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Asymptotic Distribution of Eigenvalues of Non-Symmetric Elliptic Operators

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Introduction.

In this paper we study asymptotic formulas of distributions of eigenvalues of operators associated with strongly elliptic sesquilinear forms which have non-symmetric top terms.

For operators associated with symmetric forms, Maruo-Tanabe [6] and Tsujimoto [10, 11] gave remainder estimates depending upon the smoothness of the coefficients. Maruo [7] refined and extended the results of [6] to the forms which have symmetric top terms and non-symmetric lower terms. For differential operators, Robert [9] obtained the same results as [7].

As for operators associated with the forms with non-symmetric top terms, however, only the result by Watanabe [13] seems to have been given. He assumed C^h-smoothness for the coefficients of the top terms to clarify his intention to give the formula for eigenvalues which distribute in a sector of the complex plane.

The purpose of this paper is to establish an asymptotic formula with the optimal remainder estimate in the case of C^{1+h}-smoothness. In the symmetric case, our result coincides with Maruo's formula.

Now, we explain notations before stating our result. Let $\Omega\subset \boldsymbol{R}^n$ $(n\geqq 2)$ be a bounded domain possessing the restricted cone property (see [1, p. 11]). Set $\Omega_\varepsilon=\{x\in\Omega;\ \delta(x)\geqq\varepsilon\}$ with $\varepsilon>0$ where $\delta(x)=\min\{1, \mathrm{dist}(x, \partial\Omega)\}$ for $x\in\Omega$. We impose on Ω the following condition: There exists a constant $C>0$ such that for any $\varepsilon>0$

(0.1) $$\int_{\Omega_\varepsilon}\delta^{-1}(x)dx<C|\log\varepsilon|\ ;$$

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$$\int_{\Omega\setminus\Omega_\varepsilon} dx < C\varepsilon\ . \tag{0.2}$$

We write $x=(x_1, \cdots, x_n)\in \boldsymbol{R}^n$, $x^\alpha = x^{\alpha_1}\cdots x^{\alpha_n}$, $D_k=-i\partial/\partial x_k$, $D^\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n}$ for a multi-index $\alpha=(\alpha_1, \cdots, \alpha_n)$ of nonnegative integers. For an integer $m\geqq 0$, $H_m(\Omega)$ denotes the usual Sobolev space of order m with norm $\|\ \ \|_m$. Let V be a closed subspace of $H_m(\Omega)$ containing $\overset{0}{H}_m(\Omega)$, which is the closure of $C_0^\infty(\Omega)$ in $H_m(\Omega)$. Let B be an integro-differential sesquilinear form on $V\times V$ of order m $(2m>n)$ with bounded coefficients:

$$B[u, v]=\int_\Omega \sum_{|\alpha|,|\beta|\leqq m} a_{\alpha\beta}(x)D^\alpha u\overline{D^\beta v}dx\ .$$

We impose on B the following conditions (0.3)~(0.5).

(0.3) For $|\alpha|=|\beta|=m$, $a_{\alpha\beta}(x)$ belongs to $C^{1+h}(\Omega)$ for some $0<h\leqq 1/2$.

(0.4) There exists a constant $\delta_0>0$ such that for any $u\in V$

$$\operatorname{Re} B[u, u]\geqq \delta_0\|u\|_m^2\ .$$

(0.5) For some constant $0<\theta<\pi/4$

$$\{B[u, u];\ u\in V\}\subset\Gamma=\{\lambda\in \boldsymbol{C};\ |\arg\lambda|\leqq\theta\}\ .$$

Let A be the operator associated with the form B: An element u of V belongs to $D(A)$ and $Au=f\in L_2(\Omega)$ if $B[u, v]=(f, v)_{L_2}$ is valid for any $v\in V$. It is well known (see [1]) that the spectrum of A consists of discrete eigenvalues with finite multiplicity and has no accumulation point. For $t>0$, $N(t)$ denotes the number of eigenvalues of A whose real parts are smaller or equal to t with repetition according to the multiplicities.

We now state our main result.

THEOREM. *Assume* (0.1)~(0.5). *Then for any* σ *satisfying* $0<\sigma<(h+1)/(h+3)$, *the following asymptotic formula for* $N(t)$ *holds as* $t\to\infty$:

$$\left|N(t)-\left\{\operatorname{Re} C_0\left\{\frac{\sin(n(\pi-\varphi)/2m)}{n\pi/2m}-\tilde{C}_1\sin\theta\right\}\sec^{n/4m}2\theta\right\}t^{n/2m}\right| \tag{0.6}$$
$$\leqq\{\operatorname{Re} C_0\tilde{C}_2\sin\theta\sec^{n/4m}2\theta\}t^{n/2m}$$
$$+O(t^{n/2m-(h+1-\sigma(h+3))/2m})\sqrt{\sec 2\theta-1}+O(t^{(n-\sigma)/2m})\ ,$$

where

$$\varphi=\tan^{-1}(\sqrt{\sec 2\theta-1})\ ,$$
$$C_0=\int_\Omega C_0(x)dx\ ,$$

(0.7)
$$C_0(x)=\int_{R^n}(2\pi)^{-n}\Big(\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\xi^{\alpha+\beta}+1\Big)d\xi ,$$
$$\tilde{C}_1=\sqrt{2}\,\pi^{-1}\cos((1-n/2m)(\pi-\varphi)) ,$$
$$\tilde{C}_2=\sqrt{2}\cos(\theta+\pi/4)^{-1}\sin((1-n/2m)(\pi-\varphi)) .$$

We shall briefly explain our approach. Our method is based upon the resolvent kernel estimates obtained in [6] and Tauberian theorem obtained in [12]. The main difficulty is that $(Au, u)_{L_2}$ belongs to Γ and hence the results previously obtained for self-adjoint operators are not applicable directly. For example, in estimation of the kernel $K_\lambda(x, y)$ of $(A-\lambda)^{-1}$ for λ in the complement Γ^c of Γ in C, A has to be approximated by the nice operator whose numerical range is contained exactly in Γ. Hence, the greater part of this paper is occupied in this task.

This paper is organized as follows. In Section j ($j=1\sim 4$) we define approximate forms B^j of B^{j-1} and estimate $K_\lambda^j(x, x)-K_\lambda^{j-1}(x, x)$, where K_λ^j is the resolvent kernel associated with A^j and B^j, and $K_\lambda^0=K_\lambda$, $A^0=A$ and $B^0=B$. Section 5 is devoted to the proof of Theorem by summing up the results obtained in Sections $1\sim 4$.

The author wishes to express his sincere gratitude to the referee for his many valuable advices, by which for example the formula in Theorem was improved in the present form. The original version only contained the second remainder for $0<\sigma<(h+1)/(h+4)$.

§ 1. Fundamental properties of resolvent kernels.

In this section we recall some fundamental properties of resolvent kernels.

It is well known that A is a densely defined closed operator in $L_2(\Omega)$ and its adjoint A^* is given by the form $B^*[u, v]=\overline{B[v, u]}$. Denote by V^* the antidual space of V with norm $\|\ \|_{V^*}$ and by $(\ ,\ )_{V^*\times V}$ the duality between V^* and V. We extend A to a mapping from V to V^*. This extended operator denoted again by A is defined by

$$B[u, v]=(Au, v)_{V^*\times V} \qquad \text{for any} \quad v\in V .$$

By Lemma 3.1 in [13], the resolvent set $\rho(A)$ of A contains Γ^c. Therefore, by Lemma 1.2, $(A-\lambda)^{-1}$ has a kernel $K_\lambda^0(x, y)\in C(\bar{\Omega}\times\bar{\Omega})$ for $\lambda\in\Gamma^c$. We denote by $K_\lambda^1(x, y)$ the resolvent kernel of the operator A^1, which is associated with B^1, the form B under the Dirichlet boundary condition. Then the following estimate holds.

LEMMA 1.1 ([13, Lemma 4.1]). *For any $p>0$ there exists a constant*

$C_p>0$ such that for any $\lambda\in\Gamma^c$ and $x\in\Omega$

$$|K_\lambda^0(x,x)-K_\lambda^1(x,x)|\leqq C_p|\lambda|^{n/2m}d(\lambda,\Gamma)^{-1}(\delta^{-1}(x)|\lambda|^{1-n/2m}d(\lambda,\Gamma)^{-1})^p\ . \tag{1.1}$$

The estimate (1.1) follows from the following two lemmas.

LEMMA 1.2 ([6, Lemma 3.2]). *Let S be a bounded operator on V^* to V. Then S has a kernel $M(x,y)$ such that for any $f\in L_2(\Omega)$*

$$(Sf)(x)=\int_\Omega M(x,y)f(y)dy\ .$$

$M(x,y)$ is continuous in $\bar\Omega\times\bar\Omega$ and there exists a constant $C>0$ such that for any x, y in Ω

$$|M(x,y)|\leqq C\|S\|_{V^*\to V}^{(n/2m)^2}\|S\|_{V^*\to L_2}^{(1-n/2m)(n/2m)}\|S\|_{L_2\to V}^{(1-n/2m)(n/2m)}\|S\|_{L_2\to L_2}^{(1-n/2m)^2}\ , \tag{1.2}$$

where $\|\ \|_{X\to Y}$ denotes the norm of bounded operator from X to Y.

LEMMA 1.3 ([13, Lemma 3.3]). *There exists a constant $C>0$ such that for any $\lambda\in\Gamma^c$ and integer k with $0\leqq k\leqq m$*

$$\|(A-\lambda)^{-1}f\|_k\leqq C|\lambda|^{k/2m}d(\lambda,\Gamma)^{-1}\|f\|_0 \qquad \text{for}\quad f\in L_2(\Omega)\ , \tag{1.3}$$

$$\|(A-\lambda)^{-1}f\|_k\leqq C|\lambda|^{(m+k)/2m}d(\lambda,\Gamma)^{-1}\|f\|_{V^*} \qquad \text{for}\quad f\in V^*\ . \tag{1.4}$$

Applying Lemma 1.2 and Lemma 1.3 to the operator $(A^1-\lambda)^{-1}|_{V^*}-(A-\lambda)^{-1}$, we obtain (1.1).

§ 2. Approximation of coefficients by C^∞ functions.

In this section we approximate B^1 by a nice form B^2 and estimate $K_\lambda^2(x,x)-K_\lambda^1(x,x)$, where K_λ^2 is the resolvent kernel of the operator A^2 associated with B^2.

Let ϕ be a real valued even function in $C_0^\infty(t\in\boldsymbol{R};\ |t|<n^{-1/2})$ such that $\int_{\boldsymbol{R}}\phi(t)dt=1$. We put $\varphi(x)=\phi(x_1)\cdots\phi(x_n)$ for $x=(x_1,\cdots,x_n)\in\boldsymbol{R}^n$. Hereafter, we fix $x_0\in\Omega$. For $\eta>0$, we set $\varphi_\eta(x)=\eta^{-n}\varphi((x-x_0)/\eta)$. For $\delta>0$ and $a_{\alpha\beta}(x)$ with $|\alpha|=|\beta|=m$, we put

$$a_{\alpha\beta,0}(x)=\begin{cases}\sum_{|\gamma|\leqq 1}(x-x_0)^\gamma\partial_x^\gamma a_{\alpha\beta}(x_0) & \text{for}\quad |x-x_0|\leqq\delta\ ,\\ \sum_{|\gamma|\leqq 1}(\tilde x-x_0)^\gamma\partial_x^\gamma a_{\alpha\beta}(x_0) & \text{for}\quad |x-x_0|>\delta\ ,\end{cases} \tag{2.1}$$

where $\tilde x$ is the point of intersection of the sphere $|x-x_0|=\delta$ and the line segment connecting x_0 and x. Moreover, put $b_{\alpha\beta}(x)=\varphi_\eta*a_{\alpha\beta,0}(x)$, where $*$ denotes the convolution of φ_η and $a_{\alpha\beta,0}$. Then $b_{\alpha\beta}(x)$ has the following properties (cf. [6, Lemma 5.2]): $b_{\alpha\beta}(x)$ belongs to $C^\infty(\boldsymbol{R}^n)$ and if $\eta<\delta$, $b_{\alpha\beta}(x)=a_{\alpha\beta,0}(x)$ in the set $\{x\in\boldsymbol{R}^n;\ |x-x_0|<\delta-\eta\}$ and moreover, for any $x\in\boldsymbol{R}^n$

$$|b_{\alpha\beta}(x)-a_{\alpha\beta}(x_0)|\leqq\delta\sum_{|\gamma|=1}|\partial^\gamma a_{\alpha\beta}(x_0)|\ .$$

Consider the following approximate form of B^1:

$$B'_{\delta,\eta}[u,\ v]=\int_\Omega\sum_{|\alpha|=|\beta|=m}b_{\alpha\beta}(x)D^\alpha u\overline{D^\beta v}dx\qquad\text{for}\quad u,\ v\in H_m^0(\Omega)\ .$$

Then using the ellipticity of B, the Fourier transform and the interpolation inequality, we obtain that there exist positive constants C_1 and δ_1 such that for any $\delta\leqq\delta_1$, $\eta<\delta$ and $u\in H_m^0(\Omega)$

$$\text{Re}\ B'_{\delta,\eta}[u,\ u]\geqq\delta_1\|u\|_m^2-C_1\|u\|_0^2\ . \tag{2.2}$$

Now, we put

$$B^2[u,\ v]=B'_{\delta,\eta}[u,\ v]+C_1(u,\ v)\qquad\text{for}\quad u,\ v\in H_m^0(\Omega)\ . \tag{2.3}$$

Note that by definition of B^2, $B^2[u,\ u]$ does not, in general, belong to Γ for $u\in H_m^0(\Omega)$.

LEMMA 2.1. *There exist positive constants C_2 and δ_2 such that for any $\delta<\delta_2$ and $\eta<\delta$*

$$\{B^2[u,\ u];\ u\in H_m^0(\Omega)\}\subset\Gamma_\delta=\{\lambda\in\boldsymbol{C};\ |\arg\lambda|\leqq\theta_\delta\}\ ,$$

where $\theta_\delta=\cos^{-1}(\cos\theta-C_2\delta)$.

PROOF. We have only to prove the assertion for $u\in H_m^0(\Omega)$ satisfying $\|u\|_m=1$. We write

$$\begin{aligned}B^2[u,\ u]=&\left\{\int_\Omega\sum_{|\alpha|=|\beta|=m}(b_{\alpha\beta}(x)-a_{\alpha\beta}(x_0))D^\alpha u\overline{D^\beta v}dx\right\}\\&+\left\{\int_\Omega\sum_{|\alpha|=|\beta|=m}a_{\alpha\beta}(x_0)D^\alpha u\overline{D^\beta v}dx+C_1(u,\ u)\right\}=I_1+I_2\ .\end{aligned}$$

Since $\sum_{|\alpha|=|\beta|=m}a_{\alpha\beta}(x_0)\xi^{\alpha+\beta}\in\Gamma$ for almost all $x_0\in\Omega$ and all $\xi\in\boldsymbol{R}^n$ (cf. [13, Lemma 3.6]), we obtain

$$\begin{aligned}|\text{Im}\ I_2|&=\left|\text{Im}\int\sum_{|\alpha|=|\beta|=m}a_{\alpha\beta}(x_0)|\hat{u}|^2\xi^{\alpha+\beta}d\xi\right|\\&\leqq\tan\theta\ \text{Re}\int\sum_{|\alpha|=|\beta|=m}a_{\alpha\beta}(x_0)|\hat{u}|^2\xi^{\alpha+\beta}d\xi\leqq\tan\theta\ \text{Re}\ I_2\ ,\end{aligned}$$

where $\hat{u}$ denotes the Fourier transform of u. This inequality implies that I_2 belongs to Γ. By (2.2) and (2.3) we obtain that there exists a constant $C>0$ such that

$$\operatorname{Re} B^2[u, u] \geqq \operatorname{Re} I_2 - C\delta \|u\|_m^2 = \operatorname{Re} I_2 - C\delta ,$$
$$|B^2[u, u]| \leqq |I_2| + C\delta \|u\|_m^2 = |I_2| + C\delta ,$$
$$|I_2| \geqq \operatorname{Re} I_2 \geqq \delta_1 \|u\|_m^2 = \delta_1 .$$

Using the above inequalities, we obtain

$$\begin{aligned} \text{(2.4)} \qquad \operatorname{Re} B^2[u, u]/|B^2[u, u]| &\geqq (\operatorname{Re} I_2 - C\delta)/(|I_2| + C\delta) \\ &\geqq \operatorname{Re} I_2/|I_2| - |(\operatorname{Re} I_2 - C\delta)/(|I_2| + C\delta) - \operatorname{Re} I_2/|I_2|\,| \\ &\geqq \operatorname{Re} I_2/|I_2| - 2C\delta/\delta_1 \geqq \cos\theta - 2C\delta/\delta_1 . \end{aligned}$$

Choose δ so small that $\cos\theta - 2C\delta/\delta_1 > 1/\sqrt{2}$. Then we get our assertion by putting $C_2 = 2C/\delta_1$ and $\delta_2 = \min\{\delta_1, (\cos\theta - 1/\sqrt{2})\delta_1/2C\}$ in (2.4). Q.E.D.

Denote by A^2 the operator associated with B^2 under the Dirichlet boundary condition and by $K_\lambda^2(x, y)$ the resolvent kernel of A^2. By Lemma 2.1, the resolvent set $\rho(A^2)$ of A^2 contains Γ_δ^c, and just as in Lemma 1.3, there exists a constant $C>0$ such that for any $\lambda \in \Gamma_\delta^c$, $\delta < \delta_2$ and $\eta < \delta$

$$\text{(2.5)} \qquad \|(A^2 - \lambda)^{-1}\|_{(-p,q)} \leqq C|\lambda|^{(p+q)/2m} d(\lambda, \Gamma_\delta)^{-1} ,$$

where $\|\ \ \|_{(-p,q)}$ ($p, q = 0$ or m) denote the norms of bounded operators from $H_{-p}(\Omega)$ to $\overset{\circ}{H}_q(\Omega)$ and $d(\lambda, \Gamma_\delta) = \operatorname{dist}(\lambda, \Gamma_\delta)$.

In what follows, we put $\eta = \delta/2$ in (2.2). The following lemma is the aim of this section.

PROPOSITION 2.2. *For any $j \in N$ there exists a constant $C_j > 0$ such that for any $x_0 \in \Omega$, $0 < 2\rho < \delta < \delta_2$ and λ with $\rho|\lambda|^{1/2m} \geqq 1$ $(j=1)$ and $\rho d(\lambda, \Gamma_\delta)|\lambda|^{1/2m-1} \geqq 1$ $(j \geqq 2)$*

$$\begin{aligned} \text{(2.6)} \qquad |K_\lambda^1(x_0, x_0) - K_\lambda^2(x_0, x_0)| \leqq\, & C_j(\rho^{1+h} + |\lambda|^{-1/2m})|\lambda|^{1+n/2m} d(\lambda, \Gamma_\delta)^{-2} \\ & + C_j |\lambda|^{n/2m} d(\lambda, \Gamma_\delta)^{-1} (\rho^{-1} d(\lambda, \Gamma_\delta)^{-1} |\lambda|^{1-n/2m})^j . \end{aligned}$$

We can show this proposition by a similar method to that used in the proof of [6, Lemma 6.2] by replacing $d(\lambda)$ with $d(\lambda, \Gamma_\delta)$. Hence we omit the proof.

§ 3. Approximation by operators in a ball.

In this section we approximate B^2 by B^3, which is defined in the ball $\Omega_R = \{x \in \boldsymbol{R}^n; |x| < R\}$ containing $\bar{\Omega}$. Main result here is Proposition 3.4.

We begin with Agmon's lemma which will be used later.

LEMMA 3.1 ([2, Theorem 3.11]). *Let T be a bounded linear operator on $L_2(\boldsymbol{R}^n)$ whose range $R(T)$ and the range $R(T^*)$ of its adjoint operator are contained in $H_{2m}(\boldsymbol{R}^n)$, where $2m>n$. Then T has a bounded continuous kernel $N(x, y)$ with the estimate*

$$(3.1)\qquad |N(x, y)|\leqq C(\|T\|_{L_2\to H_{2m}}+\|T^*\|_{L_2\to H_{2m}})^{n/2m}\|T\|_{L_2\to L_2}^{1-n/2m} ,$$

where C is a constant depending only on m and n.

Since $b_{\alpha\beta}(x)$ belongs to $C^\infty(\boldsymbol{R}^n)$, the following form B^3 is well defined for $u, v\in H_m^0(\Omega_R)$:

$$B^3[u, v]=\int_{\Omega_R}\sum_{|\alpha|=|\beta|=m} b_{\alpha\beta}(x)D^\alpha u\overline{D^\beta v}dx+C_1(u, v) .$$

Denote by A^3 and $(A^3)^*$ the associated operators with B^3 and its adjoint operator, respectively. Then, by a well known regularity theorem for elliptic operators, $D(A^3)=D((A^3)^*)=H_m^0(\Omega_R)\cap H_{2m}(\Omega_R)$ and for any $u\in H_m^0(\Omega_R)\cap H_{2m}(\Omega_R)$

$$A^3u=\sum_{|\alpha|=|\beta|=m} D^\beta(b_{\alpha\beta}(x)D^\alpha u)+C_1 ,$$
$$(A^3)^*u=\sum_{|\alpha|=|\beta|=m} D^\alpha(\overline{b_{\alpha\beta}(x)}D^\beta u)+C_1 .$$

LEMMA 3.2. *Let $K_1>0$ be a given constant and T_λ be a bounded linear operator on $L_2(\Omega)$ whose range and the range of its adjoint operator T_λ^* are contained in $H_{2m}(\Omega)$ for λ with $d(\lambda, \Gamma_\delta)\geqq K_1|\lambda|^{1-1/2m}$. Suppose that there exists a constant $C>0$ such that for any $\delta<\delta_2$*

$$\|T_\lambda\|_{L_2\to L_2}\leqq Cd(\lambda, \Gamma_\delta)^{-1} ,$$
$$\|T_\lambda\|_{L_2\to H_{2m}}\ (\textit{resp.}\ \|T_\lambda^*\|_{L_2\to H_{2m}})\leqq C|\lambda|d(\lambda, \Gamma_\delta)^{-1} .$$

Furthermore, for any $f\in L_2(\Omega)$ let T_λ satisfy

$$(3.2)\qquad (A^3-\lambda)T_\lambda f=0 ,\qquad ((A^3)^*-\lambda)T_\lambda^* f=0 .$$

Let $N_\lambda(x, y)$ be the kernel of T_λ and $\delta(x, y)=\min\{1, \delta(x), \delta(y)\}$. Then for any $p>0$ there exists a constant $C_p>0$ such that for any $\delta<\delta_2$

$$(3.3)\qquad |N_\lambda(x, y)|\leqq C_p|\lambda|^{n/2m}d(\lambda, \Gamma_\delta)^{-1}(|\lambda|^{1-1/2m}\delta(x, y)^{-1}d(\lambda, \Gamma_\delta)^{-1})^p .$$

We can prove this lemma as in the proof of [3, Lemma 5.1]. Hence we omit the proof.

For $u\in H_m^0(\Omega)$, let $\tilde{u}=u$ in Ω and $\tilde{u}=0$ in $\Omega_R\backslash\Omega$. Then $\tilde{u}\in H_m^0(\Omega_R)$. Furthermore, for $f\in H_{-m}(\Omega_R)$, $rf\in H_{-m}(\Omega)$ stands for the restriction of f on $H_m^0(\Omega)$. Take $\xi_\omega(x)\in C_0^\infty(\{x\in\boldsymbol{R}^n; |x-x_0|<\omega=\delta(x_0)\})$ such that $\xi_\omega(x_0)=1$.

For $H_{-m}(\Omega_R)$, we set

$$G_{\lambda_\omega}f=\xi_\omega\{((A^2-\lambda)^{-1}(rf))^{\sim}-(A^3-\lambda)^{-1}f\}\ .$$

LEMMA 3.3. *Let $\tilde{p}$, $q=0$ or m. Then for any $p>0$ there exists a constant $L_p>0$ such that for any $x_0\in\Omega$, $\delta<\delta_2$, $\lambda\in\Gamma_\delta^c$ with $\omega|\lambda|^{1/2m}\geqq 1$*

$$\|G_{\lambda_\omega}\|_{(-\tilde{p},q)}\leqq L_p(|\lambda|^{1-1/2m}\omega^{-1}d(\lambda,\Gamma_\delta)^{-1})^p|\lambda|^{(\tilde{p}+q)/2m}d(\lambda,\Gamma_\delta)^{-1}\ . \tag{3.4}$$

PROOF. Let $u=((A^2-\lambda)^{-1}(rf))^{\sim}-(A^3-\lambda)^{-1}f$ and $v=\xi_\omega u$ for $f\in H_{-m}(\Omega_R)$. Noting that the support of v is contained in Ω, we obtain

$$B^3[v,v]-\lambda(v,v)=B^3[v,v]-B^3[u,\xi_\omega v]\ ,$$

which together with (2.5) yields (3.4). Q.E.D.

Combining Lemmas 1.2 and 3.3, we get the following proposition.

PROPOSITION 3.4. *For any $p>0$, there exists a constant $L_p>0$ such that for any $x_0\in\Omega$, $\delta<\delta_2$, and $\lambda\in\Gamma_\delta^c$ with $\delta(x_0)|\lambda|^{1/2m}\geqq 1$*

$$\begin{aligned}&|K_\lambda^3(x_0,x_0)-K_\lambda^2(x_0,x_0)|\\&\quad\leqq L_p|\lambda|^{n/2m}d(\lambda,\Gamma_\delta)^{-1}(|\lambda|^{1-1/2m}\delta(x_0)^{-1}d(\lambda,\Gamma_\delta)^{-1})^p\ .\end{aligned} \tag{3.5}$$

§ 4. Approximation by operator in $\boldsymbol{R}^n$.

This section is devoted to the study of the resolvent kernel of the operator A^4, which is defined by the differential operator A^3 and whose domain $D(A^4)=H_{2m}(\boldsymbol{R}^n)$. The final objects in this section are formulated in Propositions 4.3 and 4.6.

To obtain the asymptotic formula of K_λ^4, we shall employ the method used by Nagase [8] and construct a parametrix of A^4. The calculation in this section is rather complicated since the estimates of derivatives of $b_{\alpha\beta}$ depend on parameter δ and (A^4u,u) only belongs to the sector Γ_δ. For pseudo-differential operators we shall refer the reader to the book [5] for example.

By definition, A^4 is a closed operator on $L_2(\boldsymbol{R}^n)$ and its adjoint operator coincides with its formally adjoint operator with the domain $D((A^4)^*)=H_{2m}(\boldsymbol{R}^n)$. Furthermore, the resolvent set $\rho(A^4)$ contains Γ_δ^c. Applying Lemma 3.1 to A^4, we obtain that for $\lambda\in\Gamma_\delta^c$ the kernel $K_\lambda^4(x,y)$ of $(A^4-\lambda)^{-1}$ exists. It is easy to see that $(A^4-\lambda)^{-1}$ and $((A^4)^*-\lambda)^{-1}$ satisfy the estimates (1.3) and (1.4) with Γ replaced by Γ_δ. Therefore, by (3.1) we obtain for any $x,\ y\in\boldsymbol{R}^n$

$$|K_\lambda^4(x, y)| \leqq C|\lambda|^{n/2m} d(\lambda, \Gamma_\delta)^{-1} . \tag{4.1}$$

We write

$$A^4 u = \sum_{j=1}^{2m} A_j(x, D)u ,$$

where $A_j(x, \xi)$ is a homogeneus polynomials in ξ of degree j. For $\lambda \in \boldsymbol{C}$ and $x, \xi \in \boldsymbol{R}^n$, define the symbols $q_k(\lambda, x, \xi)$ $(k=0, 1, 2, \cdots)$ by

$$(A_{2m}(x, \xi)-\lambda)q_0(\lambda, x, \xi)=0 ,$$

$$(A_{2m}(x, \xi)-\lambda)q_k(\lambda, x, \xi)+\sum_{\substack{i+j+|\mu|=k \\ i\neq k}} \frac{1}{\mu!} A_{2m-j}^{(\mu)}(x, \xi) q_{i(\mu)}(\lambda, x, \xi)=0 ,$$

where $A_{(\nu)}^{(\mu)} = \partial_\xi^\mu D_x^\nu A$. Furthermore denote

$$R_N(\lambda, x, \xi)=\sum_{\substack{k+j+|\mu|\geqq N+1 \\ k\leqq N}} \frac{1}{\mu!}\{(A_{2m}^{(\mu)}(x, \xi)-\lambda)q_{k(\mu)}(\lambda, x, \xi)$$
$$+A_{2m-j}^{(\mu)}(x, \xi)q_{k(\mu)}(\lambda, x, \xi)\} .$$

The following lemma is a variant of [8, Lemma 2.1].

LEMMA 4.1. *Let $K_1>0$ be a given constant and δ_2 be the constant defined in Lemma 2.1. Then there exists a positive constant C such that for any $\delta<\delta_2$, $0<\tau<1$, $x, \xi \in \boldsymbol{R}^n$ and $|\lambda|>1$ with $d(\lambda, \Gamma_\delta)\geqq K_1|\lambda|^{1-\tau}$*

$$|A_{2m}(x, \xi)-\lambda| \geqq C(|\xi|^{2m}+|\lambda|)^{1-\tau} . \tag{4.2}$$

PROOF. Let θ_δ be the positive number defined in Lemma 2.1 and $c_0>0$ satisfy $|A_{2m}(x, \xi)|\geqq c_0|\xi|^{2m}$ for any $x, \xi \in \boldsymbol{R}^n$. Note that c_0 can be chosen independently of $\delta<\delta_2$. We fix $x, \xi \in \boldsymbol{R}^n$. We set

$$s_1=\{\lambda \in \boldsymbol{C};\ \arg\lambda=\arg A_{2m}(x, \xi)\} ,$$
$$D_1=\{\lambda \in \Gamma_\delta^c;\ |\arg\lambda-\arg A_{2m}(x, \xi)|\geqq \pi/2\} .$$

For $\lambda \in \Gamma_\delta^c \cap D_1^c$, let $P\lambda$ be the orthogonal projection of λ on s_1. We set

$$D_2=\{\lambda \in \Gamma_\delta^c;\ |\arg\lambda-\arg A_{2m}(x, \xi)|<\pi/2,\ |P\lambda|<c_0|\xi|^{2m}/2\} ,$$
$$D_3=\{\lambda \in \Gamma_\delta^c;\ |\arg\lambda-\arg A_{2m}(x, \xi)|<\pi/2,\ |P\lambda|\geqq c_0|\xi|^{2m}/2\} .$$

First case; $\lambda \in D_1$. Then we obtain

$$|A_{2m}(x, \xi)-\lambda| \geqq (|A_{2m}(x, \xi)|^2+|\lambda|^2)^{1/2}$$
$$\geqq \frac{|A_{2m}(x, \xi)|+|\lambda|}{\sqrt{2}} \geqq C(|\xi|^{2m}+|\lambda|) \geqq C(|\xi|^{2m}+|\lambda|)^{1-\tau} .$$

Second case; $\lambda \in D_2$. Then we obtain

$$|A_{2m}(x, \xi)-\lambda| = (|A_{2m}(x, \xi)-P\lambda|^2+|P\lambda-\lambda|^2)^{1/2}$$
$$\geqq \frac{|A_{2m}(x, \xi)-P\lambda|+|P\lambda-\lambda|}{\sqrt{2}} \geqq \frac{|A_{2m}(x, \xi)|-|P\lambda|+|P\lambda-\lambda|}{\sqrt{2}}$$
$$\geqq \frac{1}{\sqrt{2}}\left\{\left(|A_{2m}(x, \xi)|-|P\lambda|-\frac{c_0|\xi|^{2m}}{4}\right)+\left(\frac{c_0|\xi|^{2m}}{4}+|P\lambda-\lambda|\right)\right\}.$$

By definition of D_2, we obtain that

$$|A_{2m}(x, \xi)|-|P\lambda|-\frac{c_0|\xi|^{2m}}{4} \geqq c_0|\xi|^{2m}-\frac{c_0|\xi|^{2m}}{2}-\frac{c_0|\xi|^{2m}}{4} \geqq \frac{c_0|\xi|^{2m}}{4},$$
$$|P\lambda-\lambda|+\frac{c_0|\xi|^{2m}}{4} \geqq \frac{|P\lambda-\lambda|}{2}+\frac{c_0|\xi|^{2m}}{4} \geqq \frac{|P\lambda-\lambda|+|P\lambda|}{2} \geqq \frac{|\lambda|}{2}.$$

Combining the above inequalities, we also obtain (4.2) in this case.
Last case; $\lambda \in D_3$. Then,

$$|A_{2m}(x, \xi)-\lambda| \geqq d(\lambda, \Gamma_\delta) \geqq K_1|\lambda|^{1-\tau} \geqq K_1\frac{|P\lambda|^{1-\tau}}{2}+K_1\frac{|\lambda|^{1-\tau}}{2}$$
$$\geqq K_1\frac{(c_0|\xi|^{2m}/2)^{1-\tau}}{2}+K_1\frac{|\lambda|^{1-\tau}}{2} \geqq C(|\xi|^{2m}+|\lambda|)^{1-\tau}.$$

Thus we get (4.2). Q.E.D.

Here, following Nagase's method [8, § 3], we prepare the estimates of $q_k^{(\mu)}$ and $R_k^{(\mu)}$. Hereafter, we fix an arbitrary ε with $0<\varepsilon<1/4m$.

LEMMA 4.2. *Let $K_1>0$ be a given constant. Then for all $k \in \boldsymbol{N} \cup \{0\}$, μ and ν, there exists a constant $C_{\mu,\nu,k}>0$ such that for any $x, \xi \in \boldsymbol{R}^n$, $\delta<\delta_2$ and $|\lambda|>1$ with $d(\lambda, \Gamma_\delta) \geqq K_1|\lambda|^{1-1/4m+\varepsilon}$*

(4.3) $$|q_{k(\nu)}^{(\mu)}(\lambda, x, \xi)| \leqq C_{\mu,\nu,k}\delta^{-(k+|\nu|)}(|\xi|^{2m}+|\lambda|)^{(1/4m-\varepsilon)(2k+|\mu|+|\nu|)-(k+|\mu|)/2m},$$

(4.4) $$|R_{k(\nu)}^{(\mu)}(\lambda, x, \xi)| \leqq C_{\mu,\nu,k}\delta^{-(k+2m+|\nu|)}(|\xi|^{2m}+|\lambda|)^{(1/4m-\varepsilon)(2k+|\mu|+|\nu|+2m)-(k+|\mu|)/2m+2}.$$

PROOF. By definition of $b_{\alpha\beta}$ and mathematical induction on k and $|\mu|+|\nu|$, we can easily obtain for $k=1, 2, \cdots$

(4.5) $$q_{k(\nu)}^{(\mu)}(\lambda, x, \xi) = \sum_{j=0}^{2k+|\mu|+|\nu|}\frac{p_{k,j,\mu,\nu}(x, \xi)}{(A_{2m}(x, \xi)-\lambda)^{j+1}},$$

where $p_{k,j,\mu,\nu}(x, \xi)$ are homogeneous polynomials in ξ of degree $2mj-k-|\mu| \geqq 0$ whose coefficients have the forms $\prod_s D^{\gamma_s}b_{\alpha\beta}(x)$ with $\sum_s|\gamma_s| \leqq k+|\nu|$, and $p_{k,j,\mu,\nu}(x, \xi)=0$ for $2mj-k-|\mu|<0$. Then using (4.2) for $\tau=1/4m-\varepsilon$ and

(4.5), we obtain

$$q^{(\mu)}_{k(\nu)}(\lambda, x, \xi)=\sum_{j=1}^{2k+|\mu|+|\nu|}\frac{|p_{k,j,\mu,\nu}(x, \xi)|}{|A_{2m}(x, \xi)-\lambda|^{j+1}}$$
$$\leqq C\delta^{-(k+|\nu|)}\sum_{j=1,2m-k-j\geqq 0}^{2k+|\mu|+|\nu|}|\xi|^{2mj-k-|\mu|}(|\xi|^{2m}+|\lambda|)^{-(1-1/4m+\varepsilon)(j+1)}$$
$$\leqq C\delta^{-(k+|\nu|)}(|\xi|^{2m}+|\lambda|)^{(1/4m-\varepsilon)(2k+|\mu|+|\nu|)-(k+|\mu|)/2m} .$$

Thus, we get (4.3).

We can show (4.4) easily by using (4.3) and (4.5). Q.E.D.

The following is the most important step of this paper, which is obtained from the result of Nagase [8, Theorem] by replacement of $(0, \infty)$ with Γ_δ.

PROPOSITION 4.3. *Let $K_1>0$ be a given constant and $C_0(x)$ be the function defined by* (0.7). *Then for any large $N\in \boldsymbol{N}$ and $k\in \boldsymbol{N}$ with $k>n$ there exists a constant $C_{N,k}>0$ such that for any $x_0\in \boldsymbol{R}^n$, $\delta<\delta_2$ and $|\lambda|>1$ with $d(\lambda, \Gamma_\delta)\geqq K_1|\lambda|^{1-1/4m+\varepsilon}$*

$$|K^4_\lambda(x_0, x_0)-C_0(x_0)(-\lambda)^{n/2m-1}|\leqq C_{N,k}\Big\{\sum_{j=1}^{N}|\lambda|^{(n-j)/2m-1}\delta^{-j} \tag{4.6}$$
$$+|\lambda|^{-(2N+2m)\varepsilon+11/2-k(1/4m-\varepsilon)}\delta^{-(N+2m)}d(\lambda, \Gamma_\delta)^{-1}\Big\} .$$

PROOF. It is known from [8, § 4] that

$$K^4_\lambda(x_0, x_0)=\sum_{j=1}^{N}(-\lambda)^{(n-j)/2m-1}(2\pi)^{-n}\int q_j(-1, x, \xi)d\xi \tag{4.7}$$
$$-\int K^4_\lambda(x_0, z)R'_N(\lambda, z, z-x_0)dz ,$$

where $R'_N(\lambda, x_0, z)=(2\pi)^{-n}\int e^{iz\cdot\xi}R_N(\lambda, x_0, \xi)d\xi$ (cf. [8, § 4]). For any multi-index μ, we obtain

$$z^\mu R'_N(\lambda, x, z)=(2\pi)^{-n}z^\mu\int e^{iz\cdot\xi}R_N(\lambda, x, \xi)d\xi$$
$$=(-i)^{|\mu|}(2\pi)^{-n}\int e^{iz\cdot\xi}R^{(\mu)}_N(\lambda, x, \xi)d\xi .$$

Then by (4.4), there exists a constant $C_\mu>0$ such that

$$|z^\mu R'_N(\lambda, x_0, z)|$$
$$\leqq C_\mu\delta^{-(N+2m)}\int(|\xi|^{2m}+|\lambda|)^{(1/4m-\varepsilon)(2N+2m+|\mu|)-(N+|\mu|)/2m+2}d\xi$$
$$\leqq C_\mu\delta^{-(N+2m)}\int(|\xi|^{2m}+|\lambda|)^{-(2N+2m)\varepsilon+5/2-(1/4m+\varepsilon)|\mu|}d\xi .$$

Choose N so large that the right hand side of the above inequality is integrable in ξ. Then

$$|z^\mu R'_N(\lambda, x_0, z)| \leqq C_\mu \delta^{-(N+2m)} |\lambda|^{-(2N+2m)\varepsilon+7/2-(1/4m+\varepsilon)|\mu|} .$$

Therefore, we obtain

$$|R'_N(\lambda, x_0, z)| \leqq C_\mu \delta^{-(N+2m)} |\lambda|^{-(2N+2m)\varepsilon-(1/4m+\varepsilon)|\mu|+9/2} (1+|z|)^{-|\mu|} .$$

Then by (4.1) we obtain for any $k=|\mu|>n$

$$\left|\int K_\lambda^4(x_0, z) R'_N(\lambda, z, z-x_0) dz\right| \leqq C_{N,k} \delta^{-(N+2m)} |\lambda|^{-(2N+2m)\varepsilon-(1/4m+\varepsilon)k+11/2} d(\lambda, \Gamma_\delta)^{-1} \int |z-x_0|^{-k} dz . \tag{4.8}$$

Since $(2\pi)^{-n}\int q_1(-1, x_0, \xi) d\xi = C_0(x_0)$, we get (4.6) by combining (4.7) and (4.8). Q.E.D.

LEMMA 4.4. *For any $p>0$, there exists a constant $C_p>0$ such that for any $x_0 \in \Omega$, $\delta<\delta_2$, and λ with $d(\lambda, \Gamma_\delta) \geqq |\lambda|^{1-1/2m}$*

$$|K_\lambda^4(x_0, x_0) - K_\lambda^3(x_0, x_0)| \leqq C_p |\lambda|^{n/2m} d(\lambda, \Gamma_\delta)^{-1} (|\lambda|^{1-1/2m} d(\lambda, \Gamma_\delta)^{-1} \delta^{-1}(x_0))^p . \tag{4.9}$$

PROOF. For $f \in L_2(\Omega_R)$, let $f_1=f$ in Ω and $f_1=0$ in $R^n \setminus \Omega$. Put $T_\lambda f = (A^3-\lambda)^{-1} f - (A^4-\lambda)^{-1} f_1|_\Omega$. Then clearly T_λ satisfies (3.2). Hence, by definition of A^4, (2.5) and (3.3), we get (4.9). Q.E.D.

We will here study $d(\lambda, \Gamma_\delta)$ in comparison with $d(\lambda, \Gamma)$.

LEMMA 4.5. *Let $K_1>0$ be a given constant. Then there exists a constant $C>0$ such that for any $\delta<\delta_2$ and $\lambda \in \Gamma_\delta^c$ with $d(\lambda, \Gamma_\delta) \geqq K_1 |\lambda|^{1-1/4m+\varepsilon}$*

$$|d(\lambda, \Gamma)^{-1} - d(\lambda, \Gamma_\delta)^{-1}| \leqq C_3 \delta |\lambda|^{1/4m-\varepsilon} d(\lambda, \Gamma)^{-1} . \tag{4.10}$$

PROOF. We have only to estimate $d(\lambda, \Gamma) - d(\lambda, \Gamma_\delta)$. By simple calculation we obtain, for $t=|\arg \lambda|$, that

$$\begin{aligned} &|d(\lambda, \Gamma) - d(\lambda, \Gamma_\delta)| \\ &\leqq \begin{cases} |\lambda|\,|\sin(t-\theta) - \sin(t-\theta_\delta)| & \text{for } \theta_\delta < t < \pi/2+\theta , \\ |\lambda|\,|1-\sin(t-\theta_\delta)| & \text{for } \pi/2+\theta \leqq t \leqq \pi/2+\theta_\delta , \\ 0 & \text{for } \pi/2+\theta_\delta < t \leqq \pi . \end{cases} \\ &\leqq |\lambda|\,|\theta-\theta_\delta| . \end{aligned} \tag{4.11}$$

Choose $\kappa>0$ so small that $\kappa \leqq \theta \leqq \pi/4-\kappa$. Then there exists a constant

$C_\kappa>0$ such that for any α and β in $[\kappa, \pi/4-k]$

$$|\cos^{-1}\alpha-\cos^{-1}\beta|\leqq C_\kappa|\alpha-\beta| .$$

Furthermore, choose $\delta_2>0$ so small that for any $0<\delta<\delta_2$, θ_δ belongs to $[\kappa, \pi/4-\kappa]$. Then we obtain

$$(4.12)\qquad |\theta-\theta_\delta|\leqq|\theta-\cos^{-1}(\cos\theta-C_2\delta)|\leqq C_\kappa C_2\delta .$$

Using (4.11) and (4.12), we can get (4.10) easily. Q.E.D.

For a given constant $K_1>0$, choose K_2 so large that $K_2-C_3\geqq K_1$. Then putting $\delta=|\lambda|^{-(1/4m-\varepsilon)}$ in (4.10), we obtain by Lemma 4.5 that for $|\lambda|\gg 1$ with $d(\lambda, \Gamma)\geqq K_2|\lambda|^{1-1/4m+\varepsilon}$

$$d(\lambda, \Gamma_\delta)\geqq d(\lambda, \Gamma)-|d(\lambda, \Gamma)-d(\lambda, \Gamma_\delta)|\geqq(K_2-C_3)|\lambda|^{1-1/4m+\varepsilon}\geqq K_1|\lambda|^{1-1/4m+\varepsilon} .$$

Therefore, putting $\delta=|\lambda|^{-(1/4m-\varepsilon)}$ in (1.1), (2.6), (3.5), (4.6) and (4.9) and $k=N>n$ in (4.6), we get the following proposition.

PROPOSITION 4.6. *Let $|\lambda|\gg 1$ satisfy $d(\lambda, \Gamma)\geqq K_2|\lambda|^{1-1/4m+\varepsilon}$. Then for any $p\geqq 0$ and $j\in \boldsymbol{N}\cup\{0\}$ there exist positive constants C_p and L_j such that for any $x_0\in\Omega$ and $0<\rho<|\lambda|^{-(1/4m-\varepsilon)/2}$ with $\rho|\lambda|^{1/2m}\geqq 1$ $(j=1)$ and $\rho^{-1}d(\lambda, \Gamma)^{-1}|\lambda|^{1-1/2m}\leqq 1$ $(j\geqq 2)$*

$$(4.13)\qquad \begin{aligned}&|K_\lambda^0(x_0, x_0)-C_0(x_0)(-\lambda)^{n/2m-1}|\\ &\leqq C_p|\lambda|^{n/2m}d(\lambda, \Gamma)^{-1}(|\lambda|^{1-1/2m}\delta^{-1}(x_0)d(\lambda, \Gamma)^{-1})^p\\ &\quad+L_j|\lambda|^{n/2m}d(\lambda, \Gamma)^{-1}(|\lambda|^{1-1/2m}\rho^{-1}d(\lambda, \Gamma)^{-1})^j\\ &\quad+\sum_{k=1}^{N}|\lambda|^{(n-k)/2m-1+(1/4m-\varepsilon)k}+C_j(\rho^{1+h}+|\lambda|^{-1/2m})|\lambda|^{n/2m+1}d(\lambda, \Gamma)^{-2}\\ &\quad+C_{N,\varepsilon}|\lambda|^{-2(2N+2m)\varepsilon+11/2}d(\lambda, \Gamma)^{-1} .\end{aligned}$$

§ 5. Asymptotic formula for the kernel and eigenvalues.

This section is devoted to the proof of the theorem. To this end, we prove the following proposition.

PROPOSITION 5.1. *Let $0<\varepsilon'<\varepsilon<1/4m$, $\sigma'=1/2-2m\varepsilon'$ and $\sigma=1/2-2m\varepsilon$. Then for any large $N\in\boldsymbol{N}$* and *$j\in\boldsymbol{N}$, there exists a constant $C>0$ such that for any $|\lambda|\gg 1$ with $d(\lambda, \Gamma)\geqq K_2|\lambda|^{1-1/4m+\varepsilon}$ and $0<\eta\ll 1$*

$$(5.1)\qquad \begin{aligned}&\left|\sum_{q=1}^{\infty}\frac{1}{\lambda_q-\lambda}-C_0(-\lambda)^{n/2m-1}\right|\\ &\leqq C|\lambda|^{(n-1/2)/2m-\varepsilon}d(\lambda, \Gamma)^{-1}+C|\lambda|^{(n-1)/2m+1+\eta}d(\lambda, \Gamma)^{-2}\\ &\quad+C|\lambda|^{n/2m-1-(h+1-\sigma(h+3))/2m}(|\lambda|^{1-\sigma'/2m}d(\lambda, \Gamma)^{-1})^2\\ &\quad+C_j|\lambda|^{n/2m}d(\lambda, \Gamma)^{-1}(|\lambda|^{1-\sigma'/2m}d(\lambda, \Gamma)^{-1})^j\\ &\quad+C_N|\lambda|^{-2(2N+2m)\varepsilon+11/2}d(\lambda, \Gamma)^{-1} .\end{aligned}$$

PROOF. It is well known that

$$\int_{\Omega} K_\lambda^0(x, x)dx = \sum_{q=1}^{\infty} \frac{1}{\lambda_q - \lambda} . \tag{5.2}$$

Set

$$\Omega_0 = \{x \in \Omega;\ \delta(x) \leqq |\lambda|^{-1/2m}\} ,$$
$$\Omega_1 = \{x \in \Omega;\ |\lambda|^{-1/2m} < \delta(x) < |\lambda|^{-(1-\sigma')/2m}\} ,$$
$$\Omega_2 = \{x \in \Omega;\ \delta(x) \geqq |\lambda|^{-(1-\sigma')/2m}\} .$$

First, for Ω_0, using (0.2) and (1.2)～(1.4) we obtain

$$\begin{aligned} &\left| \int_{\Omega_0} (K_\lambda^0(x, x) - C_0(x)(-\lambda)^{n/2m-1})dx \right| \\ &\quad \leqq C\int_{\Omega_0} (|\lambda|^{n/2m} d(\lambda, \Gamma)^{-1} + |\lambda|^{n/2m-1})dx \leqq C|\lambda|^{(n-1)/2m} d(\lambda, \Gamma)^{-1} . \end{aligned} \tag{5.3}$$

Next, for Ω_1, put $\rho=\delta(x)$ and $p=j=1$ in (4.13). Then we obtain by (0.1)

$$\begin{aligned} &\left| \int_{\Omega_1} (K_\lambda^0(x, x) - C_0(x)(-\lambda)^{n/2m-1})dx \right| \\ &\quad \leqq C|\lambda|^{(n-1/2)/2m-1-\varepsilon} + (C_1 + L_1)|\lambda|^{1+(n-1)/2m} d(\lambda, \Gamma)^{-2} \log|\lambda| \\ &\qquad + C_1(|\lambda|^{-1/2m} + |\lambda|^{-(1-\sigma')(1+h)/2m})|\lambda|^{1+n/2m} d(\lambda, \Gamma)^{-2} \\ &\qquad + C_N|\lambda|^{-2(2N+2m)+11/2} d(\lambda, \Gamma)^{-1} . \end{aligned} \tag{5.4}$$

Finally, put $\rho=|\lambda|^{-(1-\sigma')/2m}$ and $p=j$ in (4.13). Then we obtain

$$\begin{aligned} &\left| \int_{\Omega_2} (K_\lambda^0(x, x) - C_0(x)(-\lambda)^{n/2m-1})dx \right| \\ &\quad \leqq C|\lambda|^{(n-1/2)/2m-1-\varepsilon} + C_j|\lambda|^{n/2m} d(\lambda, \Gamma)^{-1}(|\lambda|^{1-\sigma'/2m} d(\lambda, \Gamma)^{-1})^j \\ &\qquad + C_j(|\lambda|^{-(1+h)(1-\sigma')/2m} + |\lambda|^{-1/2m})|\lambda|^{n/2m+1} d(\lambda, \Gamma)^{-2} \\ &\qquad + C_N|\lambda|^{-(2N+2m)\varepsilon+11/2} d(\lambda, \Gamma)^{-1} . \end{aligned} \tag{5.5}$$

Combining (5.2)～(5.5), we get (5.1). Q.E.D.

In order to prove the theorem, we use the following Tauberian theorem.

LEMMA 5.2 ([12, Theorem II]). *Let $\{\lambda_j\}$ $(j=1, 2, \cdots)$ be a sequence in the sector $\Gamma=\{\lambda \in C;\ |\arg \lambda| \leqq \theta\}$ for some $0<\theta<\pi/4$. Suppose that the sequence has no accumulation points and that*

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j - \lambda} = a(-\lambda)^{-\alpha} + O(|\lambda|^{-\beta}(|\lambda|^{\gamma} d(\lambda, \Gamma)^{-1})^{p}) \tag{5.6}$$

as $|\lambda|\to\infty$ *with* $d(\lambda, \Gamma)\geqq|\lambda|^{\gamma}$, *where* $0<\alpha<\beta<1$, $0<\gamma<1$, $p>0$ *and* $\mathrm{Re}\, a\geqq 0$. *Then as* $t\to\infty$

$$(5.7)\qquad \Big|\sum_{\mathrm{Re}\,\lambda_j\leq t} 1-\mathrm{Re}\, a\Big\{\frac{\sin((1-\alpha)(\pi-\varphi))}{(1-a)\pi} - \sqrt{2}\cos(\alpha(\pi-\varphi))\sin\theta\Big\}\sec^{(1-\alpha)/2} 2\theta\frac{t^{1-\alpha}}{\pi}\Big|$$
$$\leqq\sqrt{2}\ \mathrm{Re}\, a\,(\cos(\theta+\pi/4))^{-1}\sin(\alpha(\pi-\alpha))\sin\theta\sec^{(1-\alpha)/2}2\theta)t^{1-\alpha} + O(t^{1-\beta})\sqrt{\sec(2\theta-1)}+O(t^{\gamma-\alpha}),$$

where $\varphi=\tan^{-1}(\sqrt{\sec(2\theta-1)})$.

Now we are ready to prove our main theorem by using Proposition 5.1 and Lemma 5.2.

PROOF OF THEOREM. We apply Lemma 5.2 to (5.1). We put $\alpha=1-n/2m$, $\gamma=1-\sigma/2m$ for $\sigma=1/2-2m\varepsilon$. Comparing (5.1) with (5.6), we obtain the following five β_i ($i=1, 2, \cdots, 5$):

$$-\beta_1=\frac{n-1}{2m}-1+\frac{\sigma}{m},$$
$$-\beta_2=\frac{n-1}{2m}-1+\frac{\sigma}{m}+\eta,$$
$$-\beta_3=\frac{n}{2m}-1-\frac{(h+1-\sigma(h+3))}{2m},$$
$$-\beta_4=\frac{n}{2m}-1+\frac{\sigma}{2m}-\frac{(\sigma-\sigma')j}{2m},$$
$$-\beta_5=-2(2N+2m)\varepsilon+\frac{9}{2}+\frac{\sigma}{2m}.$$

Choose sufficiently large j and N, and a sufficiently small η. Then we see in application of Lemma 5.2 to (5.1) that remainder terms $O(t^{1-\beta_2})$, $O(t^{1-\beta_4})$ and $O(t^{1-\beta_5})$ are negligible. Hence, consider β_1 and β_3. The conditions $0<\alpha<\beta_i<1$ ($i=1, 3$) are satisfied if $0<\sigma<1/2$ and $0<\sigma<(h+1)/(h+3)$, and also $1-\beta_1<\gamma-\alpha$ if $0<\sigma<1/3$. Moreover, $1-\beta_1\leqq 1-\beta_3$ is satisfied if $\sigma\geqq h/(1+h)$. Since $0<h\leqq 1/2$, one of the inequalities $1-\beta_1<\gamma-\alpha$ and $1-\beta_1<1-\beta_3$ holds. Hence $O(t^{1-\beta_1})$ is negligible.

Thus, we get the remainder terms $O(t^{n/2m-(h+1-\sigma(h+3))/2m})$ and $O(t^{(n-\sigma)/2m})$ for any $0<\sigma<(h+1)/(h+3)$. Q.E.D.

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