$, M cannot be biholomorphic to $CP^1 \times CP^1$. Since blowing up a point of CP^2 diminishes the signature by one, if $\operatorname{sign}(M)=1$ then M is biholomorphic to the complex projective space CP^2. Let ϕ_0 be the Kaehler form of $CP^m(1)$. If $j^*: H^2(CP^m, \boldsymbol{Z}) \to H^2(M, \boldsymbol{Z})$ is the homomorphism corresponding to the immersion j, then

$$[\phi]=j^*([\phi_0]) \in H^2(M, \boldsymbol{Z}) .$$

Let φ_0 be the Kaehler form of M corresponding to the Fubini-Study metric g_0 of constant holomorphic sectional curvature 1. Since $[\phi]$, $[\varphi_0] \in H^2(M, \boldsymbol{Z})$ and $H^2(M, \boldsymbol{Z})=H^2(CP^2, \boldsymbol{Z}) \cong \boldsymbol{Z}$, we have

$$[\phi]=s[\varphi_0] \qquad \text{for some positive integer } s .$$

Thus we have

$$[\phi]^2=s^2[\varphi_0]^2 \tag{4.2}$$

and

$$[\phi]c_1=s[\varphi_0]c_1 . \tag{4.3}$$

From (4.2) and (4.3) we obtain respectively

$$\operatorname{vol}(M, g)=s^2 \operatorname{vol}(M, g_0) \quad \text{and} \quad \int_M \tau *1=s\int_M \tau_0 *1=6s \operatorname{vol}(M, g_0) .$$

Consequently we have

$$\int_M \tau *1=\frac{6}{s} \operatorname{vol}(M, g) . \tag{4.4}$$

The assumption $\tau \geqq 3$ and $6-\tau=|\sigma|^2 \geqq 0$, together with (4.4), imply $3 \leqq 6/s \leqq 6$ and so either $s=1$ or $s=2$. If $s=1$ we have $\tau=6$ and hence $\sigma=0$. Therefore M is totally geodesic. If $s=2$ we have $\tau=3$. On the other hand it is well-known that every Kaehler metric with constant scalar curvature τ on CP^2 is of constant holomorphic sectional curvature $\tau/6$. Then, from (3.1)-(3.3) and (3.6) we have $\bar{\nabla}\sigma=0$ and by Lemma 2.1 we conclude that M is congruent to the Veronese imbedding $CP^2(1/2)$.

PROOF OF COROLLARY 1. If M has positive total scalar curvature, as before we have $\operatorname{sign}(M)=2-b_2$. Therefore A) of the Corollary 1 follows from Theorem 1.

If M has $\operatorname{sign}(M) \leqq 0$, from Theorem 1 we obtain $\int_M |\sigma|^4 *1 \geqq 4 \operatorname{vol}(M)$. Moreover $\int_M |\sigma|^4 *1=4 \operatorname{vol}(M)$ implies $\operatorname{sign}(M)=0$ and the equality in (1.1). Therefore M is necessarily congruent to CQ^2.

PROOF OF COROLLARY 2. If $\tau\geqq 4$, then from inequality (1.1) we have

$$32\pi^2 \operatorname{sign}(M)\geqq\int_M (4-|\sigma|^4) *1=\int_M (\tau-4)(2+|\sigma|^2)*1\geqq 0 .$$

On the other hand, as in the proof of Theorem 2 we have $\operatorname{sign}(M)=2-b_2$. Therefore $b_2=1$ or $b_2=2$. If $b_2=1$, then $\operatorname{sign}(M)=1$ and $\tau\geqq 4>3$, so Theorem 2 implies that $M=CP^2(1)$. If $b_2=2$, then the equality holds in (1.1) and $\operatorname{sign}(M)=0$, so Theorem 1 implies that $M=CQ^2$.

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