Interpolation Theory on Sobolev Spaces over the Abstract Wiener Space

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(Communicated by Y. Ito)

Abstract. In recent years, the theory of Sobolev spaces over the abstract Wiener space was constructed in the study of stochastic differential equations ([9, 11, 15]). In the present paper, we prove an interpolation theorem on Sobolev spaces over the abstract Wiener space.

§1. Introduction.

The following interpolation theorem on Sobolev spaces over the Euclidean space R^n with the ordinary Lebesgue measure

$$W^{s,p}(R^n) = \{f \in L^1_{\mathrm{loc}}(R^n) \mid \|(I-\Delta)^s f\|_p < \infty\}$$

($s \in R$, $1 < p < \infty$, and Δ is the Laplace operator) is well-known:

THEOREM A (cf. Theorem 6.4.5 of [1]). *If s_0, $s_1 \in R$ and $1 < p_0$, $p_1 < \infty$,*

$$[W^{s_0,p_0}, W^{s_1,p_1}]_\theta = W^{s,p}$$

where

$$s = (1-\theta)s_0 + \theta s_1, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad 0 < \theta < 1.$$

Here, $[X, Y]_\theta$ is an interpolation space between the Banach spaces X, Y (about this notation, see [1] Chapter 4).

Usually, Theorem A is proved by using the "Littlewood-Paley theory" as follows.

LEMMA B. *Let $1 < p < \infty$ and $s \in R$. Then,*

$$W^{s,p}(R) \sim L^p(l_2^s)$$

where $L^p(l_2^s)$ is the space of l_2^s-valued L^p-functions and

$$l_2^s = \left\{ (a_n)_{n=-\infty}^{\infty} \mid a_n \in C \text{ for any } n \text{ and } \|(a_n)\|_{l_2^s} = \left[\sum_{n=-\infty}^{\infty} (2^{sn} a_n)^2 \right]^{1/2} < \infty \right\}$$

(see Chapter 7 of [2] and Theorem 6.4.3 of [1]).

In this paper, we shall consider the Sobolev spaces over the abstract Wiener space (B, H, μ), where B is a separable real Banach space with the norm $\|\cdot\|_B$, H is a separable real Hilbert space with the inner product $\langle \cdot, \cdot \rangle_H$ and the norm $|\cdot|_H = \sqrt{\langle \cdot, \cdot \rangle_H}$, which is densely and continuously imbedded into B, and μ is a normal mean zero Gaussian measure on B i.e. a Borel probability measure on B which satisfies

(1.1) $$\int_B e^{i\varphi(w)} d\mu(w) = \exp\left(-\frac{1}{2} |\varphi|_H^2 \right) \quad \text{for every } \varphi \in B^* .$$

As is known, one of the most important examples of the abstract Wiener space is the Gaussian-Euclidean space, the case $B = H = R^n$, Euclidean space, and μ is the Gaussian measure

$$d\mu(x) = \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2} |x|^2 \right) dx$$

where dx is the Lebesgue measure on R^n.

However, the Gaussian-version of Lemma B is unknown. So, in this paper, we shall show the similar theory over the abstract Wiener space (B, H, μ) straightforwards. In § 2, we shall state some definitions and basic theorems. In § 3, we shall show the L^p-boundedness of "Bessel potential" operators $(I-L)^{iy}$, where $y \in R$ and L is the Ornstein-Uhlenbeck operator which plays a role of Laplacian operator over the abstract Wiener spaces. And in § 4, we shall show an interpolation theorem on the Sobolev spaces over the abstract Wiener space (B, H, μ) by using the results of § 3.

Here the author wishes to express his thanks to Professor S. Koizumi and Professor S. Ishikawa for their valuable advices and encouragement, and to the referee for his valuable advice.

§ 2. Basic notations.

Let (B, H, μ) be the abstract Wiener space. We shall state several definitions.

DEFINITION 2.1. Let $\{\varphi_n\}_{n=1}^{\infty} \subset B^*$ be complete orthonormal system in H, and let p be a polynomial of several variables with complex coefficients. Then a function on B,

(2.1) $f(w) = p(\varphi_1(w), \varphi_2(w), \cdots, \varphi_n(w))$,

is said to be a polynomial on B. We call the degree of p the one of f and a family of all polynomials is denoted by P. It is known that P is a dense subset of $L^p(B, \mu)$ $(1 \leq p < \infty)$ (see [5]).

DEFINITION 2.2 (The Ornstein-Uhlenbeck semigroup). For $t \geq 0$, we define an operator T_t by

(2.2) $T_t f(w) = \int_B f(e^{-t}w + \sqrt{1 - e^{-2t}}v) d\mu(v)$ for every $f \in P$.

The family $\{T_t\}_{t \geq 0}$ is a C_0-contraction symmetric semigroup in $L^p(B, \mu)$ for every $1 \leq p < \infty$ (cf. [6]). We call it the Ornstein-Uhlenbeck semigroup, and the infinitesimal generator L of it is called the Ornstein-Uhlenbeck operator. As is known, $L^2(B, \mu)$ is decomposed into orthogonal direct sum;

$$L^2(B, \mu) = \bigoplus_{n=0}^{\infty} C_n ,$$

called Wiener-Itô decomposition, where $C_0 = \{$the set of all constant functions$\}$ and

$C_n = \{L^2$-closure of the set of all polynomials of degree $n\} \ominus (C_0 \oplus \cdots \oplus C_{n-1})$

for $n \geq 1$. Here we define that J_n is the orthogonal projection from $L^2(B, \mu)$ to C_n $(n = 0, 1, 2, \cdots)$. From the property of C_n, we have

(2.3) $Lf = \sum_{n=0}^{\infty} (-n) J_n f$ $(f \in P)$

and hence

(2.4) $T_t f = \sum_{n=0}^{\infty} e^{-nt} J_n f$ $(f \in P)$.

We shall consider multiplier operator $M_m : P \to P$ given by

(2.5) $M_m f = \sum_{n=0}^{\infty} m(n) J_n f$ for every $f \in P$

where $m(n)$ is a complex sequence. The L^p-boundedness of such M_m is an interesting problem. P. A. Meyer and I. Shigekawa showed it in a certain case.

THEOREM 2.1 (P. A. Meyer-I. Shigekawa [11], [13]). *If $m(n)$ is a real sequence and there exist a function $h(\cdot)$ which is analytic on some*

neighbourhood of the origin and a positive constant α such that $m(n) = h(n^{-\alpha})$, then M_m can uniquely extend to a bounded linear operator on L^p for each $1 < p < \infty$.

From (2.3), it is natural to define

$$(2.6) \qquad (I-L)^s f = \sum_{n=0}^{\infty} (1+n)^s J_n f \qquad \text{for every } f \in P, \ s \in C,$$

and to call it "Bessel potential" operator. Now we define the Sobolev spaces over the abstract Wiener space.

DEFINITION 2.3. For $s \in R$, $1 < p < \infty$, let $W^{s,p}$ be the completion of P by the norm $\|\cdot\|_{s,p}$, where

$$\|f\|_{s,p} = \|(I-L)^{s/2} f\|_p \qquad \text{for every } f \in P.$$

We shall state a property of $W^{s,p}$ (see [13]).

PROPOSITION 2.2. *The dual space $W^{s,p}$ is $W^{-s,p'}$ where $1/p + 1/p' = 1$ $(1 < p < \infty, \ s \in R)$.*

§3. The Marcinkiewicz-Stein multiplier theorem.

E. M. Stein proved the compact Lie group version of the "Marcinkiewicz multiplier theorem" by using the Littlewood-Paley theory [12]. He also said that it can be extended to the general case, but the proof of it is informal. In this section, we prove the abstract Wiener space version of "Marcinkiewicz theorem" rigorously and prove, as a corollary, the L^p-boundedness of the "Bessel potential" operator $(I-L)^{iy}$, where $y \in R$.

Let $0 < \beta \leq 1$ and we define the operator $T_t^{(\beta)}: P \to P$ by

$$(3.1) \qquad T_t^{(\beta)} f = \sum_{n=0}^{\infty} e^{-n^\beta t} J_n f \qquad \mu\text{-a.e. for every } f \in P.$$

It is known that $T_t^{(\beta)}$ is a C_0-semigroup and has four more properties as follows.

1. (Contractivity)
For $1 \leq p \leq \infty$ and for every $f \in L^p(B, \mu)$,

$$(3.2) \qquad \|T_t^{(\beta)} f\|_p \leq \|f\|_p.$$

2. (Self-adjointness)
For every $f, g \in L^2(B, \mu)$,

$$(3.3) \qquad \int_B (T_t^{(\beta)} f(w)) g(w) d\mu(w) = \int_B f(w)(T_t^{(\beta)} g(w)) d\mu(w).$$

3.   (Positivity)

If $f(w) \geqq 0$ for almost all $w \in B$, $T_t^{(\beta)} f(w) \geqq 0$ for almost all $w \in B$.

4.   (Conservativity)

$$T_t^{(\beta)} 1 = 1$$

In general the C_0-contraction semigroup which satisfies these four properties is called the symmetric diffusion semigroup. For such a semigroup, the Littlewood-Paley theory was constructed.

THEOREM 3.1 (E. M. Stein [12]).   *Let* $0 < \beta \leqq 1$. *For* $f \in P$, *we define*

$$(3.4) \qquad g_{k,\beta}(f)(w) = \left[\int_0^\infty t^{2k-1} \left| \frac{\partial^k}{\partial t^k} T_t^{(\beta)} f(w) \right|^2 dt \right]^{1/2} \qquad (k = 1, 2, \cdots) .$$

Then, for $1 < p < \infty$, *there exist some constants* $a_{p,k,\beta}$, $A_{p,k,\beta}$ *which depend only on* p, k, β *such that*

$$(3.5) \qquad a_{p,k,\beta} \|(I - J_0) f\|_p \leqq \|g_{k,\beta}(f)\| \leqq A_{p,k,\beta} \|f\|_p .$$

Now we consider the Marcinkiewicz-Stein multiplier theorem.

DEFINITION 3.1.   Let $a(\cdot)$ be a bounded complex function on $[0, \infty)$. Then we define the complex function $m(\cdot)$ as follows;

$$(3.6) \qquad m(\lambda) = \begin{cases} 0 , & \text{for } \lambda = 0 \\ \lambda \int_0^\infty e^{-\lambda s} a(s) ds , & \text{for } \lambda > 0 , \end{cases}$$

and call it a function of Laplace transform type.

THEOREM 3.2.   *Let* $m(\cdot)$ *be a function of Laplace transform type and* $0 < \beta \leqq 1$. *Then the multiplier operator*

$$(3.7) \qquad M_m f(w) = \sum_{n=0}^\infty m(n^\beta) J_n f(w) , \qquad f \in P$$

is L^p *bounded* $(1 < p < \infty)$, *i.e.,*

$$(3.8) \qquad \|M_m f\|_p \leqq A_p \|f\|_p \qquad \text{for every } f \in P$$

where the positive constant A_p *depends only upon* p.

PROOF.   Let us take $f \in P$ and fix it, and let us put

$$(3.9) \qquad F(w) = - \int_0^\infty \frac{\partial}{\partial t} T_t^{(\beta)} f(w) \cdot a(t) dt .$$

Then $F(w) = M_m f(w)$. In fact, since $J_n f = 0$ for all but finite n, we have

$$\int_0^\infty \frac{\partial}{\partial t} T_t^{(\beta)} f(w) a(t) dt = \sum_{n=0}^\infty \int_0^\infty -n^\beta e^{-n^\beta t} J_n f(w) a(t) dt$$

$$= \sum_{n=0}^\infty -m(n^\beta) J_n f(w) \ .$$

Applying the operator $T_{t_1}^{(\beta)}$ to the both sides of (3.9), we have

(3.10)
$$T_{t_1}^{(\beta)} F(w) = -\int_0^\infty T_{t_1}^{(\beta)} \left(\frac{\partial}{\partial t} T_t^{(\beta)} f(w) \right) \cdot a(t) dt$$

$$= -\int_0^\infty \frac{\partial}{\partial t} T_{t+t_1}^{(\beta)} f(w) \cdot a(t) dt \ ,$$

and applying the operator $\sum_{n=0}^\infty (-n^\beta) J_n$ to the both sides of (3.10), we get

$$\left| \frac{\partial}{\partial t_1} T_{t_1}^{(\beta)} F(w) \right| \leq \int_0^\infty \left| \frac{\partial^2}{\partial t^2} T_{t+t_1}^{(\beta)} f(w) \right| |a(t)| dt$$

$$\leq K \int_0^\infty \left| \frac{\partial^2}{\partial t^2} T_{t+t_1}^{(\beta)} f(w) \right| dt$$

$$= K \int_{t_1}^\infty \left| \frac{\partial^2}{\partial t^2} T_t^{(\beta)} f(w) \right| dt$$

where $K = \sup_{t \geq 0} |a(t)|$. Hence

$$(g_{1,\beta}(M_m f))^2 = \int_0^\infty t_1 \left| \frac{\partial}{\partial t_1} T_{t_1}^{(\beta)} F(w) \right|^2 dt_1$$

$$\leq K^2 \int_0^\infty t_1 \left(\int_{t_1}^\infty \left| \frac{\partial^2}{\partial t^2} T_t^{(\beta)} f(w) \right| dt \right)^2 dt_1 \ .$$

Applying the Schwarz inequality,

$$\leq K^2 \int_0^\infty t_1 \left(\int_{t_1}^\infty t^2 \left| \frac{\partial^2}{\partial t^2} T_t^{(\beta)} f(w) \right|^2 dt \frac{1}{t_1} \right) dt_1$$

$$= K^2 \int_0^\infty \left(\int_0^t t^2 \left| \frac{\partial^2}{\partial t^2} T_t^{(\beta)} f(w) \right|^2 dt_1 \right) dt$$

$$= K^2 \int_0^\infty t^3 \left| \frac{\partial^2}{\partial t^2} T_t^{(\beta)} f(w) \right|^2 dt$$

$$= K^2 (g_{2,\beta}(f))^2 \ .$$

Therefore we have

(3.11)
$$|g_{1,\beta}(M_m f)(w)| \leq K |g_{2,\beta}(f)(w)| \ ,$$

and hence we have, by Theorem 3.1,

$$\|(I-J_0)M_m f\|_p \leqq (a_{p,1,\beta})^{-1}\|g_{1,\beta}(M_m f)\|_p$$

$$\leqq (a_{p,1,\beta})^{-1}K\|g_{2,\beta}(f)\|_p$$

$$\leqq \frac{A_{p,2,\beta}}{a_{p,1,\beta}}K\|f\|_p .$$

On the other hand, $(I-J_0)M_m f = M_m f$ because $m(0)=0$. Hence we obtain

$$\|M_m f\|_p \leqq A_p \|f\|_p$$

where the positive constant A_p depends only on p. □

E. M. Stein showed the L^p-boundedness of the "Riesz potential" operator $(-L)^{iy}$ $(y \in R)$ by using this theorem. In the same way, using this theorem and the Meyer-Shigekawa theorem (Theorem 2.1), we have the L^p-boundedness of the "Bessel potential" operator as follows.

THEOREM 3.3. *The "Bessel potential" operator* $(I-L)^{iy}$ *is a bounded operator in* $L^p(B, \mu)$ $(1<p<\infty)$ *for every* $y \in R$.

PROOF. For every $\lambda>0$ and $y \in R$,

$$(3.12) \qquad (1+\lambda)^{iy} = \frac{1+\lambda}{\Gamma(1-iy)}\int_0^\infty e^{-(\lambda+1)s}s^{-iy}ds$$

$$= \frac{1}{\Gamma(1-iy)}\left(1+\frac{1}{\lambda}\right)\cdot\lambda\int_0^\infty e^{-\lambda s}(e^{-s}\cdot s^{-iy})ds .$$

Now we denote

$$(3.13) \qquad m_1(\lambda) = 1+\frac{1}{\lambda} ,$$

$$(3.14) \qquad m_2(\lambda) = \lambda\int_0^\infty e^{-\lambda s}(e^{-s}\cdot s^{-iy})ds .$$

By Theorem 2.1, the multiplier operator

$$f \longrightarrow \sum_{n=0}^\infty m_1(n)J_n f \qquad (f \in P)$$

is L^p-bounded $(1<p<\infty)$.

On the other hand, since $|e^{-s}\cdot s^{-iy}|\leqq 1$ for every $s>0$, the multiplier operator

$$f \longrightarrow \sum_{n=1}^\infty m_2(n)J_n f \qquad (f \in P)$$

is also L^p-bounded $(1<p<\infty)$ by Theorem 3.2. Now we can denote

(3.15) $$(I-L)^{iy}f = J_0 f + \frac{1}{\Gamma(1-iy)} \sum_{n=1}^{\infty} m_1(n)m_2(n)J_n f$$

for every $f \in P$.   Hence we get

(3.16) $$\|(I-L)^{iy}f\|_p \le C_p \cdot K_y \|f\|_p$$

where C_p depends only on p, and K_y depends only on y.   □

REMARK. K_y may be regarded as a continuous function of $y \in R$, and

(3.17) $$A \le K_y \Big/ \Big(\frac{e^{(\pi/2)y}}{1+|y|}\Big) \le B$$

where the positive constants A, B are independent of $y \in R$.

§4. Interpolation theory on Sobolev spaces over the abstract Wiener space.

In this section, we shall prove an interpolation theorem on Sobolev spaces over the abstract Wiener space.

MAIN THEOREM.   *Let* $1 < p_0$, $p_1 < \infty$ *and* s_0, $s_1 \in R$.   *Then*

$$[W^{s_0, p_0}, W^{s_1, p_1}]_\theta = W^{s, p} \qquad for\ every\ \theta \in [0, 1]$$

where $s = (1-\theta)s_0 + \theta s_1$, $1/p = (1-\theta)/p_0 + \theta/p_1$.   (About these notations, see [9] Appendix B.)

We divide the proof of the theorem into 2 parts, Theorem 4.1 and Theorem 4.2.   Without loss of generality, we can assume $s_0 \le s_1$.

THEOREM 4.1.   *Under the same conditions as the Main Theorem,*

$$W^{s, p} \subset [W^{s_0, p_0}, W^{s_1, p_1}]_\theta .$$

In order to prove this theorem, we shall show the following lemmas.

LEMMA 4.1.   *Let* $D = \{z \in C \mid 0 \le \mathrm{Re}.z \le 1\}$ *and* $1/p_z = (1-z)/p_0 + z/p_1$.   *For every* $f \in P$, *the function* Ψ *defined by*

(4.1) $$\Psi(z) = |(I-L)^{s/2}f|^{p/p_z}$$

is an L^{p_0}-*valued continuous function on* D.

PROOF.   Let $z \in D$ and $z + \Delta z \in D$.   Then we have

$$\|\Psi(z+\varDelta z)-\Psi(z)\|_{p_0}^{p_0}=\int_B|\ |(I-L)^{s/2}f(w)^{p/p_0+p(1/p_1-1/p_0)(z+\varDelta z)}$$
$$-|(I-L)^{s/2}f(w)|^{p/p_0+p(1/p_1-1/p_0)z}|^{p_0}d\mu(w)\ .$$

Now the integrand is dominated by the function

$$\Psi_0(w)=[2\times\max\{1,\ |(I-L)^{s/2}f|\}]^q \qquad \text{where}\ \ q=\max\{p_0,\ p_1\}$$

and Ψ_0 is integrable on B.   Hence, by the Lebesgue dominated convergence theorem, we have

$$\lim_{|\varDelta z|\to 0}\|\Psi(z+\varDelta z)-\Psi(z)\|_{p_0}^{p_0}=0\ .$$

Therefore $\Psi(z)$ is continuous on D.   □

LEMMA 4.2.   *For every $\varepsilon>0$ and $z\in D$, there exists $\delta>0$ and $f_{\varepsilon,z}\in P$ such that*

(4.2) $\|\Psi(z+\varDelta z)-f_{\varepsilon,z}\|_{p_0}<\varepsilon \qquad$ *for every $|\varDelta z|<\delta$.*

PROOF.   Fix $z\in D$.   For every $\varepsilon>0$, there exists $\delta>0$ such that

$$\|\Psi(z+\varDelta z)-\Psi(z)\|_{p_0}<\varepsilon/2$$

for every $\varDelta z\in C$ which satisfies $|\varDelta z|<\delta$ and $z+\varDelta z\in D$.   On the other hand, since $\Psi(\cdot)\in L^{p_0}$, there exists $f_{\varepsilon,z}\in P$ such that

$$\|\Psi(z)-f_{\varepsilon,z}\|_{p_0}<\varepsilon/2\ .$$

Hence we have proved this lemma.   □

LEMMA 4.3.   *Let $s_z=(1-z)s_0+zs_1$.   For $f\in P$, $z\in D$, the function Φ defined by*

(4.3) $\Phi(z)=e^{z^2-\theta^2}(I-L)^{-s_z/2}[\text{sign}\{(I-L)^{s/2}f\}\cdot|(I-L)^{s/2}f|^{p/p_z}]$

is a W^{s_0,p_0}-valued continuous function on D where $\text{sign}\,\zeta=\zeta/|\zeta|$ for every $\zeta\in C$.

PROOF.   Since $f\in P$ and $s_0\leqq s_1$, it is easy to show $\Phi(z)\in W^{s_0,p_0}$ for every $z\in D$.   The continuity of Φ is equivalent to the one of the function

(4.4) $\Phi_0(z)=(I-L)^{-s_z/2}[|(I-L)^{s/2}f|^{p/p_z}]\ .$

Now we show the continuity of Φ_0.   In the following, we denote $\varDelta z=\varDelta x+i\varDelta y$ where $\varDelta x$, $\varDelta y\in R$.   Fix any $\varepsilon>0$ and let $f_{\varepsilon,z}$ and $\delta>0$ satisfy Lemma 4.2.   And take $|\varDelta z|<\delta$.   Now,

$$\|\Phi_0(z+\Delta z)-\Phi_0(z)\|_{s_0,p_0}$$

$$= \|(I-L)^{-(s_z+\Delta z)/2}\{|(I-L)^{s/2}f|^{p/(p_z+\Delta z)}\}-(I-L)^{-s_z/2}\{|(I-L)^{s/2}f|^{p/p_z}\}\|_{s_0,p_0}$$

$$\leqq \|(I-L)^{-(s_z+\Delta z)/2}\{|(I-L)^{s/2}f|^{p/(p_z+\Delta z)}\}-(I-L)^{-(s_z+\Delta z)/2}f_{\varepsilon,z}\|_{s_0,p_0}$$

$$+\|(I-L)^{-(s_z+\Delta z)/2}f_{\varepsilon,z}-(I-L)^{-s_z/2}f_{\varepsilon,z}\|_{s_0,p_0}$$

$$+\|(I-L)^{-s_z/2}f_{\varepsilon,z}-(I-L)^{-s_z/2}\{|(I-L)^{s/2}f|^{p/(p_z+\Delta z)}\}\|_{s_0,p_0}$$

$$+\|(I-L)^{-s_z/2}\{|(I-L)^{s/2}f|^{p/(p_z+\Delta z)}\}-(I-L)^{-s_z/2}\{|(I-L)^{s/2}f|^{p/p_z}\}\|_{s_0,p_0}$$

$$=A_1+A_2+A_3+A_4 , \quad \text{say} .$$

We have

$$A_1 = \|(I-L)^{(s_0-s_x+\Delta z)/2}(I-L)^{i(y+\Delta y)(s_0-s_1)/2}\{|(I-L)^{s/2}f|^{p/(p_z+\Delta z)}-f_{\varepsilon,z}\}\|^{p_0}$$

$$\leqq C_{p_0}\cdot K_{y+\Delta y}\cdot\varepsilon$$

by Theorem 3.3 and Lemma 4.2. From (3.17), for small $\delta>0$, we can take $K_{y+\Delta y}$ as

$$K_{y+\Delta y}\leqq 2K_y .$$

Since $f_{\varepsilon,z}\in P$, $J_n f_{\varepsilon,z}=0$ for all but finitely many n. Hence we have

$$A_2 = \|[(I-L)^{-(s_z+\Delta z)/2}-(I-L)^{-s_z/2}]f_{z,\varepsilon}\|_{s_0,p_0}$$

$$\leqq \sum_{n=0}^{\infty}|(1+n)^{-s_0/2}\{(1+n)^{(s_0-s_1)(z+\Delta z)/2}-(1+n)^{(s_0-s_1)z/2}\}|\ \|J_n f_{z,\varepsilon}\|_{p_0}$$

$$<\varepsilon$$

for sufficiently small $\delta>0$. Similarly to the first term, we can get $A_3<\varepsilon$, and we have

$$A_4 = \|(I-L)^{(s_0-s_x)/2}(I-L)^{iy(s_0-s_1)/2}\{|(I-L)^{s/2}f|^{p/(p_z+\Delta z)}-|(I-L)^{s/2}f|^{p/p_z}\}\|_{p_0}$$

$$\leqq C_p\cdot K_y\cdot\varepsilon$$

by Theorem 3.3 and Lemma 4.1. Hence Φ_0 is continuous. □

LEMMA 4.4. *For the function Φ in Lemma 4.3, there exists a constant $M>0$ such that*

(4.5) $\|\Phi(z)\|_{s_0,p_0}\leqq M$ *for every $z\in D$* .

PROOF. We have

$$\|\Phi(z)\|_{s_0,p_0}$$

$$= \|e^{z^2-\theta^2}(I-L)^{-s_z/2}[\text{sign}\{(I-L)^{s/2}f\}\cdot|(I-L)^{s/2}f|^{p/p_z}]\|_{s_0,p_0}$$

$$= e^{x^2-y^2-\theta^2}\|(I-L)^{(s_0-s_x)/2}(I-L)^{iy(s_0-s_1)/2}[\text{sign}\{(I-L)^{s/2}f\}\cdot|(I-L)^{s/2}f|^{p/p_z}]\|_{p_0}$$

$$\leqq e^{x^2-y^2-\theta^2}K_y\| \,|(I-L)^{s/2}f|^{p/p_s}\|_{p_0} \quad \text{(by Theorem 3.3)}$$

$$=e^{x^2-y^2-\theta^2}K_y\left[\int_B |(I-L)^{s/2}f|^{p\cdot p_0/p_s}d\mu\right]^{1/p_0}$$

$$\leqq e^{x^2-y^2-\theta^2}K_y\left[\int_B |(I-L)^{s/2}f|^{p\cdot p_0}d\mu\right]^{1/p_x\cdot p_0}.$$

Since K_y is continuous, and by (3.17), the function

$$e^{x^2-y^2-\theta^2}K_y$$

is bounded on D (where $z=x+iy$). Therefore we get

$$\|\varPhi(z)\|_{s_0,p_0}\leqq C\|f\|_{s,p\cdot p_0}^{p/p_x}$$
$$\leqq C(1+\|f\|_{s,p\cdot p_0})=M$$

for every $z\in D$. □

LEMMA 4.5. $\varPhi$ is a W^{s_0,p_0}-valued holomorphic function on D_0, the interior of D.

PROOF. From the general theory of vector-valued holomorphic function (cf. [6]) and Proposition 2.2, it is sufficient to show that the function

(4.6) $$F(z)=\int_B \varPhi(z)(w)g(w)d\mu(w)$$

is holomorphic on D_0 for every $g\in W^{-s_0,p_0'}$ where $1/p_0+1/p_0'=1$. Now we fix such g. Since P is a dense set of $W^{-s_0,p_0'}$, for every $k\in N$, there exists $g_k\in P$ such that

$$\|g-g_k\|_{-s_0,p_0'}<\frac{1}{k}.$$

Then

$$F(z)=\int_B \varPhi(z)(w)g_k(w)d\mu(w)+\int_B \varPhi(z)(w)[g(w)-g_k(w)]d\mu(w).$$

By Lemma 4.4 and Proposition 2.2, we have

$$\int_B \varPhi(z)(w)[g(w)-g_k(w)]d\mu(w)\leqq M\times\frac{1}{k}\to 0 \qquad \text{uniformly in } z \text{ on } D.$$

Hence the sequence of functions $\{F_k\}$;

(4.7) $$F_k(z)=\int_B \varPhi(z)(w)g_k(w)d\mu(w),$$

converges to $F(z)$ uniformly on D. Therefore, it is sufficient to show the holomorphicity of each $F_k(z)$ on D. From Lemma 4.3 and the fact $g_k \in W^{-s_0,p_0'}$, each $F_k(z)$ is continuous on D. Now, let Γ be any closed curve in D_0. Then we have

$$\int_\Gamma F_k(z)dz$$

$$= \int_\Gamma \int_B e^{z^2-\theta^2}(I-L)^{-sz/2}[\mathrm{sign}\{(I-L)^{s/2}f(w)\}$$

$$\times |(I-L)^{s/2}f(w)|^{p/p_s}]g_k(w)d\mu(w)dz$$

$$= \int_\Gamma \int_B e^{z^2-\theta^2} \mathrm{sign}\{(I-L)^{s/2}f(w)\}|(I-L)^{s/2}f(w)|^{p/p_0+p(1/p_1-1/p_0)z}$$

$$\times (I-L)^{-sz/2}g_k(w)d\mu(w)dz$$

$$= \int_B \mathrm{sign}\{(I-L)^{s/2}f(w)\}\int_\Gamma e^{z^2-\theta^2}|(I-L)^{s/2}f(w)|^{p/p_0+p(1/p_1-1/p_0)z}$$

$$\times \sum_{n=0}^{\infty}(1+n)^{-s_0+z(s_1-s_0)}J_n f_k(w)dzd\mu(w)$$

$$= 0 .$$

Here in the last formula of the above estimation, we use the following facts: $J_n f_k = 0$ for all but finite n and $\int_\Gamma (\cdot)dz = 0$ for other n. Hence, by Morera's theorem, each F_k is holomorphic on D_0. $\qquad\square$

PROOF OF THEOREM 4.1. Fix $f \in P$. Since $W^{s_0,p_0} \subset W^{s_0,p_0} + W^{s_1,p_1}$, Φ, defined in Lemma 4.3, is continuous on D and holomorphic on D_0 in the topology of $W^{s_0,p_0} + W^{s_1,p_1}$. Similar to Lemma 4.4, we get

(4.8) $$\sup_y \|\Phi(iy)\|_{s_0,p_0} \leq C\|f\|_{s,p}^{p/p_0} .$$

By Lemma 4.3, $\Phi(iy)$ is a continuous function of y. In the same way, we can show

(4.9) $$\sup_y \|\Phi(1+iy)\|_{s_1,p_1} \leq C\|f\|_{s,p}^{p/p_1}$$

and the continuity of $\Phi(1+iy)$ of y. Hence we get

$$f \in [W^{s_0,p_0}, W^{s_1,p_1}]_\theta ,$$

and we have

$$\|f\|_{[W^{s_0,p_0},W^{s_1,p_1}]_\theta}$$

$$\leq \max\{\sup_y \|\Phi(iy)\|_{s_0,p_0}, \sup_y \|\Phi(1+iy)\|_{s_1,p_1}\} \qquad \text{(see [8] Appendix B)}$$

$$\leq C(\|f\|_{s,p}^{p/p_0} + \|f\|_{s,p}^{p/p_1}) .$$

Therefore we get

$$\|f\|_{[W^{s_0,p_0},W^{s_1,p_1}]_\theta} \leq C\|f\|_{s,p} .$$

Now we complete the proof.   □

THEOREM 4.2. *Under the same conditions as the Main Theorem,*

$$[W^{s_0,p_0}, W^{s_1,p_1}]_\theta \subset W^{s,p} .$$

PROOF. P is a dense subset of $W^{s,p}$, $W^{s_0,p_0} \cap W^{s_1,p_1}$, and $[W^{s_0,p_0}, W^{s_1,p_1}]_\theta$. Hence it is sufficient to show that there exists a constant $C'>0$ such that

$$\|f\|_{s,p} \leq C'\|f\|_{[W^{s_0,p_0},W^{s_1,p_1}]_\theta} \qquad \text{for every } f \in P .$$

Now, let $g \in P \subset W^{-s,p'}$ such that $\|g\|_{-s,p'} \leq 1$. From Theorem 4.1 and Proposition 2.2,

$$W^{-s,p'} \subset [W^{-s_0,p_0'}, W^{-s_1,p_1'}]_\theta = ([W^{s_0,p_0}, W^{s_1,p_1}]_\theta)^* ,$$

hence the functional

$$f \to \int_B f(w)g(w)d\mu(w)$$

is a bounded linear functional on $[W^{s_0,p_0}, W^{s_1,p_1}]_\theta$. Hence

$$\left| \int_B f(w)g(w)d\mu(w) \right| \leq \|f\|_{[W^{s_0,p_0},W^{s_1,p_1}]_\theta} \|g\|_{[W^{-s_0,p_0'},W^{-s_1,p_1'}]_\theta}$$

$$\leq \|f\|_{[W^{s_0,p_0},W^{s_1,p_1}]_\theta} \times C\|g\|_{-s,p'} \qquad \text{(by Theorem 4.1)}$$

$$\leq C\|f\|_{[W^{s_0,p_0},W^{s_1,p_1}]_\theta} .$$

Here C is independent of g. Therefore we take the supremum with respect to g of both sides at the above inequality and get

$$\|f\|_{s,p} \leq C\|f\|_{[W^{s_0,p_0},W^{s_1,p_1}]_\theta} \qquad \text{for every } f \in P .$$

Now we complete the proof.   □

References

[1] J. BERGH and J. LÖFSTRÖM, *Interpolation Spaces*, Springer-Verlag, 1976.

[2] R. E. EDWARDS and G. I. GAUDRY, *Littlewood-Paley and Multiplier Theory*, Springer-Verlag, 1977.

[3] L. GROSS, Abstract Wiener spaces, Proceedings of Fifth Berkeley Symp. Math. Statist. Prob. II, part 1 (1965), 31–41, Univ. Calif. Press, Berkeley.

[4] ————, Logarithmic Sobolev inequalities, Amer. J. Math., **97** (1975), 1061–1083.

[5] T. HIDA, *Brownian Motion*, Springer-Verlag, 1980.

[6] N. NELSON, The free Markoff field, J. Funct. Anal., **12** (1973), 211–227.

[7] W. RUDIN, *Functional Analysis*, McGraw-Hill, 1981.

[8] ————, *Real and Complex Analysis*, McGraw-Hill, 1986.

[9] C. SADOSKY, *Interpolation of Operators and Singular Integrals*, Marcel Dekker, 1979.

[10] I. SHIGEKAWA, Derivatives of Wiener functionals and absolute continuity of induced
measures, J. Math. Kyoto Univ., **20** (1980), 263–289.

[11] ————, De Rham-Hodge-Kodaira's decomposition of an abstract Wiener space, J. Math.
Kyoto Univ., **26** (1986), 191–202.

[12] E. M. STEIN, *Topics in Harmonic Analysis Related to the Littlewood-Paley Theory*,
Princeton Univ. Press, 1974.

[13] H. SUGITA, Sobolev spaces of Wiener functional and Malliavin calculus, J. Math. Kyoto
Univ., **25** (1985), 31–48.

[14] ————, Characterization of Sobolev spaces over an abstract Wiener space, J. Math.
Kyoto Univ., **25** (1985), 717–725.

[15] H. TRIEBEL, *Interpolation Theory, Function Spaces, Differential Operators*, North-
Holland, 1978.

[16] S. WATANABE, *Lecture Note of Stochastic Differential Equation and Malliavin Calculus*,
Tata Inst. Fund. Res., Springer-Verlag, 1984.

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