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A Sum Formula for Casson's λ-Invariant

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Dedicated to Professor Itiro Tamura on his 60th birthday

A. Casson [1] defined an integer valued invariant $\lambda(M)$ for an oriented homology 3-sphere M.

In [4] J. Hoste gave a formula to calculate $\lambda(M)$ from a special framed link description of M. He required the framed link to satisfy the condition that linking numbers of any two components of the link are zero.

In this note, we give a sum formula to calculate Casson's λ-invariant for an oriented homology 3-sphere which is constructed by gluing two knot exteriors in homology 3-spheres with some diffeomorphism between their boundaries. Our result is just the λ-invariant version of C. Gordon's theorem [2, Theorem 2] for μ-invariant.

§ 1. Preliminaries.

Casson proved the following theorem.

THEOREM 1 (Casson). *Let M be an oriented homology 3-sphere. There exists an integer valued invariant $\lambda(M)$ with the following properties.*

(1) *If $\pi_1(M)=1$, then $\lambda(M)=0$.*

(2) *$\lambda(-M)=-\lambda(M)$, where $-M$ denotes M with the opposite orientation.*

(3) *Let K be a knot in M and $(K_n; M)$ be the oriented homology 3-sphere obtained by performing $1/n$-Dehn surgery on M along K, $n\in \mathbf{Z}$. $\lambda(K_{n+1}; M)-\lambda(K_n; M)$ is determined independently of n.*

(4) *$\lambda(M)$ reduces,* mod 2, *to the Rohlin invariant $\mu(M)$.*

By the property (3), $\lambda'(K; M)=\lambda(K_{n+1}; M)-\lambda(K_n; M)$ is well defined. By the induction on n, we have:

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COROLLARY 1.

$$\lambda(K_n; M)=\lambda(M)+n\lambda'(K, M) .$$

As Alexander polynomial of a knot K, we consider only normalized Alexander polynomial $\Delta_{K;M}(t)$, that is, $\Delta_{K;M}(t)$ has the form $a_n t^{-n}+\cdots+a_1 t^{-1}+a_0+a_1 t+\cdots+a_n t^n$ and $\Delta_{K;M}(1)=1$. Casson's second theorem shows that the λ-invariant is related to the Alexander polynomial.

THEOREM 2 (Casson).

$$\lambda'(K; M)=\frac{1}{2}\Delta''_{K;M}(1) ,$$

where $\Delta''_{K;M}(t)$ is the second derivative of the normalized Alexander polynomial of K.

We begin with the following trivial lemma.

LEMMA 1. *Let M and M' be homology 3-spheres and K be a knot in M. Let K_0 be the knot in $M\sharp M'$ which corresponds to K. Then $\Delta_{K_0;M\sharp M'}(t)=\Delta_{K;M}(t)$.*

PROOF. Let F be a Seifert surface of K. Then we obtain a Seifert surface F_0 of K_0 which corresponds to F. Their Seifert forms are naturally isomorphic and Lemma 1 follows.

LEMMA 2. *Let K^* be a 0-parallel knot of K in M. Let K^*_N be the knot in $N=(K_n; M)$ which corresponds to K^*. Then $\Delta_{K^*_N;N}(t)=\Delta_{K;M}(t)$.*

PROOF. Let $N(K)$ be a tubular neighbourhood of K in M, and $E=\overline{M-N(K)}$. We consider K^* as a 0-parallel knot of K which lies on $\partial N(K)$. N is represented as $N=E\cup_h V$ with a solid torus V and a diffeomorphism $h\colon \partial E\to\partial V$. Since $K^*\subset E$, we can consider a knot K^*_N in N which corresponds to K^*. Let F be a Seifert surface of K^*. We can assume that $F\subset E$. Hence we obtain a Seifert surface F_N of K^*_N which corresponds to F. Since the homomorphism $H_1(F)\to H_1(E)$ induced from inclusion is zero map, for any 1-cycle z on F, there is a 2-chain c which lies on E and $\partial c=z$. The corresponding fact holds for F_N. Hence K^* and K^*_N have isomorphic Seifert forms. This implies $\Delta_{K^*;M}(t)=\Delta_{K^*_N;N}(t)$. Since K^* is isotopic to K in M, $\Delta_{K^*;M}(t)=\Delta_{K;M}(t)$. We obtain the lemma.

LEMMA 3. *Let M and M' be oriented homology 3-spheres. Then $\lambda(M\sharp M')=\lambda(M)+\lambda(M')$.*

PROOF. Suppose that M is obtained by Dehn surgery on a framed link $L=\{K_1, K_2, \cdots, K_n\}$. We can assume that the linking number $\mathrm{lk}(K_i, K_j)=0$ for every pair K_i, K_j $(i\neq j)$ of components of L and (the framing of K_i)$=\varepsilon_i=\pm 1$ $(i=1, \cdots, n)$. Let N_j be the manifold obtained by the Dehn surgery on the framed link $\{K_1, \cdots, K_j\}$. We can regard $K_{j+1}, \cdots, K_n$ as knots in N_j.

First we see that the framings of $K_{j+1}, \cdots, K_n$ in N_j is the same as those of $K_{j+1}, \cdots, K_n$ in S^3. Since the linking number of any pair of components of L is zero, for any component K_k, there exists a Seifert surface F_k such that $F_k\cap K_i=\emptyset$ $(i\neq k)$. Hence F_k can be also regarded as a Seifert surface of surgered manifold N_j. This means that the 0-framings of K_k in M and N_j coincide $(j<k)$. Hence the framing of K_k in N_j is $\varepsilon_k=\pm 1$.

Thus we obtain that $N_{j+1}=((K_{j+1})_{\varepsilon_{j+1}}; N_j)$ is also a homology 3-sphere. By the induction on j, we obtain

$$\lambda(M)=\sum_{j=1}^{n}\varepsilon_j\frac{1}{2}\Delta''_{K_j;N_{j-1}}(1)\ . \tag{1}$$

Next we regard the framed link L as the framed link in $S^3\# M'$, which we will denote by $L^*=\{K_1^*, \cdots, K_n^*\}$. Similarly we regard K_{j+1} as a knot in $N_j\# M'$, which we will denote by K_{j+1}^*. By Lemma 1, we have

$$\Delta_{K_{j+1};N_j}(t)=\Delta_{K^*_{j+1};N_j\# M'}(t)\ . \tag{2}$$

Since $M\# M'$ can be obtained from $S^3\# M'$ by the sequence of surgeries on $K_1^*, K_2^*, \cdots, K_n^*$, we obtain

$$\begin{aligned}\lambda(M\# M')&=\sum_{j=1}^{n}\varepsilon_j\frac{1}{2}\Delta''_{K^*_j;N_{j-1}\# M'}(1)+\lambda(M')\\ &=\sum_{j=1}^{n}\varepsilon_j\frac{1}{2}\Delta''_{K_j;N_{j-1}}(1)+\lambda(M') \qquad \text{(from (2))}\\ &=\lambda(M)+\lambda(M') \qquad \text{(from (1))}\ .\end{aligned}$$

This completes the proof.

§2. Homology spheres constructed from knot exteriors.

We will study oriented homology 3-spheres which are constructed by C. Gordon [2].

For $i=1, 2$, let K_i be an oriented knot in an oriented homology 3-sphere M_i with the exterior X_i. We always identify ∂X_i with $S^1\times\partial D^2$ and parametrize ∂X_i by an angular coordinate (θ, ϕ). If $A=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ is a 2×2

integral matrix with $\det A=-1$, then A determines an orientation reversing diffeomorphism $h: \partial X_1 \to \partial X_2$ by $h(\theta, \phi)=(\alpha\theta+\beta\phi, \gamma\theta+\delta\phi)$. We denote the naturally oriented closed 3-manifold obtained by gluing two knot exteriors with h, $X_1 \cup_h X_2$ by $M(K_1, K_2; A)$ or $M(K_1, K_2; \alpha, \beta, \gamma, \delta)$. By the explicit computation of the first homology group of $M(K_1, K_2; \alpha, \beta, \gamma, \delta)$, it is known that $M(K_1, K_2; \alpha, \beta, \gamma, \delta)$ becomes a homology 3-sphere if and only if $|\gamma|=1$.

In the following section, we always assume $|\gamma|=1$. Since $\det A=\alpha\delta-\beta\gamma=\alpha\delta-\pm\beta=-1$, $\beta=\pm(\alpha\delta+1)$ is determined by α, δ and $\varepsilon=$(the sign of γ). The oriented homology 3-sphere $M(K_1, K_2; \alpha, \pm(\alpha\delta+1), \gamma, \delta)$ will be denoted by $M^{\varepsilon}(K_1, K_2; \alpha, \delta)$.

§ 3. Calculation of Casson's λ-invariant.

Let M_i be an oriented homology 3-sphere and K_i be an oriented knot in M_i $(i=1, 2)$. In the sense of the normalized Alexander polynomial, the following equation holds:

LEMMA 4.

$$\Delta_{K_1\# K_2; M_1\# M_2}(t)=\Delta_{K_1; M_1}(t)\cdot\Delta_{K_2; M_2}(t) .$$

PROOF. Let F_i be an oriented Seifert surface of K_i in M_i with genus h_i for $i=1, 2$. Then $F=F_1 \natural F_2$ is a Seifert surface of genus h_1+h_2 of $K_1\# K_2$ in $M_1\# M_2$. The normalized Alexander polynomial of $K_1\# K_2$ in $M_1\# M_2$ is given as follows:

$$\Delta_{K_1\# K_2; M_1\# M_2}(t)=t^{-(h_1+h_2)}\det(V-tV^T) ,$$

where V is a Seifert matrix of F. Since $V=\begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}$ for Seifert matrix V_i of K_i, $i=1, 2$, we have $\Delta_{K_1\# K_2; M_1\# M_2}(t)=t^{-(h_1+h_2)}\det(V-tV^T)=t^{(-h_1)}\det(V_1-tV_1^T)\cdot t^{(-h_2)}\det(V_2-tV_2^T)=\Delta_{K_1; M_1}(t)\cdot\Delta_{K_2; M_2}(t)$. This completes the proof.

For a knot K in a homology 3-sphere M and its normalized Alexander polynomial $\Delta_{K; M}(t)$, it holds that $\Delta'_{K; M}(1)=0$. Computing second derivatives of the equation of Lemma 4, we have:

LEMMA 5.

$$\lambda'(K_1\# K_2; M_1\# M_2)=\lambda'(K_1; M_1)+\lambda'(K_2; M_2) .$$

By Corollary 1, $\lambda((K_1\# K_2)_n; M_1\# M_2)=\lambda(M_1\# M_2)+n\lambda'(K_1\# K_2; M_1\# M_2)$. Using Lemma 3 and Lemma 5, we obtain:

COROLLARY 2.

$$\lambda((K_1\# K_2)_n;\ M_1\# M_2)=\lambda((K_1)_n;\ M_1)+\lambda((K_2)_n;\ M_2)\ .$$

Our result is as follows:

THEOREM 3. *Let K_i be an oriented knot in an oriented homology 3-sphere M_i, $i=1, 2$. Then*

$$\lambda(M^{\varepsilon}(K_1,\ K_2;\ \alpha,\ \delta))=\lambda(M_1)+\lambda(M_2)-\varepsilon\delta\lambda'(K_1;\ M_1)+\varepsilon\alpha\lambda'(K_2;\ M_2)\ .$$

REMARK. It is known that $\lambda'(K;M)=(1/2)\Delta''_{K;M}(1)$ reduces, mod 2, to the Arf invariant $c(K;M)$. The theorem above is λ-invariant version of Gordon's formula [2, Theorem 2] for μ-invariant of the oriented homology 3-sphere $M^{\varepsilon}(K_1, K_2; \alpha, \delta)$.

In the proof of Theorem 3, we need the following lemma.

LEMMA 6. *Under the same assumption as in Theorem 3,*

$$((K_1\# K_2)_{\mp 1};\ M_1\# M_2)\cong M(K_1,\ K_2;\ -1,\ 0,\ \pm 1,\ 1)\ .$$

REMARK. Gordon [3] noted that the same conclusion holds in the case of a knot in S^3. The following proof is essentially due to Gordon.

PROOF. Let X be the exterior of $K=K_1\# K_2$ in $M=M_1\# M_2$. Then there exists an annulus A in ∂X such that A is a meridional annulus in ∂X_i and that $X\cong X_1\cup_A X_2$. Let λ_i, $\mu_i\in H_1(\partial X_i)$ (resp. λ, $\mu\in H_1(\partial X)$) be a longitude-meridian pair of K_i, $i=1, 2$ (resp. K), U be a solid torus and λ_0, $\mu_0\in H_1(\partial U)$ be a longitude-meridian pair of U.

$(K_{\mp 1}; M)$ is the oriented homology 3-sphere $X\cup_f U$, where $f\colon \partial U\to\partial X$ is an orientation preserving diffeomorphism which satisfies $f_*(\mu_0)=\mp\mu+\lambda$, $f_*(\lambda_0)=-\mu$. Since $f_*^{-1}(\mu)=-\lambda_0$, and $f_*^{-1}(\lambda)=\mu_0\mp\lambda_0$, $X\cup_f U\cong(X_1\cup_A X_2)\cup_f U\cong(X_1\cup_{A'} U)\cup X_2$, where A' is the annulus $\partial X_1-\operatorname{int} A$ in ∂X_1 and longitudinal annulus in ∂U. Hence $X_1\cup_{A'} U\cong X_1$. Moreover the computation yields that the gluing diffeomorphism $h\colon \partial X_1\to\partial X_2$ is given by $h_*(\lambda_1)=-\lambda_2\pm\mu_2$ and $h_*(\mu_1)=\mu_2$. It follows that $((K_1\# K_2)_{\mp 1}; M_1\# M_2)\cong X_1\cup_h X_2\cong M(K_1, K_2; -1, 0, \pm 1, 1)$.

PPOOF OF THEOREM 3. Let $A=\begin{pmatrix}\alpha & \pm(\alpha\delta+1)\\ \pm 1 & \delta\end{pmatrix}$ be a 2×2 integral matrix, K_i be a knot in an oriented homology 3-sphere M_i and K_i^* be a 0-parallel knot of K_i, $i=1, 2$. Then K_i^* can be considered as a knot in $N_i=((K_i)_{n_i}; M_i)$, where $n_1=\pm(1-\delta)$ and $n_2=\pm(\alpha+1)$. Let $M=((K_1^*\# K_2^*)_{\mp 1}; N_1\# N_2)$. By Lemma 6, $M\cong M(K_1^*, K_2^*; -1, 0, \pm 1, 1)\cong X_1^*\cup_h X_2^*$, where X_i^*

is the exterior of K_i^* in N_i for $i=1, 2$.

Next we examine the exterior X_i^* as follows. We choose regular neighbourhoods $N(K_i)$ and $N(K_i^*)$ so that $N(K_i)\subset N(K_i^*)$ and $N(K_i)\cap K_i^*=\emptyset$. Let $Y_i=\overline{N(K_i^*)-N(K_i)}$, then $Y_i\cong S^1\times\partial D^2\times I$ and $Y_i\supset K_i^*$. By X_i (resp. $\tilde{X}_i^*$) we denote the exterior of K_i (resp. K_i^*) in M_i (resp. N_i). Then $X_i=\tilde{X}_i^*\cup_{\mathrm{id}}Y_i$, where id: $\partial N(K_i^*)$ (=the outer boundary of Y_i)$\to\partial\tilde{X}_i^*$. Note that $X_i\cong\tilde{X}_i^*$, since K_i^* is a 0-parallel knot of K_i.

We consider $N_i=((K_i)_{n_i}; M_i)$ as follows. For a solid torus V_i and an orientation preserving diffeomorphism $h_i: \partial V_i\to\partial X_i=\partial N(K_i)$ (= the inner boundary of Y_i) given by $\begin{pmatrix}1 & n_i\\ 0 & 1\end{pmatrix}$, $N_i=X_i\cup_{h_i}V_i=\tilde{X}_i^*\cup_{\mathrm{id}}(Y_i\cup_{h_i}V_i)$. Since $K_i^*\subset Y_i$, we choose a small regular neighbourhood $N'(K_i^*)$ of K_i^* such that $N'(K_i^*)\subset Y_i$. Let $Y_i^*=(Y_i\cup_{h_i}V_i)-N'(K_i^*)$ and λ_{1i}, μ_{1i} (resp. λ_i, μ_i and λ_i^*, μ_i^*) be a longitude-meridian pair of $\partial N(K_i^*)$ (resp. $\partial N(K_i)$ and $\partial N'(K_i^*)$). By the definition of h_i, h_i maps a longitude l_i of V_i to λ_i, and l_i is isotopic to λ_{1i} and λ_i^* in Y_i. Moreover in $Y_i\cup_{h_i}V_i$, $\mu_i'=\mu_{1i}+n_i\lambda_{1i}$ bounds a disk which is obtained by attaching a meridian disk in V_i to an annulus consisting of parallel curves in Y_i by h_i. Thus we parametrize $Y_i\cup_{h_i}V_i\cong S^1\times D^2$ so that K_i^* (resp. μ_i') corresponds to $S^1\times 0$ (resp. $\mathrm{pt}\times\partial D^2$). Hence $Y_i^*\cong S^1\times\partial D^2\times I$ and the identification f_i of the outer boundaries of Y_i^* and Y_i is given by $\begin{pmatrix}1 & n_i\\ 0 & 1\end{pmatrix}$. Therefore

$$\begin{aligned} X_i^* &= \overline{N_i-N'(K_i^*)} \\ &= \tilde{X}_i^*\cup_{\mathrm{id}}\overline{(Y_i\cup_{\mathrm{id}}V_i)-N'(K_i^*)} \\ &\cong X_i\cup_{f_i}Y_i^* \,. \end{aligned}$$

Finally $M=X_1^*\cup_h X_2^*\cong(X_1\cup_{f_1}Y_1^*)\cup_h(Y_2^*\cup_{f_2}X_2)\cong X_1\cup_g X_2$, where $g: \partial X_1\to\partial X_2$ is the composition $f_2\circ h\circ f_1^{-1}$ given by

$$\begin{pmatrix}1 & n_2\\ 0 & 1\end{pmatrix}\begin{pmatrix}-1 & 0\\ \pm 1 & 1\end{pmatrix}\begin{pmatrix}1 & -n_1\\ 0 & 1\end{pmatrix}=\begin{pmatrix}-1\pm n_2 & n_1+n_2\mp n_1n_2\\ \pm 1 & \mp n_1+1\end{pmatrix}=\begin{pmatrix}\alpha & \pm(\alpha\delta+1)\\ \pm 1 & \delta\end{pmatrix}=A\,.$$

That is $M^{\bullet}(K_1, K_2; \alpha, \delta)\cong((K_1^*\# K_2^*)_{\mp 1}; N_1\# N_2)$. Applying lemmas, we can compute $\lambda(M^{\bullet}(K_1, K_2; \alpha, \delta))$ as follows:

$$\begin{aligned} \lambda(M^{\bullet}(K_1, K_2; \alpha, \delta)) &= \lambda((K_1^*\# K_2^*)_{\mp 1}; N_1\# N_2) \\ &= \lambda(N_1\# N_2)\mp\lambda'(K_1^*\# K_2^*; N_1\# N_2) \\ &= \lambda(N_1)+\lambda(N_2)\mp\lambda'(K_1^*; N_1)\mp\lambda'(K_2^*; N_2) \\ &= \lambda(M_1)+n_1\lambda'(K_1; M_1)+\lambda(M_2)+n_2\lambda'(K_2; M_2) \\ &\quad \mp\lambda'(K_1^*; N_1)\mp\lambda'(K_2^*; N_2)\,. \end{aligned}$$

By Lemma 2 and Theorem 2, $\lambda'(K_i^*; N_i)=\lambda'(K_i; M_i)$. Hence

$$\lambda(M^\varepsilon(K_1, K_2; \alpha, \delta))=\lambda(M_1)+\lambda(M_2)+(n_1\mp1)\lambda'(K_1; M_1)+(n_2\mp1)\lambda'(K_2; M_2)$$
$$=\lambda(M_1)+\lambda(M_2)-\varepsilon\delta\lambda'(K_1; M_1)+\varepsilon\alpha\lambda'(K_2; M_2) .$$

This completes the proof of Theorem 3.

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