TOKYO J. MATH.
VOL. 12, NO. 1, 1989

Time Reversal of Random Walks in One-Dimension

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Introduction.

Given a one-dimensional random walk S_n with $S_0=0$, we consider the time reversal

$$(1)\qquad (0, S_{\tau-1}-S_\tau, S_{\tau-2}-S_\tau, \cdots, S_1-S_\tau, -S_\tau)$$

where τ denotes the time of first entry into the open negative half line $(-\infty, 0)$ for the random walk S_n. We then take independent copies $w_1, w_2, \cdots$ of the (finite length) path-valued random variable (1) and define a new process $\{W_n, n\geqq 0\}$ by (1.2) (see §1). The purpose of this paper is to prove that, under the assumption that $\tau<\infty$ a.s., $\{W_n, n\geqq 0\}$ is a Markov process on $[0, \infty)$ with transition function (1.3) which is of a form of a superharmonic transform of the dual random walk. Golosov obtained a similar result in the study of random walks in random environment (see Lemma 6 of [2]); however, it was assumed in [2] that the random walk has zero expectation and finite variance, and the transition function of the process W_n whose Markovian property is our concern was given in a form which is somewhat different from ours (see the final remark in §5). Our only assumption is that the random walk enters the open half line $(-\infty, 0)$ almost surely. Our method is quite elementary. It is not clear whether the present problem can be discussed in the framework of the general theory of time reversal of Markov processes due to Hunt [3] and Nagasawa [6].

This work was motivated by the study of the probability law of a valley which appeared in the investigation of limiting behavior of random walks and diffusion processes in one-dimensional random environment (cf. [2] [4] [5] [8] [9]).

§1. Main theorem.

Given real valued i.i.d. random variables X_k, $k\geqq 1$, we consider the

Received July 7, 1988

random walk

$$S_0=0\ , \qquad S_n=X_1+\cdots+X_n \quad (n\geqq 1)$$

and denote by τ the time of first entry of the random walk into the open half line $(-\infty, 0)$. We assume throughout the paper that

$$P\{\tau<\infty\}=1\ . \tag{1.1}$$

Let $w_1, w_2, \cdots$ be independent copies of

$$(0, S_{\tau-1}-S_\tau, S_{\tau-2}-S_\tau, \cdots, S_1-S_\tau, -S_\tau)$$

which is regarded as a random variable with values in

$$\mathscr{W}=\left\{w=(w(0), w(1), \cdots, w(l)) : \begin{array}{l} w(0)=0 \\ 0<w(l)=\min\limits_{1\leq k\leq l} w(k),\ l\geqq 1 \end{array}\right\}.$$

Writing $w_k=(w_k(0), w_k(1), w_k(2), \cdots w_k(l_k))$, $k\geqq 1$, we define a process $\{W_n, n\geqq 0\}$ as follows:

$$W_n=\begin{cases} w_1(n) & \text{for} \quad 0\leqq n\leqq l_1\ , \\ w_1(l_1)+w_2(n-l_1) & \text{for} \quad l_1<n\leqq l_1+l_2\ , \\ \quad\vdots & \quad\vdots \\ \sum\limits_{j=1}^{k-1} w_j(l_j)+w_k\Big(n-\sum\limits_{j=1}^{k-1} l_j\Big) & \text{for} \quad \sum\limits_{j=1}^{k-1} l_j<n\leqq\sum\limits_{j=1}^{k} l_j\ , \\ \quad\vdots & \quad\vdots \end{cases} \tag{1.2}$$

We also define $\hat{p}_\xi(x, dy)$ by

$$\hat{p}_\xi(x, dy)=\frac{1}{\xi(x)}P\{x-X_1\in dy\}\xi(y)\mathbf{1}_{(0,\infty)}(y)\ , \tag{1.3}$$

where

$$\xi(x)=\begin{cases} 1 & \text{for} \quad x=0\ , \\ E\Big\{\sum\limits_{n=0}^{\tau}\mathbf{1}_{[0,x)}(S_n)\Big\} & \text{for} \quad x>0 \end{cases} \tag{1.4}$$

wherein $\mathbf{1}_A$ denotes the indicator function of a set A. We easily see that $\xi(x)<\infty$, $x\geqq 0$ (see the remark at the end of the next section). It will be proved that $\hat{p}_\xi(x, dy)$ is a transition function on $[0, \infty)$ (see Lemma 1). Now we can state our main theorem.

THEOREM. *Under the assumption* (1.1), $\{W_n, n\geqq 0\}$ *is a Markov* *process on* $[0, \infty)$ *with transition function* $\hat{p}_\xi(x, dy)$.

§ 2. Transition function.

For $x \in \boldsymbol{R}$ we write $S_n^x = x + S_n$. Let

$$\tau^x = \min\{n \geqq 1 : S_n^x < 0\}, \quad x \geqq 0,$$

and put

$$G(x, A) = E\left\{\sum_{n=0}^{\tau^x} 1_A(S_n^x)\right\}, \qquad x \geqq 0, \quad A \in \mathscr{B}([0, \infty)), \tag{2.1}$$

$$p(x, dy) = P\{x + X_1 \in dy\}, \qquad \hat{p}(x, dy) = P\{x - X_1 \in dy\}. \tag{2.2}$$

Then $\xi(x) = G(0, [0, x))$ for $x > 0$. In this section we prove the following lemma.

LEMMA 1. $\hat{p}_\xi(x, dy)$ *is a Markov transition function on* $[0, \infty)$.

PROOF. We are going to prove

$$\hat{p}_\xi(x, [0, \infty)) = \hat{p}_\xi(x, (0, \infty)) = 1, \quad x \geqq 0.$$

For this it is enough to prove that

$$\int_{(0,\infty)} \hat{p}(x, dy)\xi(y) = \xi(x), \quad x \geqq 0. \tag{2.3}$$

The proof is divided into two steps.

Step 1. $\int_{[0,\infty)} G(0, dx) P\{-X_1 \in (x, \infty)\} = 1.$

In fact, we have

$$\begin{aligned}
1 &= P\{\tau < \infty\} \\
&= P\{\tau = 1\} + \sum_{n=1}^{\infty} P\{\tau = n+1\} \\
&= \int_{(-\infty,0)} p(0, dy) \\
&\quad + \sum_{n=1}^{\infty} \int_{[0,\infty)} p(0, dx_1) \int_{[0,\infty)} p(x_1, dx_2) \cdots \int_{[0,\infty)} p(x_{n-1}, dx_n) \int_{(-\infty,0)} p(x_n, dy) \\
&= \int_{[0,\infty)} G(0, dx) \int_{(-\infty,0)} p(x, dy) \\
&= \int_{[0,\infty)} G(0, dx) P\{-X_1 \in (x, \infty)\}.
\end{aligned}$$

Step 2. $\int_{(0,\infty)}\hat{p}(x, dy)\xi(y)=\xi(x),\ x\geqq 0.$
In fact, the left hand side of the above is equal to

$$\begin{aligned}
&\int_{(0,\infty)}P\{x-X_1\in dy\}\int_{[0,y)}G(0, dz)\\
&\quad=\int_{[0,\infty)}G(0, dz)\int_{(z,\infty)}P\{x-X_1\in dy\}\\
&\quad=\int_{[0,\infty)}G(0, dz)[P\{-X_1\in(z, \infty)\}+P\{-X_1\in(z-x, z]\}]\\
&\quad=1+\int_{[0,\infty)}G(0, dz)P\{-X_1\in(z-x, z]\}\ ,
\end{aligned}$$

where we used the result of step 1; also notice that the second term vanishes if $x=0$. The last line of the above equalities can be written as

$$\begin{aligned}
&1+\int_{[0,\infty)}G(0, dz)P\{z+X_1\in[0, x)\}\\
&\quad=1+\int_{[0,x)}p(0, dx_1)\\
&\qquad+\sum_{n=1}^{\infty}\int_{[0,\infty)}p(0, dx_1)\int_{[0,\infty)}p(x_1, dx_2)\cdots\int_{[0,\infty)}p(x_{n-1},dx_n)\int_{[0,x)}p(x_n, dy)\\
&\quad=G(0, [0, x))=\xi(x)\ .
\end{aligned}$$

The proof of the lemma is finished.

REMARK. There are several ways of proving $\xi(x)<\infty$, $x\geqq 0$. Here is a proof based on the identity (2.3) that was proved without using the finiteness of $\xi(x)$. The assumption (1.1) implies that $\hat{p}(x, (x+\varepsilon, \infty))=P\{-X_1>\varepsilon\}>0$ for some $\varepsilon>0$. Therefore the identity (2.3) with $x=0$ implies that $\xi(x_1)<\infty$ for some $x_1>\varepsilon$, and again the identity (2.3) with $x=x_1$ implies that $\xi(x_2)<\infty$ for some $x_2>x_1+\varepsilon$. Repeating this argument, we see that there exists a sequence $\{x_n\}$ such that $x_0=0$, $x_n-x_{n-1}>\varepsilon$ and $\xi(x_n)<\infty$ for all $n\geqq 1$. This combined with the monotonicity of $\xi(x)$ proves the finiteness of $\xi(x)$ for all $x\geqq 0$.

§ 3. Proof of the theorem in a special case.

In this section we give a proof of the theorem in the special case where

(3.1) $$P\{X_k\in \boldsymbol{Z}\}=1\ .$$

In this case the space $\mathscr{W}$ consists of the paths of the form $w=$

$(w(0), w(1), \cdots, w(l))$ where $w(k) \in \boldsymbol{Z}$ $(0 \leqq k \leqq l)$, $w(0)=0$, $0<w(l)=\min_{1\leqq k\leqq l} w(k)$ and $l \geqq 1$. We denote by μ the probability law of $(0, S_{\tau-1}-S_\tau, S_{\tau-2}-S_\tau, \cdots, S_1-S_\tau, -S_\tau)$; of course, μ is a probability measure on $\mathscr{W}$. Put

$$p(x, y)=P\{x+X_1=y\}\ , \quad \hat{p}(x, y)=p(y, x)\ , \qquad x, y \in \boldsymbol{Z}$$

and let us prepare a simple lemma.

LEMMA 2. *If* $a_1, a_2, \cdots, a_l \in \boldsymbol{Z}$ $(l \geqq 1)$ *satisfy*

$$\min_{1\leqq k\leqq l} a_k = a_l > 0\ , \tag{3.2}$$

then

$$\mu\{w=(0, a_1, \cdots, a_l)\}=\hat{p}(0, a_1)\hat{p}(a_1, a_2)\cdots \hat{p}(a_{l-1}, a_l)\ . \tag{3.3}$$

PROOF. Since the event

$$\Gamma=\{\tau=l,\ S_{l-k}-S_l=a_k\ (1\leqq k\leqq l)\}$$

is the same as the event $\{S_{l-k}-S_l=a_k\ (1\leqq k\leqq l)\}$, the left hand side of (3.3) equals

$$\begin{aligned} P\{\Gamma\} &= p(0, a_{l-1}-a_l)p(a_{l-1}-a_l, a_{l-2}-a_l)\cdots p(a_1-a_l, -a_l) \\ &= p(a_l, a_{l-1})p(a_{l-1}, a_{l-2})\cdots p(a_1, 0) \\ &= \text{the right hand side of (3.3)}\ . \end{aligned}$$

In what follows x, x_j, y, a are always assumed to be *integers*. For $x, y \geqq a$ we put

$$g_a(x, y)=\delta_{x,y}+\sum_{n=0}^{\infty}\sum_{\substack{x_0=x\\ x_1,\cdots,x_n\geqq a}} p(x_0, x_1)p(x_1, x_2)\cdots p(x_n, y)\ ,$$

$$\hat{g}_a(x, y)=\delta_{x,y}+\sum_{n=0}^{\infty}\sum_{\substack{x_0=x\\ x_1,\cdots,x_n\geqq a}} \hat{p}(x_0, x_1)\hat{p}(x_1, x_2)\cdots \hat{p}(x_n, y)\ .$$

Then it is clear that

$$g_a(x, y)=\hat{g}_a(y, x)\ , \qquad x, y \geqq a\ , \tag{3.4a}$$

$$g_a(x, y)=g_{a+b}(x+b, y+b)\ , \qquad x, y\geqq a,\ \forall b \in \boldsymbol{Z}\ , \tag{3.4b}$$

$$\xi(x)=G(0, [0, x))=\sum_{0<a\leqq x} g_a(a, x)\ , \qquad x \geqq 1\ . \tag{3.4c}$$

For $w=(w(0), w(1), \cdots, w(l)) \in \mathscr{W}$ we define $l(w)$ by

$$l(w)=l\ . \tag{3.5}$$

Given *positive* integers $a_1, \cdots, a_m$ $(m \geqq 1)$ we put

$$a^* = \min_{1 \leq k \leq m} a_k \tag{3.6}$$

and for an integer a with $0 < a \leqq a^*$ we consider the events

$$\Lambda_n(a_1, \cdots, a_m; a) = \left\{ w \in \mathscr{W} : \begin{array}{l} w(0)=0 \\ w(k)=a_k \ (1 \leqq k \leqq m) \\ l(w)=m+n \\ w(m+n)=a \end{array} \right\}, \qquad n \geqq 0 ,$$

$$\Lambda(a_1, \cdots, a_m; a) = \bigcup_{n=0}^{\infty} \Lambda_n(a_1, \cdots, a_m; a) .$$

LEMMA 3. *For positive integers $a, a_1, \cdots, a_m$ $(m \geqq 1)$ with $0 < a \leqq a^*$ where a^* is defined by* (3.6) *we have*

$$\mu\{\Lambda(a_1, \cdots, a_m; a)\} = \left\{ \prod_{j=1}^{m} \hat{p}(a_{j-1}, a_j) \right\} \cdot g_a(a, a_m) , \qquad a_0 = 0 . \tag{3.7}$$

PROOF. The identity (3.7) is a consequence of the following (3.8), (3.9) and (3.10):

$$\mu\{\Lambda_0(a_1, \cdots, a_m; a)\} = \begin{cases} \prod_{k=1}^{m} \hat{p}(a_{k-1}, a_k) & \text{if } a = a_m \text{ (and hence } = a^*) \\ 0 & \text{otherwise .} \end{cases} \tag{3.8}$$

$$\begin{aligned} &\mu\{\Lambda_n(a_1, \cdots, a_m; a)\} \\ &= \sum_{a_{m+1}, a_{m+2}, \cdots, a_{m+n-1} \geqq a} \mu\{w = (0, a_1, \cdots, a_m, a_{m+1}, \cdots, a_{m+n-1}, a)\} \\ &= \sum_{a_{m+1}, a_{m+2}, \cdots, a_{m+n-1} \geqq a} \hat{p}(0, a_1)\hat{p}(a_1, a_2) \cdots \hat{p}(a_{m+n-1}, a) \qquad \text{(by (3.3))} \\ &= \left\{ \prod_{j=1}^{m} \hat{p}(a_{j-1}, a_j) \right\} \cdot g_n , \qquad n \geqq 1 , \end{aligned} \tag{3.9}$$

where

$$g_n = \begin{cases} \hat{p}(a_m, a) & \text{if } n=1 , \\ \sum_{a_{m+1}, a_{m+2}, \cdots, a_{m+n-1} \geqq a} \hat{p}(a_m, a_{m+1})\hat{p}(a_{m+1}, a_{m+2}) \cdots \hat{p}(a_{m+n-1}, a) & \text{if } n \geqq 2 . \end{cases}$$

$$\delta_{a_m, a} + \sum_{n=1}^{\infty} g_n = \hat{g}_a(a_m, a) = g_a(a, a_m) . \tag{3.10}$$

We now proceed to the proof of the theorem assuming (1.1) and (3.1). Let $w_1, w_2, \cdots$ be i.i.d. random variables with values in $\mathscr{W}$ and with common probability distribution μ and define a process $\{W_n, n \geqq 0\}$

by (1.2). Given integers

$$a_0=0, \quad a_1>0, \quad \cdots, \quad a_m>0 \ (m\geqq 1) ,$$

we consider the events

$$\Lambda=\{W_k=a_k \ (1\leqq k\leqq m)\} ,$$
$$\Lambda_a=\{W_k=a_k \ (1\leqq k\leqq m), \ W_m^*=a\} ,$$

where $W_m^*=\min_{n\geqq m} W_n$. Then $\Lambda=\cup_{0<a\leqq a_m}\Lambda_a$ (the case $a=0$ is excluded because $W_n\geqq 1$ for all $n\geqq 1$). Let $0<a\leqq a_m$ and define $m(0)>m(1)>m(2)>\cdots>m(\alpha)=0$ as follows:

$$\begin{aligned} m(0)&=m , \\ m(1)&=\max\{n<m\colon a_n<a\} , \\ m(2)&=\max\{n<m(1)\colon a_n<a_{m(1)}\} , \\ &\vdots \\ m(\alpha)&=\max\{n<m(\alpha-1)\colon a_n<a_{m(\alpha-1)}\} . \end{aligned}$$

Then it is clear that

$$\begin{aligned} &(0, a_1, a_2, \cdots, a_{m(\alpha-1)})\in\mathscr{W} , \\ &(0, a_{m(\alpha-1)+1}-a_{m(\alpha-1)}, a_{m(\alpha-1)+2}-a_{m(\alpha-1)}, \cdots, a_{m(\alpha-2)}-a_{m(\alpha-1)})\in\mathscr{W} , \\ &\vdots \\ &(0, a_{m(2)+1}-a_{m(2)}, a_{m(2)+2}-a_{m(2)}, \cdots, a_{m(1)}-a_{m(2)})\in\mathscr{W} . \end{aligned}$$

Therefore, the event Λ_a can be expressed as

$$\begin{aligned} \Lambda_a=&\left[\bigcap_{k=1}^{\alpha-1}\{w_k=(0, a_{m(\alpha-k+1)+1}-a_{m(\alpha-k+1)}, \cdots, a_{m(\alpha-k)}-a_{m(\alpha-k+1)})\}\right] \\ &\cap\{w_\alpha\in\Lambda(a_{m(1)+1}-a_{m(1)}, \cdots, a_m-a_{m(1)}; a-a_{m(1)})\} , \end{aligned}$$

and consequently an application of Lemma 2 and Lemma 3 yields

$$\begin{aligned} P\{\Lambda_\alpha\}&=\left\{\prod_{k=1}^{\alpha}\prod_{j=1}^{m(\alpha-k)-m(\alpha-k+1)}\hat{p}_{kj}\right\}\cdot g_{a-a_{m(1)}}(a-a_{m(1)}, a_m-a_{m(1)}) \\ &=\left\{\prod_{k=1}^{m}\hat{p}(a_{k-1}, a_k)\right\}\cdot g_a(a, a_m) , \end{aligned}$$

where

$$\hat{p}_{kj}=\hat{p}(a_{m(\alpha-k+1)+j-1}-a_{m(\alpha-k+1)}, a_{m(\alpha-k+1)+j}-a_{m(\alpha-k+1)}) .$$

Thus we have

$$P\{\Lambda\}=\left\{\prod_{k=1}^{m}\hat{p}(a_{k-1}, a_k)\right\}\cdot\sum_{0<a\leqq a_m} g_a(a, a_m)$$
$$=\left\{\prod_{k=1}^{m}\hat{p}(a_{k-1}, a_k)\right\}\cdot\xi(a_m) \qquad \text{(by (3.4c))}$$
$$=\prod_{k=1}^{m}\hat{p}_\xi(a_{k-1}, a_k)\ ,$$

which proves the theorem in the special case.

REMARK. From what we have proved it follows that the theorem holds for $\lambda\boldsymbol{Z}$-valued random walks satisfying the condition (1.1) where $\lambda>0$ is a constant.

§4. Proof of the theorem in general case.

To prove the theorem in a general situation we approximate $\{S_n, n\geqq 0\}$ by a sequence of random walks $\{S_{N,n}, n\geqq 0\}$ with values in $2^{-N}\boldsymbol{Z}$, $N\geqq 1$.

For integers N, $k\geqq 1$ we put

$$A_{N,k}=\{|X_k|\leqq N\}\ .$$

To define $S_{N,n}$ we need another sequence of events $B_{N,k}$. Enlarging the basic probability space if necessary, we choose a sequence of events $B_{N,k}$, N, $k=1, 2, \cdots$, such that

(i) for each integer $N\geqq 1$ the random variables $1_{B_{N,k}}$, $k\geqq 1$, are i.i.d.,

(ii) $P\{B_{N,k}\}<1$ for each N and $\lim_{N\to\infty} P\{B_{N,k}\}=1$,

(iii) $\{B_{N,k}; N, k\geqq 1\}$ is independent of $\{X_k, k\geqq 1\}$.

For an integer $N\geqq 1$ we define a function φ_N by

$$\varphi_N(x)=(j+1)2^{-N} \qquad \text{for } j2^{-N}\leqq x<(j+1)2^{-N}\ , \quad j=0, \pm 1, \cdots .$$

Then $\varphi_N(x)\downarrow x$ as $N\uparrow\infty$. Now we are in position to define $S_{N,n}$. Put $\Gamma_{N,k}=A_{N,k}\cap B_{N,k}$ and define $X_{N,k}$ by

$$X_{N,k}(\omega)=\begin{cases}\varphi_N(X_k(\omega)) & \text{if } \omega\in\Gamma_{N,k}\ ,\\ -c_N & \text{otherwise}\ ,\end{cases}$$

where c_N is a constant of the form $j2^{-N}$ which is chosen so that $E\{X_{N,k}\}\leqq 0$ holds. Such a constant c_N exists because $P\{\Gamma_{N,k}^c\}>0$ by (ii). Let

$$S_{N,0}=0\ , \qquad S_{N,n}=X_{N,1}+\cdots+X_{N,n}\ , \quad n\geqq 1\ ,$$
$$\tau_N=\min\{n\geqq 1: S_{N,n}<0\}\ .$$

Then $\{S_{N,n}, n\geqq 0\}$ is a random walk on $2^{-N}\boldsymbol{Z}$ satisfying the condition

$P\{\tau_N<\infty\}=1$ which is a consequence of $E\{X_{N,n}\}\leqq 0$. Therefore the result in the special case can be applied for $\{S_{N,n}, n\geqq 0\}$.

LEMMA 4. *$\xi_N(x)$ converges to $\xi(x)$ boundedly on any bounded subset of $[0, \infty)$ as $N\to\infty$, where $\xi(x)$ is defined by* (1.4) *and*

$$\xi_N(x)=\begin{cases}1 & \textit{for } x=0\ ,\\ E\left\{\sum_{n=0}^{\tau_N} 1_{[0,x)}(S_{N,n})\right\} & \textit{for } x>0\ .\end{cases}$$

PROOF. From the definition of $X_{N,k}$ it is clear that

$$(4.1)\qquad \begin{cases}X_{N,k}=\varphi_N(X_k) & \text{for } 1\leqq\forall k\leqq n\ ,\\ 0\leqq S_{N,n}-S_n\leqq n2^{-N}\ ,\\ \tau_N\geqq\tau\ ,\end{cases}$$

holds on the set $\tilde{\Gamma}_{N,n}=\{\tau=n\}\cap\{\cap_{k=1}^{n}\Gamma_{N,k}\}$. Therefore

$$(4.2)\qquad \begin{cases}X_{N,k}=\varphi_N(X_k) & \text{for } 1\leqq\forall k\leqq\tau\ ,\\ 0\leqq S_{N,n}-S_n\leqq n2^{-N} & \text{for } 1\leqq\forall n\leqq\tau\ ,\\ \tau_N=\tau\ ,\end{cases}$$

holds on the set $\Gamma_N=\{\cup_{n=1}^{\infty}\tilde{\Gamma}_{N,n}\}\cap\{S_\tau<-\tau 2^{-N}\}$. Since $P\{\Gamma_{N,k}\}\to 1$ as $N\to\infty$ for each fixed k and τ is finite a.s., it is easy to see that

$$(4.3)\qquad \lim_{N\to\infty} P\{\Gamma_N\}=1\ .$$

The assumption (1.1) implies that there exists $\delta>0$ such that $P\{X_1<-\delta\}\geqq\delta$. Then from the definition of $X_{N,k}$ it follows that

$$P\{X_{N,k}<-\delta/2\}\geqq P\{(X_1<-\delta)\cap\Gamma_{N,1}\}\geqq\delta/2$$

for all sufficiently large N, say for $N\geqq N_0$. Let $x>0$ be given and put

$$\nu=[2x/\delta]+1\ ,$$
$$\sigma_N^a=\min\{n\geqq 1: a+S_{N,n}\notin[0, x)\}\ ,\quad 0\leqq a<x\ .$$

Then for $N\geqq N_0$ we have

$$(4.4)\qquad \begin{aligned}P\{\sigma_N^a>\nu\}&\leqq 1-P\{\sigma_N^a\leqq\nu\}\\ &\leqq 1-P\{X_{N,k}\leqq-\delta/2,\ 1\leqq\forall k\leqq\nu\}\\ &\leqq 1-(\delta/2)^\nu<1\ ,\qquad 0\leqq a<x\ .\end{aligned}$$

Note that (4.4) implies that there exist constants $c>0$ and $\theta\in[0, 1)$ (which depend on x) such that

$$P\{\sigma_N^a > n\} \leqq c\theta^n \qquad \text{for} \quad \forall n \geqq 1,\ 0 \leqq \forall a < x,\ \forall N \geqq N_0\,,$$

from which it follows that

$$(4.5) \qquad \begin{cases} M_1 \equiv \sup\limits_{\substack{0 \leqq a < x \\ N \geqq N_0}} E\{\sigma_N^a\} < \infty\,, \\ M_2 \equiv \sup\limits_{\substack{0 \leqq a < x \\ N \geqq N_0}} [E\{(\sigma_N^a)^2\}]^{1/2} < \infty\,. \end{cases}$$

For typographical convenience we often write $S_N(n)$ instead of $S_{N,n}$. We define $\sigma_{N,k}$, $k \geqq 0$, as follows:

i) $\sigma_{N,0} = 0$, $\sigma_{N,1} = \sigma_N^0$.

ii) If $\sigma_{N,j}$, $0 \leqq j \leqq k$ $(k \geqq 1)$, are defined, we define $\sigma_{N,k+1}$ by

$$\sigma_{N,k+1} = \begin{cases} \sigma'_{N,k+1} + \sigma''_{N,k+1} & \text{if } S_N(\sigma_{N,k}) \geqq x\,, \\ \tau_N & \text{otherwise}\,, \end{cases}$$

where

$$\sigma'_{N,k+1} = \min\{n \geqq \sigma_{N,k} : S_{N,n} \leqq x\}\,,$$

$$\sigma''_{N,k+1} = \begin{cases} \min\{n \geqq 1 : S_N(\sigma'_{N,k+1} + n) \notin [0, x)\} & \text{if} \quad S_N(\sigma'_{N,k+1}) \in [0, x)\,, \\ 0 & \text{if} \quad S_N(\sigma'_{N,k+1}) < 0\,. \end{cases}$$

Then we have

$$(4.6) \qquad \begin{aligned} & E\Big\{\sum_{k=0}^{\tau_N} \mathbf{1}_{[0,x)}(S_{N,k})\,;\ \tau_N > n\Big\} \\ & = \sum_{k=1}^{\infty} E\Big\{\sum_{\sigma_{N,k-1} \leqq j < \sigma_{N,k}} \mathbf{1}_{[0,x)}(S_{N,j})\,;\ \tau_N > n,\ \sigma_{N,k-1} < \tau_N\Big\} \\ & = \sum_{k=1}^{\infty} E\Big\{\sum_{\sigma'_{N,k} \leqq j \leqq \sigma_{N,k}} \mathbf{1}_{[0,x)}(S_{N,j})\,;\ \tau_N > n,\ \sigma_{N,k-1} < \tau_N\Big\} \\ & \hspace{18em} (\text{we put } \sigma'_{N,1} = 0) \\ & \leqq \sum_{k=1}^{m} E\Big\{\sum_{\sigma'_{N,k} \leqq j \leqq \sigma_{N,k}} \mathbf{1}_{[0,x)}(S_{N,j})\,;\ \tau_N > n,\ \sigma_{N,k-1} < \tau_N\Big\} \\ & \quad + \sum_{k=m+1}^{\infty} E\Big\{\sum_{\sigma'_{N,k} \leqq j \leqq \sigma_{N,k}} \mathbf{1}_{[0,x)}(S_{N,j})\,;\ \sigma_{N,k-1} < \tau_N\Big\} \\ & \leqq \sum_{k=1}^{m} E\Big\{\Big|\sum_{\sigma'_{N,k} \leqq j \leqq \sigma_{N,k}} \mathbf{1}_{[0,x)}(S_{N,j})\Big|^2 ;\ \sigma_{N,k-1} < \tau_N\Big\}^{1/2} P\{\tau_N > n\}^{1/2} \\ & \quad + \sum_{k=m+1}^{\infty} M_1 \rho^{k-1} \qquad (m \geqq 1 \text{ being arbitrary}) \\ & \leqq m M_2 P\{\tau_N > n\}^{1/2} + M_1 \rho^m (1-\rho)^{-1} \qquad (N \geqq N_0)\,, \end{aligned}$$

where

$$\rho \equiv \sup_{\substack{0 \leqq a < x \\ N \geqq N_0}} P\{a+S_{N,n} \text{ hits } (x, \infty) \text{ before hitting } (-\infty, 0)\}$$
$$=1-\inf_{N \geqq N_0} P\{x+S_{N,n} \text{ hits } (-\infty, 0) \text{ before hitting } (x, \infty)\}$$
$$\leqq 1-\inf_{N \geqq N_0} P\{X_{N,h} \leqq -\delta/2,\ 1 \leqq \forall k \leqq \nu\} \leqq 1-(\delta/2)^{\nu} < 1 .$$

For $x>0$ we put

$$\xi^{(n)}(x)=E\left\{\sum_{k=0}^{\tau} \mathbf{1}_{[0,x)}(S_k)\ ;\ \tau \leqq n\right\} ,$$
$$\tilde{\xi}^{(n)}(x)=E\left\{\sum_{k=0}^{\tau} \mathbf{1}_{[0,x)}(S_k)\ ;\ \tau > n\right\} ,$$
$$\xi_N^{(n)}(x)=E\left\{\sum_{k=0}^{\tau_N} \mathbf{1}_{[0,x)}(S_{N,k})\ ;\ \tau_N \leqq n\right\} ,$$
$$\tilde{\xi}_N^{(n)}(x)=E\left\{\sum_{k=1}^{\tau_N} \mathbf{1}_{[0,x)}(S_{N,k})\ ;\ \tau_N > n\right\} .$$

Then we have $\xi(x)=\xi^{(n)}(x)+\tilde{\xi}^{(n)}(x)$ and a similar formula for $\xi_N(x)$. Since the probability that (4.2) holds tends to 1 as $N \to \infty$ by virtue of (4.3), we have

$$\lim_{N \to \infty} \xi_N^{(n)}(x)=\xi^{(n)}(x) \qquad \text{for each fixed } n ,$$

while (4.6) implies

$$\tilde{\xi}_N^{(n)}(x) \leqq m M_2\{P(\tau_N > n)\}^{1/2}+M_1\rho^m(1-\rho)^{-1} , \qquad N \geqq N_0$$

for any $m \geqq 1$. Therefore

$$\lim_{n \to \infty} \overline{\lim_{N \to \infty}}\, \tilde{\xi}_N^{(n)}(x)=0 ,$$

and consequently we have

$$\begin{aligned}\overline{\lim_{N \to \infty}}\, |\xi_N(x)-\xi(x)| &\leqq \overline{\lim_{N \to \infty}}\, |\xi_N^{(n)}(x)-\xi^{(n)}(x)| \\ &\quad + \overline{\lim_{N \to \infty}}\, \tilde{\xi}_N^{(n)}(x)+\tilde{\xi}^{(n)}(x) \\ &= \overline{\lim_{N \to \infty}}\, \tilde{\xi}_N^{(n)}(x)+\tilde{\xi}^{(n)}(x) \\ &\to 0 \qquad \text{as} \quad n \to \infty .\end{aligned}$$

The estimate (4.6) also implies that $\xi_N(x)$, $N \geqq 1$, are bounded by a constant which depends monotonically on x. We have therefore proved the lemma.

Now we are in the final stage of the proof of the theorem. Let f_k,

$1 \leqq k \leqq m$, ($m \geqq 1$ being arbitrary) be continuous functions on $[0, \infty)$ with compact supports and vanishing at $x=0$ and put

$$I = E\left\{\prod_{k=1}^{m} f_k(W_k)\right\},$$

$$I_N = E\left\{\prod_{k=1}^{m} f_k(W_{N,k})\right\},$$

where $\{W_{N,n}, n \geqq 0\}$ is defined from $\{S_{N,n}, n \geqq 0\}$ in a way similar to (1.2). For simplicity we also put

$$\tilde{p}(x, dy) = \hat{p}_\xi(x, dy), \qquad \tilde{p}_N(x, dy) = \hat{p}_{\xi_N}(x, dy),$$

where $\hat{p}_{\xi_N}(x, dy)$ is defined in a way similar to (1.3). Then by the result in the special case we can write

$$\begin{aligned} (4.7) \qquad I_N &= \int_{(0,\infty)} \tilde{p}_N(0, dx_1) f_1(x_1) \int_{(0,\infty)} \tilde{p}_N(x_1, dx_2) f_2(x_2) \cdots \int_{(0,\infty)} \tilde{p}_N(x_{m-1}, dx_m) f_m(x_m) \\ &= E\{f_1(-S_{N,1}) f_2(-S_{N,2}) \cdots f_m(-S_{N,m}) \xi_N(-S_{N,m})\}, \end{aligned}$$

where we put, for $x<0$, $\xi_N(x) = f_k(x) = 0$ $(1 \leqq k \leqq m)$. Since $\xi_N(x)\mathbf{1}_{(0,\infty)}(x)$ and $\xi(x)\mathbf{1}_{(0,\infty)}(x)$ are left continuous nondecreasing functions, (4.2), (4.3) and Lemma 4 imply that $\xi_N(-S_{N,m})$ converges to $\xi(-S_m)$ in probability as $N \to \infty$, and consequently the second expression of (4.7) tends, as $N \to \infty$, to

$$\begin{aligned} (4.8) \qquad & E\{f_1(-S_1) f_2(-S_2) \cdots f_m(-S_m) \xi(-S_m)\} \\ &= \int_{(0,\infty)} \tilde{p}(0, dx_1) f_1(x_1) \int_{(0,\infty)} \tilde{p}(x_1, dx_2) f_2(x_2) \cdots \int_{(0,\infty)} \tilde{p}(x_{m-1}, dx_m) f_m(x_m). \end{aligned}$$

Therefore $I = \lim_{N\to\infty} I_N =$ the right hand side of (4.8). This completes the proof of the theorem in the general case.

§5. Examples.

EXAMPLE 1. Simple random walk. We consider the case where $X_1, X_2, \cdots$ are i.i.d. with

$$P\{X_k = 1\} = p, \qquad P\{X_k = -1\} = 1 - p = q.$$

We assume that $0 < p \leqq 1/2$, so that the condition (1.1) is satisfied. An easy calculation shows that

$$g_0(0, x) = r^x/q, \qquad x \geqq 0,$$

where $r = p/q$, and consequently for $x \geqq 1$

$$\xi(x)=\begin{cases}\dfrac{x}{q} & \text{if} \quad p=\dfrac{1}{2}\,,\\[2mm] \dfrac{1}{q}\cdot\dfrac{1-r^x}{1-r} & \text{if} \quad 0<p<\dfrac{1}{2}\,.\end{cases}$$

Therefore, if $p=1/2$, then

$$\hat{p}_\xi(x, y)=\begin{cases}\dfrac{x-1}{2x} & \text{for} \quad x\geqq 2,\ y=x-1\,,\\[2mm] \dfrac{x+1}{2x} & \text{for} \quad x\geqq 2,\ y=x+1\,,\\[2mm] 0 & \text{for} \quad x\geqq 2,\ y\neq x\pm 1\,;\end{cases}$$

if $0<p<1/2$, then

$$\hat{p}_\xi(x, y)=\begin{cases}p\cdot\dfrac{1-r^{x-1}}{1-r^x} & \text{for} \quad x\geqq 2,\ y=x-1\,,\\[2mm] q\cdot\dfrac{1-r^{x+1}}{1-r^x} & \text{for} \quad x\geqq 2,\ y=x+1\,,\\[2mm] 0 & \text{for} \quad x\geqq 2,\ y\neq x\pm 1\,;\end{cases}$$

in either case

$$\hat{p}_\xi(0, 1)=1\,, \qquad \hat{p}_\xi(0, y)=0 \quad \text{for} \ 0\leqq y\neq 1\,,$$
$$\hat{p}_\xi(1, 2)=1\,, \qquad \hat{p}_\xi(1, y)=0 \quad \text{for} \ 0\leqq y\neq 2\,.$$

Let $\hat{X}_1, \hat{X}_2, \cdots$ be i.i.d. random variables with $\hat{X}_k \overset{d}{=} -X_1$ where $\overset{d}{=}$ means the equality in distribution, and consider the random walk $\hat{S}_n = \hat{X}_1+\cdots+\hat{X}_n$ ($\hat{S}_0=0$). Let $\{W_n, n\geqq 0\}$ be the Markov chain (1.2) and let $\{\hat{V}_n, n\geqq 0\}$ be defined by

$$\hat{V}_n=\begin{cases}0 & \text{for} \quad n=0\\ 1+\hat{U}_{n-1} & \text{for} \quad n\geqq 1\,,\end{cases}$$

where $\hat{U}_n=\hat{S}_n-2\min_{0\leq k\leq n}\hat{S}_k$, $n\geqq 0$. Then by virtue of Lemma 2 it is not hard to see that

$$\{W_n, n\geqq 0\}\overset{d}{=}\{\hat{V}_n, n\geqq 0\}\,,$$

from which one can obtain the following Pitman's theorem for the random walk $\hat{S}_n$ ([7]): $\{\hat{U}_n, n\geqq 0\}$ is a Markov chain on $\{0, 1, \cdots\}$ with transition function

$$p^*(x, y)=\begin{cases} 1 & \text{for } x=0,\ y=1\ , \\ p\cdot\dfrac{1-r^x}{1-r^{x+1}} & \text{for } x\geqq 1,\ y=x-1\ , \\ q\cdot\dfrac{1-r^{x+2}}{1-r^{x+1}} & \text{for } x\geqq 1,\ y=x+1\ , \\ 0 & \text{otherwise}\ . \end{cases}$$

(Note that, in the case $p=1/2$, $p(1-r^x)(1-r^{x+1})^{-1}$ and $q(1-r^{x+2})(1-r^{x+1})^{-1}$ in the above must be replaced by $2^{-1}(x+1)^{-1}x$ and $2^{-1}(x+1)^{-1}(x+2)$, respectively.)

The above Pitman's theorem for $\hat{U}_n$ holds also for $1/2<p<1$ as can be proved directly (e.g. see [10]). Now suppose a constant b is given and, for each $\varepsilon>0$ which is assumed to be small, consider the random walk $\hat{S}_n^{(\varepsilon)}$ with

$$P\{\hat{X}_k=1\}=\frac{1+b\varepsilon}{2}\ , \qquad P\{\hat{X}_k=-1\}=\frac{1-b\varepsilon}{2}\ .$$

Taking the weak limit of scaled $\varepsilon\hat{S}^{(\varepsilon)}_{[t/\varepsilon^2]}$ and $\varepsilon\hat{U}^{(\varepsilon)}_{[t/\varepsilon^2]}$ as $\varepsilon\downarrow 0$, we can obtain the following result: If $B(t)$ is a Brownian motion with constant drift, i.e., a diffusion process with generator $(1/2)(d^2/dx^2)+b(d/dx)$ starting from 0, then $B(t)-2\min_{0\leq s\leq t}B(s)$ is a diffusion process on $[0, \infty)$ with generator $(1/2)(d^2/dx^2)+b\coth(bx)(d/dx)$ starting from 0. This result was obtained also by Pitman.

EXAMPLE 2. Let f be the common probability density of the i.i.d. random variables defining the random walk S_n. We consider the following two cases:

Case (i) (Bilateral exponential distribution):

$$f(x)=\begin{cases} \dfrac{ab}{a+b}e^{ax} & \text{for } x<0\ , \\ \dfrac{ab}{a+b}e^{-bx} & \text{for } x>0\ . \end{cases} \tag{5.1}$$

Case (ii) (Modified bilateral exponential distribution):

$$f(x)=\begin{cases} \dfrac{a^2}{a+b}e^{ax} & \text{for } x<0\ , \\ \dfrac{a^2}{a+b}e^{-bx} & \text{for } x>0\ . \end{cases} \tag{5.2}$$

Here a and b are positive constants. If $a=b$, the two cases coincide.

It is assumed that $a<b$ in the case (i), so that $E\{X_k\}=b^{-1}-a^{-1}<0$. In the case (ii) $E\{X_k\}=0$ holds always. Therefore, the condition (1.1) is satisfied in either case. Let $G(0, dx)$ be defined by (2.1) and, for an interval $[r_1, r_2]$ containing 0, put

$$T=\min\{n\geqq 1: S_n \notin [r_1, r_2]\} .$$

We apply the Fourier method as explained in Feller [1: p. 600]. After somewhat messy computation we obtain the following result.

Case (i):

$$G(0, dx)=\delta_0(dx)+ae^{-(b-a)x}dx . \tag{5.3}$$

$$\begin{cases} P\{S_T<r_1\}=e^{ar_1}\cdot\dfrac{a^{-1}be^{-ar_2}-e^{-br_2}}{a^{-1}be^{-(r_2-r_1)a}-ab^{-1}e^{-(r_2-r_1)b}} , \\ P\{S_T>r_2\}=e^{-br_2}\cdot\dfrac{e^{ar_1}-ab^{-1}e^{br_1}}{a^{-1}be^{-(r_2-r_1)a}-ab^{-1}e^{-(r_2-r_1)b}} . \end{cases} \tag{5.4}$$

Case (ii):

$$G(0, dx)=\delta_0(dx)+bdx . \tag{5.5}$$

$$\begin{cases} P\{S_T<r_1\}=\dfrac{r_2+b^{-1}}{r_2-r_1+a^{-1}+b^{-1}} , \\ P\{S_T>r_2\}=\dfrac{-r_1+a^{-1}}{r_2-r_1+a^{-1}+b^{-1}} . \end{cases} \tag{5.6}$$

REMARK. In the case (ii) we have for $x>0$

$$\begin{aligned} \xi(x)&=G(0, [0, x)) \\ &=\text{const.} \lim_{\lambda\to\infty} \lambda P\{S_n \text{ hits } I_\lambda \text{ before it hits } I\} \end{aligned} \tag{5.7}$$

where $I_\lambda=(-\infty, -\lambda]$ and $I=[x, \infty)$. Comparing our theorem with Lemma 6 of Golosov [2], we see that (5.7) holds in general if the random walk has zero expectation and finite variance; naturally this fact itself can be verified by a more direct method.

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