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A Non-Existence Result for Harmonic Mappings from R^n into H^n

Atsushi TACHIKAWA

Keio University

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§0. Introduction.

The purpose of this paper is to give a non-existence result for harmonic mappings defined on the whole R^n, a Euclidean n-space ($n\geqq 2$), into a real hyperbolic n-space H^n.

For harmonic mappings $U: M\to N$ (M, N: complete Riemannian manifolds) some Liouville type theorems have been proved. By S. Hildebrandt - J. Jost - K.-O. Widman [4] it has been shown that a harmonic mapping $U: M\to N$ must be a constant mapping if M is *simple* and image $U(M)$ is contained in a geodesic ball $B_R(Q)\subset N$ with $R<\pi/(2\sqrt{\kappa})$ where κ denotes the maximum of the sectional curvatures of N. Here, a Riemannian manifold is said to be *simple*, if it is topologically R^m furnished with a metric for which the associated Laplace-Beltrami operator is uniformly elliptic on R^m. (See also [1] and [6].) Moreover by L. Karp [5] it has been shown that, for a complete, noncompact Riemannian manifold M and a simply-connected Riemannian manifold N with nonpositive sectional curvature, a nonconstant harmonic mapping $U: M\to N$ satisfies a certain growth-order condition. This implies a non-existence theorem for harmonic mappings under some growth condition. On the contrary, our non-existence theorem in this paper requires no growth condition.

In order to describe our main result precisely we introduce some notations: We use a standard coordinate system $x=(x^1, \cdots, x^n)$ on R^n and a normal coordinate system $u=(u^1, \cdots, u^n)$ centered at some point P_0 on H^n. $\langle\cdot,\cdot\rangle$ and $|\cdot|$ stand for the Euclidean scalar product and norm. We shall write $(g_{ij}(u))$ for the metric tensor on H^n with respect to the normal coordinate system $(u^i)_{1\leq i\leq n}$, $(g^{ij}(u))$ for the inverse of $(g_{ij}(u))$, and the Christoffel symbols of the first and second kind of the Levi-Civita connection on H^n will be denoted by Γ_{ijk} and Γ^i_{jk}.

A mapping $U: R^n\to H^n$ is said to be a *harmonic mapping* if it is of

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class C^2 and the representation $u(x)$ of U in the coordinate systems $(x^1, \cdots, x^n)$ and $(u^1, \cdots, u^n)$ satisfies the system of quasilinear elliptic partial differential equations

$$\Delta u^i + \sum_{\alpha=1}^{n} \Gamma^i_{jk}(u) \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\alpha} = 0 \qquad \text{for} \quad 1 \leqq i \leqq n , \tag{0.1}$$

where Δ denotes the standard Laplacian on $\boldsymbol{R}^n$, i.e. $\Delta = \sum_{\alpha=1}^{n} (\partial/\partial x^\alpha)^2$.

Our main result may be stated as follows.

THEOREM 0.1. *There exists no harmonic mapping U: $\boldsymbol{R}^n \to \boldsymbol{H}^n$ which is defined on the whole $\boldsymbol{R}^n$ and a coordinate representation $u(x)$ with respect to the normal coordinate system centered at $U(0)$ satisfies*

$$\sum_{i=1}^{n} \sum_{\alpha=1}^{n} \left(\frac{\partial}{\partial x^\alpha} \left(\frac{u^i(x)}{|u(x)|} \right) \right)^2 \geqq \frac{n-1}{|x|^2} \qquad \textit{for all} \quad x \in \boldsymbol{R}^n . \tag{0.2}$$

For example, the mappings which can be written as $u(x) = (x/|x|)\mu(x)$ with μ: $\boldsymbol{R}^n \to \boldsymbol{R}$ satisfy the condition (0.2). It should be noted that rotationally symmetric mappings $u(x) = (x/|x|)\rho(|x|)$ with ρ: $\boldsymbol{R} \to \boldsymbol{R}$ satisfy the condition (0.2), and therefore Theorem 0.1 shows that there exists no rotationally symmetric harmonic mapping from $\boldsymbol{R}^n$ to $\boldsymbol{H}^n$. This contrasts with the result of [7] which asserts the existence of rotationally symmetric harmonic mappings from $\boldsymbol{R}^n$ onto a warped product manifold $\boldsymbol{R}_+ \times_f S^{n-1}$ whose warping function f: $\boldsymbol{R}_+ \to \boldsymbol{R}_+$ satisfies the following conditions;

$$f(t) > 0, \ f'(t) > 0 \quad \text{on} \quad \boldsymbol{R}_+ , \tag{0.3}$$

$$\lim_{t \to +0} \frac{f(t)}{t} = 1 , \tag{0.4}$$

and

$$f \cdot f'(t) \text{ is at most of linear growth as } t \to \infty . \tag{0.5}$$

Remark that if we take $f(t) = \sinh t$, $\boldsymbol{R}_+ \times_f S^{n-1}$ coincides with $\boldsymbol{H}^n$ and the conditions (0.3) and (0.4) are satisfied.

REMARK 0.1. For the case that $n=2$, using polar coordinates, we can replace the condition (0.2) with slight stronger but simpler one. Let (r, θ) $(0 \leqq \theta \leqq 2\pi)$ and (R, Θ) be the polar coordinate systems on $\boldsymbol{R}^2$ and $\boldsymbol{H}^2$ respectively and write a mapping u: $\boldsymbol{R}^2 \to \boldsymbol{H}^2$ as $(u^1(x), u^2(x)) = (R(r, \theta)\cos\Theta(r, \theta),\ R(r, \theta)\sin\Theta(r, \theta))$. Then it is easy to see that the condition

$$\frac{\partial \Theta}{\partial \theta} \geqq 1 \tag{0.6}$$

implies (0.2). On the other hand the condition (0.6) implies that Θ moves from 0 to 2π when θ moves from 0 to 2π. Thus the condition (0.2) can be considered as a kind of condition for nondegeneracy.

§ 1. Equation for $|u|$.

Since we are using a normal coordinate system $(u^1, \cdots, u^n)$ on $\boldsymbol{H}^n$, the coefficients $g_{ij}(u)$ of the metric tensor can be written as

$$g_{ij}(u)=\frac{u^i u^j}{|u|^2}+\frac{(\sinh|u|)^2}{|u|^2}\cdot\left(\delta_{ij}-\frac{u^i u^j}{|u|^2}\right). \tag{1.1}$$

From (1.1), by direct calculations, we get the following lemma.

LEMMA 1.1. *Let $(g_{ij}(u))$ be as above and $\Gamma^i_{jk}(u)$ be the coefficients of the second kind of Christoffel symbols of the Levi-Civita connection on $\boldsymbol{H}^n$. Then we have*

$$g_{ij}(u)(\xi^i\xi^j+u^k\Gamma^i_{km}(u)\xi^m\xi^j)=|\xi_t|^2+\left(\frac{1}{2|u|}\sinh(2|u|)\right)|\xi_n|^2 \tag{1.2}$$

$$\textit{for all } u \textit{ and } \xi\in\boldsymbol{R}^n,$$

where $\xi_t=(\langle\xi, u\rangle/|u|^2)u$ and $\xi_n=\xi-\xi_t$.

PROOF. From (1.1) we have

$$\begin{aligned}g_{ij}(u)\Gamma^i_{km}(u)=&\frac{1}{|u|^4}\cdot\left(1-\frac{(\sinh|u|)^2}{|u|^2}\right)\cdot(\delta_{km}u^j|u|^2-u^k u^j u^m)\\&+\frac{1}{|u|^3}\cdot\frac{\sinh|u|}{|u|}\cdot\frac{|u|\cosh|u|-\sinh|u|}{|u|^2}\\&\cdot\{(\delta_{kj}u^m+\delta_{jm}u^k-\delta_{km}u^j)|u|^2-u^k u^j u^m\},\end{aligned}$$

and therefore

$$g_{ij}(u)u^k\Gamma^i_{km}(u)\xi^m\xi^j=\begin{cases}0 & \text{for } \xi/\!/u\\ \left(-\dfrac{(\sinh|u|)^2}{|u|^2}+\dfrac{1}{2|u|}\sinh 2|u|\right)|\xi|^2 & \text{for } \langle\xi, u\rangle=0.\end{cases} \tag{1.3}$$

Moreover for the case that $\xi/\!/u$ and $\langle\eta, u\rangle=0$, we get

$$g_{ij}(u)u^k\Gamma^i_{km}(u)\xi^m\eta^j=0. \tag{1.4}$$

On the other hand we have

$$g_{ij}(u)\xi^i\xi^j=|\xi_t|^2+\frac{(\sinh|u|)^2}{|u|^2}|\xi_n|^2 . \tag{1.5}$$

From (1.3), (1.4) and (1.5) we obtain (1.2). □

Now, let $u: \boldsymbol{R}^n \to \boldsymbol{H}^n$ be a harmonic mapping. Then u satisfies the following equation of weak form,

$$\int_{\boldsymbol{R}^n}\sum_{\alpha=1}^n g_{ij}(u)(D_\alpha u^i D_\alpha \psi^j+\psi^k \Gamma^j_{km} D_\alpha u^m D_\alpha u^i)dx=0 \tag{1.6}$$

$$\text{for all} \quad \psi \in C_0^\infty(\boldsymbol{R}^n, \boldsymbol{R}^n) .$$

Here and in the sequel we write D_α for $\partial/\partial x^\alpha$.

Now we can prove the following proposition.

Proposition 1.1. *Let $u: \boldsymbol{R}^n \to \boldsymbol{H}^n$ be a harmonic mapping. Then $|u|$ satisfies the following elliptic equation*

$$\Delta|u|-\frac{\sinh 2|u|}{2|u|^2}(|Du|^2-|D|u||^2)=0 \tag{1.7}$$

on $\Omega=\{x \in \boldsymbol{R}^n: u(x)\neq 0\}$, *where* $|Du|^2=\sum_{i=1}^n\sum_{\alpha=1}^n(D_\alpha u^i)^2$ *and* $|D|u||^2=\sum_{\alpha=1}^n(D_\alpha|u|)^2$.

Proof. Taking $\psi=u\eta$, $\eta \in C_0^\infty(\boldsymbol{R}^n, \boldsymbol{R})$, in (1.6), we get

$$\int\sum_{\alpha=1}^n g_{ij}(u)\{u^j D_\alpha u^i D_\alpha\eta+(D_\alpha u^i D_\alpha u^j+u^k\Gamma^j_{km}(u)D_\alpha u^m D_\alpha u^i)\eta\}dx=0 . \tag{1.8}$$

Since we are using a normal coordinate system on $\boldsymbol{H}^n$, by Gauss' lemma (cf. [3], p. 136), we see that

$$g_{ij}(u)u^j=u^i .$$

Thus (1.8) becomes

$$\int\sum_{\alpha=1}^n\left\{\frac{1}{2}D_\alpha|u|^2 D_\alpha\eta+g_{ij}(u)(D_\alpha u^i D_\alpha u^j+u^k\Gamma^j_{km}D_\alpha u^m D_\alpha u^i)\eta\right\}dx=0 \tag{1.9}$$

$$\text{for all} \quad \eta \in C_0^\infty(\boldsymbol{R}^n, \boldsymbol{R}) .$$

Now, using Lemma 1.1, we obtain from (1.9) an equation for $|u|$,

$$\int\sum_{\alpha=1}^n\left\{\frac{1}{2}D_\alpha|u|^2 D_\alpha\eta+\left(|\xi|^2+\left(\frac{1}{2|u|}\sinh 2|u|\right)|\zeta|^2\right)\eta\right\}dx=0 \tag{1.10}$$

$$\text{for all} \quad \eta \in C_0^\infty(\boldsymbol{R}^n, \boldsymbol{R}) ,$$

where we are writing

$$\xi=(\xi^i_\alpha)=\left(\frac{\langle u, D_\alpha u\rangle}{|u|^2}u^i\right) \quad \text{and} \quad \zeta=(\zeta^i_\alpha)=(D_\alpha u^i)-(\xi^i_\alpha) ,$$

and therefore

$$|\xi|^2=\sum_{i=1}^{n}\sum_{\alpha=1}^{n}(\xi^i_\alpha)^2=\frac{|D|u|^2|^2}{4|u|^2} ,$$

$$|\zeta|^2=\sum_{i=1}^{n}\sum_{\alpha=1}^{n}(\zeta^i_\alpha)^2=|Du|^2-\frac{|D|u|^2|^2}{4|u|^2} .$$

Thus from (1.10), we can see that $|u|$ satisfies the following equation on Ω;

$$\frac{1}{2}\Delta|u|^2-\frac{|D|u|^2|^2}{4|u|^2}-\frac{1}{2|u|}\sinh 2|u|\left(|Du|^2-\frac{|D|u|^2|^2}{4|u|^2}\right)=0 . \tag{1.11}$$

Now, from (1.11) we obtain (1.7). □

§2. Proof of Theorem 0.1.

First of all let us consider the case that u is a rotationally symmetric mapping. Let u_s: $\boldsymbol{R}^n\to\boldsymbol{H}^n$ be a rotationally symmetric mapping which can be written as

$$u_s(x)=\frac{x}{|x|}\rho(|x|) ,$$

with the radius function ρ: $\boldsymbol{R}_+\to\boldsymbol{R}_+$. For rotationally symmetric mappings the equation (1.7) is reduced to an equation for ρ,

$$\Delta\rho-\frac{n-1}{2|x|^2}\sinh(2\rho)=0 . \tag{2.1}$$

Now, by direct calculations we get the following lemma.

LEMMA 2.1. *Let* $\rho_R(t)=2\tanh^{-1}(t/R)$. *Then* $\rho_R(|x|)$ *satisfies*

$$\Delta\rho_R-\frac{n-1}{2|x|^2}\sinh(2\rho_R)\leqq 0 \tag{2.2}$$

for all $R>0$ *and for all* $n\geqq 2$.

Using above lemma, we can prove Theorem 0.1 by a comparison theorem for elliptic equations.

PROOF OF THEOREM 0.1. Let U: $\boldsymbol{R}^n\to\boldsymbol{H}^n$ be a harmonic mapping whose coordinate representation u with respect to normal coordinate system centered at $U(0)$ satisfies (0.2). Then from (0.2) and (1.7) we can see that $|u|$ satisfies

$$\Delta|u|-\frac{n-1}{2|x|^2}\sinh 2|u|\geqq 0\,, \tag{2.3}$$

for we can see from (0.2) that

$$\frac{|Du|^2-|D|u|\,|^2}{|u|^2}=\left|D\left(\frac{u}{|u|}\right)\right|^2\geqq\frac{n-1}{|x|^2}\,.$$

Since (0.2) implies that u is not a constant mapping, we can choose a compact set $D\subset\subset \boldsymbol{R}^n-\{0\}$ on which $|u|\geqq\varepsilon_0$ for sufficiently small $\varepsilon_0>0$.

Take R_0 sufficiently large so that

$$\rho_R(|x|)<\varepsilon_0 \qquad \text{on } D \quad \text{for } R>R_0\,.$$

Remarking that $\rho_R\to\infty$ as $|x|\to R$ while u remains to be bounded on every bounded set, we can see that for every $R\geqq R_0$ there exists a bounded domain $\Omega_R\supset D$, $\not\ni 0$ such that $|u|=\rho_R$ on $\partial\Omega_R$. Now, from (2.2) and (2.3) we can use a comparison theorem (see for example [2] Theorem 10.1) to get $|u|\leqq\rho_R$ in Ω_R for all $R>R_0$. This implies that $|u|=0$ on D, which is a contradiction. Thus Theorem 0.1 is proved. □

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Present Address:
DEPARTMENT OF MATHEMATICS, FACULTY OF LIBERAL ARTS, SHIZUOKA UNIVERSITY
OHYA, SHIZUOKA 422, JAPAN