Exponentially Bounded $C$-Semigroups and Integrated Semigroups

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Introduction.

Let $X$ be a Banach space. We denote by $B(X)$ the set of all bounded linear operators from $X$ into itself.

Let $C$ be an injective operator in $B(X)$. We do not assume that the range $R(C)$ is dense in $X$. A family $\{S(t): t \geq 0\}$ in $B(X)$ is called an exponentially bounded $C$-semigroup on $X$, if

(0.1) $S(t+s)C = S(t)S(s)$ for $t, s \geq 0$ and $S(0)=C$,
(0.2) $S(\cdot)x: [0, \infty) \rightarrow X$ is continuous for $x \in X$,
(0.3) there are $M \geq 0$ and $a \in \mathbb{R} \equiv (-\infty, \infty)$ such that $\|S(t)\| \leq Me^{at}$ for $t \geq 0$.

Let us define $L_\lambda \in B(X)$ for $\lambda > a$ by

$$L_\lambda x = \int_0^\infty e^{-\lambda t}S(t)x dt \quad \text{for} \quad x \in X.$$ 

Similarly as in the case of $\overline{R(C)}=X$ (see [4]), we see that $L_\lambda$ is injective for $\lambda > a$ and the closed linear operator $Z$ defined by

(0.4) $\begin{cases} D(Z) = \{x \in X: Cx \in R(L_\lambda)\} \\ Zx = (\lambda - L_\lambda^{-1}C)x \quad \text{for} \quad x \in D(Z) \end{cases}$

is independent of $\lambda > a$. The operator $Z$ will be called the generator of $\{S(t): t \geq 0\}$.

Recently, Davies and Pang [4] introduced the notion of an exponentially bounded $C$-semigroup under the assumption that $R(C)$ is dense in $X$ and gave a characterization of the generator of an exponentially bounded $C$-semigroup. (See [3] also.) Later, the authors [6, 9, 11] gave a characterization of the complete infinitesimal generator of an exponentially

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bounded C-semigroup and then a unified treatment of the generation of semigroups of class \((C_{(k)})\) and that of semigroups of growth order \(\alpha\).

Let \(n\) be a positive integer. A family \(\{U(t): t \geq 0\}\) in \(B(X)\) is called an \(n\)-times integrated semigroup on \(X\) (see [1]), if

\[
U(\cdot)x: [0, \infty) \to X \text{ is continuous for } x \in X, \\
U(t)U(s)x = \frac{1}{(n-1)!} \left( \int_{s+t}^{s+t-r} (s+t-r)^{n-1} U(r) x dr \right. \\
\left. - \int_{0}^{l} (s+t-r)^{n-1} U(r) x dr \right) \text{ for } x \in X \text{ and } s, t \geq 0, \text{ and } U(0) = 0, \\
U(t)x = 0 \text{ for all } t > 0 \text{ implies } x = 0, \\
\text{there are } M \geq 0 \text{ and } \omega \in R \text{ such that } \|U(t)\| \leq Me^{\omega t} \text{ for } t \geq 0.
\]

For convenience we call a semigroup of class \((C_{0})\) on \(X\) also 0-times integrated semigroup on \(X\).

It is known that if \(\{U(t): t \geq 0\}\) is an \(n\)-times integrated semigroup, then there exists a unique closed linear operator \(A\) such that \((\omega, \infty) \subset \rho(A)\) (the resolvent set of \(A\)) and

\[
R(\lambda; A)x (\equiv (\lambda - A)^{-1}x) = \int_{0}^{\infty} \lambda^{-t} U(t) x dt \text{ for } x \in X \text{ and } \lambda > \omega.
\]

The operator \(A\) is called the generator of \(\{U(t): t \geq 0\}\).

In §1 we derive some results on the generator of an exponentially bounded \(C\)-semigroup. Among others, we obtain that the generator \(Z\) has the following properties ([Proposition 1.4]):

\(a_{1}\) \quad \(\lambda - Z\) is injective for \(\lambda > a\);

\(a_{2}\) \quad \(D((\lambda - Z)^{-m}) \subset R(\lambda)\) for \(\lambda > a\) and \(m \geq 1\);

\(a_{3}\) \quad \(\|(\lambda - Z)^{-m}C\| \leq \frac{M}{(\lambda - a)^{m}}\) for \(\lambda > a\) and \(m \geq 1\);

\(a_{4}\) \quad \(Cx \in D(Z)\) and \(ZCx = CZx\) for \(x \in D(Z)\).

In §2 we shall construct an exponentially bounded \(C\)-semigroup under the above conditions \((a_{1})-(a_{4})\). Our Theorem 2.1 (the first main result) shows that if \(A\) is a closed linear operator satisfying \((a_{1})-(a_{4})\) with \(Z\) replaced by \(A\), then there exists an exponentially bounded \(C_{1}\)-semigroup on \(\overline{D(A)}\) with generator \(C_{1}^{-1}A_{1}C_{1}\), where \(C_{1} = C|_{\overline{D(A)}}\) and \(A_{1}\) is the part of
A in $\overline{D(A)}$. This generalizes results in [4, 6] and will be applied to establish Theorem 3.1 (the second main result) in § 3 which clarifies the relations between exponentially bounded C-semigroups and integrated semigroups. Theorem 3.1 generalizes a result in [10].

§ 1. Exponentially bounded C-semigroups.

For simplicity, by a C-semigroup on $X$ we mean an exponentially bounded C-semigroup on $X$.

Let $\{S(t): t \geq 0\}$ be a C-semigroup on $X$ with generator $Z$. Let us define linear operators $G$ and $\mathfrak{U}$ by

\[
\begin{align*}
D(G) &= \{x \in R(C) : \lim_{t \to +0} \frac{C^{-1}S(t)x - x}{t} \text{ exists} \} \\
Gx &= \lim_{t \to +0} \frac{C^{-1}S(t)x - x}{t} \quad \text{for } x \in D(G)
\end{align*}
\]

and

\[
\begin{align*}
D(\mathfrak{U}) &= \{x \in X : \lim_{t \to +0} \frac{S(t)x - Cx}{t} \in R(C) \} \\
\mathfrak{U}x &= C^{-1}\lim_{t \to +0} \frac{S(t)x - Cx}{t} \quad \text{for } x \in D(\mathfrak{U}),
\end{align*}
\]

respectively. ($\mathfrak{U}$ is the infinitesimal generator of $\{S(t): t \geq 0\}$ in the sense of Da Prato [3].)

The relations among $G$, $\mathfrak{U}$ and $Z$ are as follows.

**Proposition 1.1.** We obtain the following (1.3) and (1.4):

(1.3) $G \subset \overline{G} \subset \mathfrak{U} = Z$, where $\overline{G}$ denotes the closure of $G$;

(1.4) $C^{-1}GC = C^{-1}\overline{G}C = C^{-1}ZC = Z$.

**Proof.** To show $\mathfrak{U} \subset Z$, let $x \in D(\mathfrak{U})$ and $\lambda > a$, where $a$ is a constant in (0.3). By $dS(t)Cx/dt = S(t)C\mathfrak{U}x$ for $t \geq 0$, we have

\[
CL_1(\lambda - \mathfrak{U})x = L_1C(\lambda - \mathfrak{U})x = \lambda L_1Cx - \int_0^\infty e^{-\lambda t} \frac{dS(t)Cx}{dt} dt = C^2x,
\]

i.e.,

\[
L_1(\lambda - \mathfrak{U})x = Cx \quad \text{for } x \in D(\mathfrak{U}) \text{ and } \lambda > a.
\]

This implies $\mathfrak{U} \subset Z$. Next, to show $Z \subset \mathfrak{U}$, let $x \in D(Z)$ and take $y \in X$ such that $Cx = L_2y$, where $\lambda > a$. Noting $C^{-1}S(h)u = S(h)C^{-1}u$ for $u \in R(C)$ and $h > 0$,
\[ h^{-1}(S(h)x-Cx) = h^{-1}(C^{-1}S(h)L_\lambda y - L_\lambda y) \]
\[ = h^{-1}(e^{\lambda h} - 1) \int_0^h e^{-\lambda t} S(t)y dt - h^{-1} \int_0^h e^{-\lambda t} S(t)y dt \]
\[ \rightarrow \lambda L_\lambda y - Cy = C(\lambda x - y) \in R(C) \quad \text{as} \quad h \rightarrow 0^+. \]

This shows that \( x \in D(\mathfrak{U}) \) and \( \mathfrak{U}x = \lambda x - y = Zx \). Therefore \( Z \subset \mathfrak{U} \) and hence \( \mathfrak{U} = Z \). Since \( G \subset \mathfrak{U} \) and \( \mathfrak{U} (= Z) \) is closed, \( G \) is closable and \( \overline{G} \subset \mathfrak{U} \). So we have (1.3).

To prove (1.4), let \( x \in D(\mathfrak{U}) \) first. Then \( \lim_{t \rightarrow 0^+}(C^{-1}S(t)Cx-Cx)/t = \lim_{t \rightarrow 0^+}(S(t)x-Cx)/t = C\mathfrak{U}x \), and hence \( Cx \in D(G) \) and \( GCx = C\mathfrak{U}x \). Therefore \( \mathfrak{U} \subset C^{-1}GC \). Now, we want to show that \( C^{-1}ZC \subset Z \). To this end, let \( x \in D(C^{-1}ZC) \), i.e., \( Cx \in D(Z) \) and \( ZCx \in R(C) \).

Then \[ L_\lambda(\lambda - C^{-1}ZC)x = L_\lambda C^{-1}(\lambda - Z)Cx = C^{-1}L_\lambda(\lambda - Z)Cx = Cx, \]
and hence \( x \in D(Z) \) and \( Zx = (\lambda - L_\lambda^{-1}C)x = C^{-1}ZCx \). Consequently, \( C^{-1}ZC \subset Z \). Combining these with (1.3), we obtain (1.4).

Q.E.D.

\( \overline{G} \) is called the complete infinitesimal generator (c.i.g.) of \( \{S(t):t \geq 0\} \). The following example shows that \( \overline{G} = Z \) does not hold in general.

**EXAMPLE.** Let \( X = C[0,1] \), and define \( C \in B(X) \) by
\[ (Cx)(t) = \int_0^t x(s)ds, \quad 0 \leq t \leq 1, \quad \text{for} \quad x \in C[0,1]. \]

Then \( C \) is injective and \( R(C) = \{x \in C'[0,1]: x(0) = 0\} \) (and hence \( R(C) \) is not dense in \( X \)). Consider the \( C \)-semigroup \( \{S(t): t \geq 0\} \) defined by \( S(t) = C \) for all \( t \geq 0 \). In this case, \( D(Z) = X \) and \( Zx = 0 \) for \( x \in X \), but \( D(G) \subset R(C) \) and hence \( D(\overline{G}) \subset \overline{R(C)} \neq X \). This shows \( \overline{G} \neq Z \).

**PROPOSITION 1.2.** We have the following (1.5)-(1.7):

\[ (\lambda - Z)L_\lambda x = Cx \quad \text{for} \quad x \in X \quad \text{and} \quad \lambda > a \]

\[ L_\lambda(\lambda - Z)x = Cx \quad \text{for} \quad x \in D(Z) \quad \text{and} \quad \lambda > a, \]

where \( a \) is a constant in (0.3);

\[ S(t)x \in D(Z) \quad \text{and} \quad ZS(t)x = S(t)Zx \quad \text{for} \quad x \in D(Z) \quad \text{and} \quad t \geq 0; \]

\[ \int_0^t S(s)xds \in D(Z) \quad \text{and} \quad S(t)x - Cx = Z\int_0^t S(s)xds \quad \text{for} \quad x \in X \quad \text{and} \quad t \geq 0. \]

**PROOF.** (1.5) and (1.6) follow from the definition of \( Z \). It is easily
seen that \( \int_{0}^{t} S(s)x ds \in D(\mathfrak{U}) \) and \( S(t)x-Cx=\mathfrak{U}\int_{0}^{t} S(s)x ds \) for \( x \in X \) and \( t \geq 0 \). By \( Z=\mathfrak{U} \), we obtain (1.7).

**Q.E.D.**

**COROLLARY 1.3.** For every \( x \in C(D(Z)) \), \( u(t)=C^{-1}S(t)x \) is a unique \( C^{1} \) (continuously differentiable) solution of the Cauchy problem

\[
\frac{du(t)}{dt}=Zu(t), \quad t \geq 0, \quad \text{and} \quad u(0)=x.
\]

Moreover, the \( u(t) \) satisfies \( \|u(t)\| \leq Me^{at}\|C^{-1}x\| \), where \( M \) and \( a \) are constants in (0.3).

**PROOF.** Let \( x \in C(D(Z)) \) and put \( u(t)=C^{-1}S(t)x \) for \( t \geq 0 \). By (1.6) and (1.7) we have that \( Zu(t)=ZS(t)y=S(t)Zy \) and

\[
u(t)=\int_{0}^{t} Zu(s)ds \quad \text{for} \quad t \geq 0,
\]

where \( y \) is an element in \( D(Z) \) such that \( x=Cy \). Therefore \( u(t) \) is a \( C^{1} \)-solution of (CP), and \( \|u(t)\|=\|S(t)y\| \leq Me^{at}\|y\|=Me^{at}\|C^{-1}x\| \). To show the uniqueness, let \( v(t) \) be a \( C^{1} \)-solution of (CP) and \( s>0 \) be arbitrarily given. Then

\[
\frac{d}{dt}S(s-t)v(t)=S(s-t)Zv(t)-ZS(s-t)v(t)=0
\]

for \( 0 \leq t \leq s \). Integrating this over \([0, s]\), we obtain \( Cv(s)=S(s)x \), i.e., \( v(s)=C^{-1}S(s)x=u(s) \) for every \( s>0 \).

**Q.E.D.**

**PROPOSITION 1.4.** \( Z \) satisfies the following (a\(_{1}\))-(a\(_{4}\)):

- (a\(_{1}\)) \( \lambda-Z \) is injective for \( \lambda>a \);
- (a\(_{2}\)) \( D((\lambda-Z)^{-m}) \supset R(C) \) for \( \lambda>a \) and \( m \geq 1 \);
- (a\(_{3}\)) \( \| (\lambda-Z)^{-m}C \| \leq M/(\lambda-a)^{m} \) for \( \lambda>a \) and \( m \geq 1 \), where \( M \) and \( a \) are constants in (0.3);
- (a\(_{4}\)) \( CX \in D(Z) \) and \( ZCx=CZx \) for \( x \in D(Z) \).

**PROOF.** (a\(_{1}\)) and (a\(_{4}\)) follow from (1.5) and (1.6), respectively. Next, using induction with respect to \( m \), we obtain (a\(_{3}\)) and

\[
(\lambda-Z)^{-m}C x = \int_{0}^{\infty} \cdots \int_{0}^{\infty} e^{-x(t_{1}+\cdots+t_{m})} S(t_{1}+\cdots+t_{m}) x dt_{1} \cdots dt_{m}
\]

for \( x \in X, \lambda>a \) and \( m \geq 1 \). Combining this with (0.3) we get (a\(_{3}\)).

**Q.E.D.**
§2. Construction of C-semigroups.

Throughout this section $A$ denotes a closed linear operator in $X$ satisfying the following conditions (which correspond to $(a_1)$–$(a_4)$ in Proposition 1.4):

(A) there exists an $a \in \mathbb{R}$ such that $\lambda - A$ is injective for $\lambda > a$;
(A) $D((\lambda - A)^{-m}) \supset R(C)$ for $\lambda > a$ and $m \geq 1$;
(A) there exists an $M \geq 0$ such that $\| (\lambda - A)^{-m} C \| \leq M/(\lambda - a)^m$ for $\lambda > a$ and $m \geq 1$;
(A) $Cx \in D(A)$ and $ACx = CAx$ for $x \in D(A)$.

It is easily seen that (A) is equivalent to the following (A):
(A) $(\lambda - A)^{-1} Cx = C(\lambda - A)^{-1} x$ for $\lambda > a$ and $x \in D((\lambda - A)^{-1})$.

The purpose of this section is to construct a C-semigroup on $\overline{D(A)}$ under these conditions. Our idea for construction is based on that of [7].

Our theorem is the following which generalizes [4, Theorem 11] and [6, Theorem 2].

**THEOREM 2.1.** Let $A$ be a closed linear operator satisfying $(A_1)$–$(A_4)$. Then for every $x \in \overline{D(A)}$, the limit

$$S_1(t)x = \lim_{n \to \infty} \left(1 - \frac{tA}{n}\right)^{-n} Cx$$

exists uniformly on every bounded subset of $[0, \infty)$. The family $\{S_1(t): t \geq 0\}$ has the following properties:

(2.1) $S_1(t): \overline{D(A)} \to \overline{D(A)}$ ;

(2.2) $S_1(t+s)Cx = S_1(t)S_1(s)x$ and $S_1(0)x = Cx$ for $x \in \overline{D(A)}$ and $t, s \geq 0$ ;

(2.3) $\| S_1(t)x \| \leq Me^{\alpha t} \| x \|$ for $x \in \overline{D(A)}$ and $t \geq 0$ ;

(2.4) $\| S_1(t)x - S_1(s)x \| \leq Me^{\alpha t} \| x \| \| Ax \| \| t - s \|$ for $x \in D(A)$ and $t, s \geq 0$ ,

and hence $S_1(\cdot)x: [0, \infty) \to \overline{D(A)}$ is continuous for $x \in \overline{D(A)}$ ;

(2.5) $S_1(t)x - Cx = A \int_{0}^{t} S_1(s)x ds$ for $x \in \overline{D(A)}$ and $t \geq 0$ ;

(2.6) $(\lambda - A)^{-1} Cx = \int_{0}^{\infty} e^{-\lambda t} S_1(t)x dt$ for $x \in \overline{D(A)}$ and $\lambda > a$ .

Therefore, setting $C_1 = C|_{\overline{D(A)}}$, $\{S_1(t): t \geq 0\}$ is a C₁-semigroup on the Banach space $\overline{D(A)}$. 
Moreover, $C_1^{-1}A_1C_1$ is the generator of the $C_1$-semigroup $\{S_1(t); t \geq 0\}$, where $A_1$ denotes the part of $A$ in $\overline{D(A)}$.

Before proving this theorem we prepare two lemmas. We define a linear subset $\tilde{\Sigma}$ of $X$ and a function $\tilde{N}(\cdot)$ on $\tilde{\Sigma}$ by

$$\tilde{\Sigma} = \{ x \in \bigcap_{\lambda > a, m \geq 0} D((\lambda - A)^{-m}) : \sup_{\lambda > a, m \geq 0} \| (\lambda - a)^m (\lambda - A)^{-m} x \| < \infty \}$$

and

$$\tilde{N}(x) = \sup_{\lambda > a, m \geq 0} \| (\lambda - a)^m (\lambda - A)^{-m} x \| \quad \text{for } x \in \tilde{\Sigma}.$$ 

Obviously, $\| x \| \leq \tilde{N}(x)$ for $x \in \tilde{\Sigma}$ and $\tilde{N}(\cdot)$ defines a norm on $\tilde{\Sigma}$. Our assumptions $(A_2)$ and $(A_3)$ imply

$$(2.7) \quad R(C) \subset \tilde{\Sigma} \quad \text{and} \quad \tilde{N}(x) \leq M \| C^{-1} x \| \quad \text{for } x \in R(C).$$

**Lemma 2.2.** The following conditions $(b_1)$-$\ (b_3)$ (which are stated in [7, § 4]) are satisfied with $Y = \tilde{\Sigma}$ and $\| \cdot \| = \tilde{N}(\cdot)$:

$(b_1)$ $Y$ is a normed space under a certain norm $\| \cdot \|$, which is stronger than the original norm $\| \cdot \|$ of $X$;

$(b_2)$ there exists a real $\omega$ such that for $\lambda > \omega$, $R(\lambda - A)$ contains $Y$, $R(\lambda) = (\lambda - A)^{-1}$ exists, and such that $Y$ is invariant under $R(\lambda)$;

$(b_3)$ there exists a constant $M \geq 0$ such that

$$\| R(\lambda)^m x \| \leq M(\lambda - \omega)^{-m} \| x \| \quad \text{for } x \in Y, \lambda > \omega \text{ and } m \geq 0.$$ 

Moreover we have

$$(2.8) \quad \tilde{N}((\lambda - a) R(\lambda) x) \leq \tilde{N}(x) \quad \text{for } x \in \tilde{\Sigma} \text{ and } \lambda > a.$$ 

**Proof.** $(b_1)$ is obvious. To prove $(b_2)$ and $(b_3)$, we first note that clearly $R(\lambda - A) \supset \tilde{\Sigma}$ and $R(\lambda) = (\lambda - A)^{-1}$ exists for $\lambda > a$, and the following equality holds:

$$(2.9) \quad R(\lambda)^m R(\mu)^n x = \sum_{l=m}^{\infty} \binom{m}{l} C_{m-l} (\mu - \lambda)^{l-m+1} R(\mu)^{l+n+1} x$$

for $x \in \tilde{\Sigma}$, $\mu > \lambda > a$, $m \geq 1$ and $n \geq 0$.

Indeed, since

$$\| C_{m-l} (\mu - \lambda)^{l-m+1} R(\mu)^{l+n+1} x \| \leq C_{m-l} (\frac{\mu - \lambda}{\mu - a})^{l-m+1} (\mu - a)^{-m-n} \tilde{N}(x) \quad \text{for } x \in \tilde{\Sigma},$$

the series of the right side in (2.9) is absolutely convergent with respect to the norm $\| \cdot \|$. Let $x \in \tilde{\Sigma}$. Then
$(\lambda - A) \sum_{l=0}^{k} (\mu - \lambda)^{l} R(\mu)^{l+n+1} x = R(\mu)^{n} x - (\mu - \lambda)^{k+1} R(\mu)^{k+n+1} x$

and

$$\| (\mu - \lambda)^{k+1} R(\mu)^{k+n+1} x \| \leq \left( \frac{\mu - \lambda}{\mu - a} \right)^{k+1} (\mu - a)^{-n} \tilde{N}(x) \to 0 \quad \text{as} \quad k \to \infty,$$

which imply that (2.9) holds for $m=1$. The conclusion follows from the induction with respect to $m$.

Next, setting $n=0$ in (2.9), we obtain

$$R(\mu)^{m} x = \sum_{l=m-1}^{\infty} c_{m-l} (\lambda - \mu)^{l-n+1} R(\mu)^{l+1} x \quad \text{for} \quad x \in \Sigma, \lambda > \mu > a \quad \text{and} \quad m \geq 1.$$ Since $R(\lambda)$ is closed, for $x \in \Sigma$, $\lambda > \mu > a$, $m \geq 1$ and $n \geq 0$

$$R(\lambda)^{n} R(\mu)^{m} x = \sum_{l=m-1}^{\infty} c_{m-l} (\lambda - \mu)^{l-n+1} R(\lambda)^{l+n+1} x.$$ (2.10)

Now, let $\mu > a$ and $x \in \Sigma$. Then, by (2.9), for $\lambda$ with $\mu > \lambda > a$ we have

$$\| (\lambda - a)^{m} R(\lambda)^{n} (\mu - a)^{n} R(\mu)^{n} x \| \leq \tilde{N}(x).$$

Consequently, for $n \geq 1$ and $\mu > a$,

$$\| (\lambda - a)^{m} R(\lambda)^{n} (\mu - a)^{n} R(\mu)^{n} x \| \leq \tilde{N}(x) \quad \text{for} \quad \lambda > a, \quad m \geq 1 \quad \text{and} \quad x \in \Sigma.$$ (i.e., $(b_{2})$ and $(b_{3})$ hold with $Y=\Sigma$, $||| \cdot ||| = \tilde{N}(\cdot)$, $\omega = a$ and $M=1$), and (2.8) holds.

In particular, $\Sigma$ is invariant under $R(\lambda)$ for $\lambda > a$ and $|| R(\lambda)^{n} x || \leq (\lambda - a)^{-n} \tilde{N}(x)$ for $x \in \Sigma$. In view of Lemma 2.2, we may employ the results given in [7, § 4]. Also, by using the argument due to [2] and [5], (2.8) implies the following
Lemma 2.3 ([5]). For \( \lambda, \mu > 0 \) with \( |\lambda a| \leq 1/2, |\mu a| \leq 1/2 \) and \( n, m \geq 0 \),

\[
\tilde{N}(J_{\lambda}^{m}x - J_{\mu}^{n}x) \leq \exp(2|m\lambda + n\mu|)((m\lambda - n\mu)^{2} + m\lambda^{2} + n\mu^{2})^{1/2}\tilde{N}(Ax)
\]

for \( x \in \Sigma_{1} \equiv \{ x \in \tilde{\Sigma} : Ax \sim Ax \in \tilde{\Sigma} \} \), where \( J_{\lambda} = (1 - \lambda A)^{-1} \).

Proof of Theorem 2.1. First, let \( x \in D(A) \). Since \( Cx \in D(A) \cap \tilde{\Sigma} \) and \( ACx = CAx \in \tilde{\Sigma} \) (and hence \( Cx \in C(D(A)) \subset \tilde{\Sigma}_{1} \)) by (A4) and (2.7), it follows from Lemma 2.3 and (2.7) that

\[
||J_{\lambda}^{[t/\lambda]}Cx - J_{\mu}^{[t/\mu]}Cx|| \leq Me^{4|a|t}((\lambda + \mu)^{2} + t(\lambda + \mu))^{1/2}||Ax||
\]

for \( t \geq 0 \). Therefore the limit \( \lim_{\lambda \to 0^{+}}J_{\lambda}^{[t/\lambda]}Cx \) exists uniformly on every bounded subset of \( [0, \infty) \). This remains true for every \( x \in \overline{D(A)} \), because \( ||J_{\lambda}^{[t/\lambda]}C|| \) are uniformly bounded on every bounded subset of \( [0, \infty) \) as \( \lambda \to 0^{+} \).

Define \( S_{1}(t) \) for \( t \geq 0 \) by

\[
S_{1}(t)x = \lim_{\lambda \to 0^{+}}J_{\lambda}^{[t/\lambda]}Cx = \lim_{n \to \infty}(1 - \frac{tA}{n})^{-n}Cx
\]

for \( x \in \overline{D(A)} \).

Clearly (2.1) and (2.3) hold, and it follows from (2.3), (2.7) and [7, Theorem 4.6] that (2.4) and (2.6) hold. By Lemma 2.3 again, for \( x \in D(A) \)

\[
||J_{\lambda}^{[t+s/\lambda]}C \cdot Cx - J_{\lambda}^{[t/\lambda]}C \cdot J_{\lambda}^{[s/\lambda]}Cx|| = ||J_{\lambda}^{[t+s/\lambda]}C^{2}x - J_{\lambda}^{[t+s/\lambda]}C^{2}x||
\]

(by \( A_{4} \))

\[
\leq e^{4|a|(t+s)}(4\lambda^{2} + 2\lambda(t+s))^{1/2}\tilde{N}(AC^{2}x) \to 0
\]

as \( \lambda \to 0^{+} \),

which implies

\[
S_{1}(t+s)Cx = S_{1}(t)S_{1}(s)x \quad \text{for} \quad x \in D(A) \quad \text{and} \quad t, s \geq 0.
\]

Therefore (2.2) holds. Next, we will prove that (2.5) holds. By virtue of [7, Lemma 4.5] and the closedness of \( A \), we have

\[
(1 - \frac{tA}{n})^{-n}x - x = \int_{0}^{t}(1 - \frac{sA}{n})^{-(n+1)}Axds = A\int_{0}^{t}(1 - \frac{sA}{n})^{-(n+1)}C^{2}xds
\]

for \( x \in \tilde{\Sigma}_{1}, \ t \geq 0 \) and integer \( n \) with \( n > |a|t \). In particular, the following holds:

\[
(1 - \frac{tA}{n})^{-n}Cx - Cx = A\int_{0}^{t}(1 - \frac{sA}{n})^{-(n+1)}C^{2}xds
\]

for \( x \in D(A), \ t \geq 0 \) and integer \( n \) with \( n > |a|t \). Letting \( n \to \infty \), and noting that
\[
\left\| \left( 1 - \frac{sA}{n} \right)^{-(n+1)} Cx - \left( 1 - \frac{sA}{n} \right)^{-n} Cx \right\| = \frac{s}{n} \left\| \left( 1 - \frac{sA}{n} \right)^{-(n+1)} CAx \right\| \leq M s \left( 1 - \frac{s|a|}{n} \right)^{-(n+1)} \frac{\|Ax\|}{n} \to 0 \quad \text{as } n \to \infty,
\]

as \( n \to \infty \), the closedness of \( A \) implies

\[
\int_{0}^{t} S_{1}(s)xds \in D(A) \quad \text{and} \quad S_{1}(t)x - Cx = A \int_{0}^{t} S_{1}(s)xds
\]

for \( x \in D(A) \) and \( t \geq 0 \). These remain true for \( x \in \overline{D(A)} \) by the closedness of \( A \) and (2.3).

Finally, we will prove that \( C_{1}^{-1}A_{1}C_{1} \) is the generator of the \( C_{1} \)-semigroup \( \{S_{1}(t): t \geq 0\} \) on \( \overline{D(A)} \). To this end, let \( Z_{1} \) be the generator of \( \{S_{1}(t): t \geq 0\} \) and let \( \mathfrak{U}_{1} \) be the operator defined by

\[
\begin{aligned}
D(\mathfrak{U}_{1}) &= \left\{ x \in \overline{D(A)} : \lim_{t \to +} \frac{S_{1}(t)x - C_{1}x}{t} \in R(C_{1}) \right\} \\
\mathfrak{U}_{1}x &= C_{1}^{-1} \lim_{t \to +} \frac{S_{1}(t)x - C_{1}x}{t} \quad \text{for } x \in D(\mathfrak{U}_{1})
\end{aligned}
\]

(see (1.2)). Then we have

(2.11) \quad \quad A_{1} \subset Z_{1}.

In fact, let \( x \in D(A_{1}) \). Noting \( (\lambda - A)^{-k}CA_{1}x = A(\lambda - A)^{-k}Cx \) for every \( k \geq 0 \) and \( \lambda > a \), it follows that

\[
A \left( 1 - \frac{tA}{n} \right)^{-n} Cx = \left( 1 - \frac{tA}{n} \right)^{-n} CA_{1}x
\]

for \( t \geq 0 \). Letting \( n \to \infty \), by the closedness of \( A \), we have

\[
S_{1}(t)x \in D(A) \quad \text{and} \quad AS_{1}(t)x = S_{1}(t)A_{1}x \in \overline{D(A)}
\]

and hence \( S_{1}(t)x \in D(A_{1}) \) and \( A_{1}S_{1}(t)x = AS_{1}(t)x = S_{1}(t)A_{1}x \) for \( t \geq 0 \). Combining this with (2.6), we have

\[
C_{1}x = (\lambda - A) \int_{0}^{\infty} e^{-\lambda t} S_{1}(t)x dt = \int_{0}^{\infty} e^{-\lambda t} S_{1}(t)(\lambda - A_{1})x dt
\]

\[
= \mathcal{L}_{2}(\lambda - A_{1})x \quad \text{for } \lambda > a,
\]

where \( \mathcal{L}_{2}z = \int_{0}^{\infty} e^{-\lambda t} S_{1}(t)z dt \) for \( z \in \overline{D(A)} \) and \( \lambda > a \). So, by the definition of generator \( Z_{1} \), we get
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$x \in D(Z_i)$ and $A_i x = (\lambda - \mathcal{L}_i^{-1}C_i)x = Z_i x$.

This proves (2.11). Next, let $x \in D(\mathfrak{U}_i)$. By (2.5)

$$A\left( t^{-1}\int_0^t S_i(s)x ds \right) = \frac{S_i(t)x - C_i x}{t} \to C_i \mathfrak{U}_i x$$

as $t \to 0^+$. Since $\lim_{t \to 0^+} t^{-1}\int_0^t S_i(s)x ds = C_i x$ and $A$ is closed, we get

$$C_i x \in D(A) \quad \text{and} \quad AC_i x = C_i \mathfrak{U}_i x \in \overline{D(A)}.$$  

This means that $C_i x \in D(A_i)$ and $A_i C_i x (= AC_i x) = C_i \mathfrak{U}_i x$, i.e., $x \in D(C_i^{-1}A_i C_i) = \{ x \in \overline{D(A)} : C_i x \in D(A_i) \}$ and $A_i C_i x \in R(C_i)$ and $\mathfrak{U}_i x = C_i^{-1}A_i C_i x$. Therefore we obtain

$$C_i x \in D(A) \quad \text{and} \quad AC_i x = C_i \mathfrak{U}_i x \in \overline{D(A)}.$$  

This means that $C_i x = C_i^{-1}Z_i C_i \supset C_i^{-1}A_i C_i$ and $\mathfrak{U}_i x = C_i^{-1}A_i C_i x$, i.e.,

$$x \in D(C_i^{-1}A_i C_i) \equiv \{ z \in \overline{D(A)} : C_i z \in D(A_i) \} \quad \text{and} \quad A_i C_i z \in R(C_i)$$

and $\mathfrak{U}_i x = C_i^{-1}A_i C_i x$.

Therefore we obtain

$$\mathfrak{U}_i \subset C_i^{-1}A_i C_i.$$  

But $\mathfrak{U}_i = Z_i = C_i^{-1}Z_i C_i \supset C_i^{-1}A_i C_i$ by Proposition 1.1 and (2.11). Combining this with (2.12), we have that $Z_i = C_i^{-1}A_i C_i$.

Q.E.D.

§ 3. C-semigroups and integrated semigroups.

The following theorem establishes the relations between C-semigroups and integrated semigroups.

**Theorem 3.1.** Let $A$ be a closed linear operator in $X$ with $\rho(A) \neq \emptyset$. Let $c \in \rho(A)$ and $n \geq 0$ be an integer. The following (i)-(iii) are mutually equivalent:

(i) $A$ is the generator of an $(n+1)$-times integrated semigroup $\{U(t) : t \geq 0\}$ on $X$ satisfying $\|U(t+h) - U(t)\| \leq K'h^\omega(t+h)$ for $t$, $h \geq 0$, where $K' \geq 0$ and $\omega \in \mathbb{R}$ are constants;

(ii) $A$ is the generator of a C-semigroup $\{S(t) : t \geq 0\}$ on $X$ with $C = R(c : A)^{n+1}$ satisfying $\|S(t+h) - S(t)\| \leq K'h^\omega(t+h)$ for $t$, $h \geq 0$, where $K \geq 0$ and $\omega \in \mathbb{R}$ are constants;

(iii) There exist $M \geq 0$ and $a \in \mathbb{R}$ such that $(a, \infty) \subset \rho(A)$ and

$$\|R(\lambda : A)^m R(c : A)^n\| \leq M / (\lambda - a)^m$$

for $\lambda > a$ and $m \geq 1$.

In this case, we have for $t \geq 0$

$$U(t)x = (c-A)^{n+1}\int_0^t \cdots \int_0^t S(t_{n+1}) x dt_{n+1} \cdots dt_1 dt$$

for $x \in X$.
\[
(=\int_0^t \int_0^{t_1} \cdots \int_0^{t_{n+1}} S(t_{n+1})(c-A)^{n+1}x dt_{n+1} \cdots dt_2 dt_1 \quad \text{for } x \in D(A^{n+1})).
\]

Moreover, if \( A \) is a closed linear operator in \( X \) with \( \rho(A) \neq \emptyset \) satisfying the equivalent conditions above and \( A_1 \) is the part of \( A \) in \( \overline{D(A)} \), then we obtain the following (c)–(c):

(c) \( A_1 \) is the generator of a \( C_1 \)-semigroup \( \{S(t): t \geq 0\} \) on \( \overline{D(A)} \) with \( C_1 = R(c: A)^{n}|_{\overline{D(A)}} \);

(c) (1) \( A_1 \) is the generator of an n-times integrated semigroup \( \{U_1(t): t \geq 0\} \) on \( \overline{D(A)} \);

(c) \( U_1(t)x = (c-A_1)^n \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} S(t_n)x dt_n \cdots dt_2 dt_1 \) for \( x \in \overline{D(A)} \) and \( t \geq 0 \)

\[
(=\int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} S(t_n)(c-A_1)^nx dt_n \cdots dt_2 dt_1 \quad \text{for } x \in D(A^n) \text{ and } t \geq 0).
\]

This is a generalization of [10, Theorem 1] (and [8, Theorem 4.6]). Indeed Theorem 3.1 leads to

**COROLLARY 3.2 (10, Theorem 1).** Let \( A \) be a densely defined closed linear operator in \( X \) with \( \rho(A) \neq \emptyset \). Let \( c \in \rho(A) \) and \( n \geq 0 \) be an integer. The following (i')–(iii') are equivalent:

(i') \( A \) is the generator of an n-times integrated semigroup \( \{\tilde{U}(t): t \geq 0\} \) on \( X \);

(ii') \( A \) is the c.i.g. of a \( C \)-semigroup \( \{\tilde{S}(t): t \geq 0\} \) on \( X \) with \( C = R(c: A)^{n} \);

(iii') there exist \( M \geq 0 \) and \( a \in \mathbb{R} \) such that \( (a, \infty) \subset \rho(A) \) and

\[
\|R(\lambda: A)^{n}R(c: A)^{n}\| \leq \frac{M}{(\lambda-a)^{n}} \quad \text{for } \lambda > a \text{ and } m \geq 1.
\]

In this case, we have for \( t \geq 0 \)

\[
(3.2) \quad \tilde{U}(t)x = (c-A)^n \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \tilde{S}(t_n)x dt_n \cdots dt_2 dt_1 \quad \text{for } x \in X
\]

\[
(=\int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \tilde{S}(t_n)(c-A_1)^nx dt_n \cdots dt_2 dt_1 \quad \text{for } x \in D(A^n) \).
\]

To derive the corollary from Theorem 3.1 we note the following which will be easily proved:

(d) If \( Z \) is the generator of a \( C \)-semigroup \( \{S(t): t \geq 0\} \) on \( X \) and \( P \in B(X) \) is an injective operator satisfying \( S(t)P = P S(t) \) for \( t \geq 0 \), then \( \{S(t)P: t \geq 0\} \) is a \( PC \)-semigroup on \( X \) and \( Z \) is the generator of \( \{S(t)P: t \geq 0\} \).
\( (d) \) If \( A \) is the generator of an \( n \)-times integrated semigroup 
\( \{U(t): t \geq 0\} \) on \( X \) and \( V(t), \ t \geq 0 \), are defined by \( V(t)x = \int_{0}^{t} U(s)x \, ds \) for \( x \in X \),
then \( \{V(t): t \geq 0\} \) is an \( (n+1) \)-times integrated semigroup on \( X \) satisfying 
\[ \|V(t+h) - V(t)\| \leq M e^{\omega(t+h)}h \] for \( t, h \geq 0 \), where \( M \geq 0 \) and \( \omega \in \mathbb{R} \) are some constants, and \( A \) is the generator of \( \{V(t): t \geq 0\} \).

**Proof of Corollary 3.2.** In this case, note that the c.i.g. \( A \) in (ii') coincides with the generator of \( \{\tilde{S}(t): t \geq 0\} \) (see [4, Theorem 35]).
Since \( A_1 = A \) and \( C_1 = R(c: A)^n \) by \( \overline{D(A)} = X \), (iii') \( \Rightarrow \) (ii') and (iii') \( \Rightarrow \) (i') follow from Theorem 3.1 (c1) and (c2), respectively.

To prove ("(i') \( \Rightarrow \) (iii')") let \( A \) be the generator of a C-semigroup 
\( \{\tilde{S}(t): t \geq 0\} \) on \( X \) with \( C = R(c: A)^n \) and let \( \|\tilde{S}(t)\| \leq \tilde{M} e^{\alpha t} \) for \( t \geq 0 \).
Define \( S(t), t \geq 0 \), by \( S(t)x = \tilde{S}(t)R(c: A)x \).
Since \( \tilde{S}(t)R(c: A) = R(c: A)\tilde{S}(t) \) for \( t \geq 0 \) by (1.6), it follows from (d) that \( \{S(t): t \geq 0\} \) is a C-semigroup on \( X \) with \( C = R(c: A)^{n+1} \) and \( A \) is the generator of \( \{S(t): t \geq 0\} \).
Moreover, \( \|S(t+h)x - S(t)x\| = \left\| \int_{t}^{t+h} \tilde{S}(s)AR(c: A)xds \right\| \) (by (1.6) and (1.7)) \( \leq \tilde{M} \|AR(c: A)\|h \times e^{\alpha(t+h)}\|x\| \) for \( x \in X \) and \( t, h \geq 0 \). Therefore \( A \) satisfies (ii) in Theorem 3.1, and hence (iii') \( \Rightarrow \) (iii) in Theorem 3.1 holds.

To show ("(i') \( \Rightarrow \) (iii')") let us define \( U(t), t \geq 0 \), by \( U(t)x = \int_{0}^{t} \tilde{U}(s)x \, ds \) for \( x \in X \). By (d), \( A \) satisfies (i) in Theorem 3.1 and then (iii') holds.
Moreover, by (3.1)
\[
\int_{0}^{t} \tilde{U}(s)x \, ds = U(t)x = (c-A)^{n+1} \int_{0}^{t} \cdots \int_{0}^{t} \tilde{S}(t_{n+1})R(c: A)x \, dt_{n+1} \cdots dt_{2} \, dt_{1}
\]
\[
= \left\{ \int_{0}^{t} (c-A)^n \int_{0}^{t} \cdots \int_{0}^{t} \tilde{S}(t_{n+1})x \, dt_{n+1} \cdots dt_{2} \right\} \, dt_{1}
\]
which implies (3.2).

**Q.E.D.**

**Remarks.** 1. Each of the equivalent conditions (i)-(iii) in Theorem 3.1 is equivalent to the following (iv) (see [1, Theorem 4.1]):

(iv) there exist \( M \geq 0 \) and \( a \in \mathbb{R} \) such that \( (a, \infty) \subset \rho(A) \) and
\[
\left\| \frac{R(\lambda: A)/\lambda^n}{k!} \right\| \leq \frac{M}{(\lambda-a)^{k+1}} \text{ for } \lambda > a \text{ and } k \geq 0 .
\]

2. In the case of \( \overline{D(A)} \neq X \) in Theorem 3.1, "generator" in (ii) can not be replaced by "c.i.g.". In fact, the operator \( A \) of Example 6.4 in [1] is the generator of a C-semigroup on \( X (=E) \) with \( C = R(c: A) \) satisfying \( \|S(t+h)-S(t)\| \leq Ke^{\omega(t+h)} \) for \( t, h \geq 0 \), but it is not the c.i.g. of any C-semigroup on \( X \) with \( C = R(c: A) \).
Proof of Theorem 3.1. We start by showing “(iii)→(ii)”.

By virtue of Theorem 2.1, there exists a $C_{1}$-semigroup $\{S_{1}(t): t \geq 0\}$ on $\overline{D(A)}$ with $C_{1}=R(c; A)^{n}|_{\overline{D(A)}}$ satisfying the following (3.3)-(3.6):

\begin{align*}
(3.3) & \quad S_{1}(t)x = \lim_{m \to \infty} \left( \frac{m}{t} \right)^{m} R(m/t; A)^{n} R(c; A)^{n} x \quad \text{for } x \in \overline{D(A)} \text{ and } t \geq 0 ; \\
(3.4) & \quad ||S_{1}(t)x|| \leq Me^{\alpha t} ||x|| \quad \text{for } x \in \overline{D(A)} \text{ and } t \geq 0 ; \\
(3.5) & \quad ||S_{1}(t+h)x-S_{1}(t)x|| \leq Me^{\alpha(|t+h)|} \Vert Ax\Vert \quad \text{for } x \in D(A) \text{ and } t, h \geq 0 ; \\
(3.6) & \quad R(\lambda; A)R(c; A)^{n}x = \int_{0}^{\infty} e^{-\lambda t} S_{1}(t)x dt \quad \text{for } x \in \overline{D(A)} \text{ and } \lambda > \alpha .
\end{align*}

Let us define $S(t) \in B(X)$, $t \geq 0$, by

\[ S(t)x = S_{1}(t)R(c; A)x \quad \text{for } x \in X . \]

Clearly, $\{S(t): t \geq 0\}$ satisfies (3.7)-(3.9):

\begin{align*}
(3.7) & \quad ||S(t)|| \leq M ||R(c; A)|| e^{\alpha t} \quad \text{for } t \geq 0 ; \\
(3.8) & \quad ||S(t+h)-S(t)|| \leq M ||AR(c; A)|| e^{\alpha(|t+h)|} \quad \text{for } t, h \geq 0 ; \\
(3.9) & \quad S(t)S(s)=S(t+s)R(c; A)^{n+1} \quad \text{for } t, s \geq 0 \text{ and } S(0)=R(c; A)^{n+1} .
\end{align*}

Therefore $\{S(t): t \geq 0\}$ is a $C$-semigroup on $X$ with $C=R(c; A)^{n+1}$. Now, let $Z$ be the generator of $\{S(t): t \geq 0\}$. We want to show

\begin{equation}
A \subset Z .
\end{equation}

To this end, let $x \in D(A)$ and $\lambda > \alpha$. Then by the closedness of $A$

\begin{align*}
L_{2}Ax = & \int_{0}^{\infty} e^{-\lambda t} S(t)Ax dt = \int_{0}^{\infty} e^{-\lambda t} S_{1}(t)R(c; A)Ax dt \\
= & \int_{0}^{\infty} e^{-\lambda t} AS_{1}(t)R(c; A)x dt = A \int_{0}^{\infty} e^{-\lambda t} S_{1}(t)R(c; A)x dt = AL_{2}x .
\end{align*}

Combining this with

\[ R(c; A)^{n+1}x = (\lambda-A) \int_{0}^{\infty} e^{-\lambda t} S_{1}(t)R(c; A)x dt = (\lambda-A)L_{2}x \quad \text{(by (3.6))} , \]

we have $Cx = R(c; A)^{n+1}x = L_{2}(\lambda-A)x$ ($\in R(L_{2})$), i.e.,

\[ x \in D(Z) \text{ and } Zx = (\lambda-L_{2}^{-1}C)x = Ax . \]

Therefore we obtain (3.10). Since $\lambda-Z$ is injective for $\lambda > \alpha$ (by Proposition 1.4 (a)) (3.10) and $\alpha, \infty) \subset \rho(A)$ imply $Z=A$. 
Next, to prove "(ii)⇒(i)" let \( A \) be the generator of a \( C \)-semigroup \( \{ S(t): t \geq 0 \} \) on \( X \) with \( C = R(c: A)^{n+1} \) satisfying \( ||S(t+h) - S(t)|| \leq Ke^{\omega(t+h)}h \) for \( t, h \geq 0 \). Let us define \( V_k(t), k \geq 0 \), by \( V_0(t) = S(t) \) and

\[
V_k(t)x = \int_0^t \cdots \int_0^{t_{k-1}} S(t_{k})x \cdots dt_{k} \cdot \cdot \cdot dt_2 dt_1 \quad \text{for } x \in X \text{ and } t \geq 0.
\]

Similarly as in [10, Lemma], for \( k = 1, 2, \ldots, n+1 \) we have the following (3.11)-(3.13):

(3.11) \( V_k(t)x \in D(A^k) \) and \( \int_0^t (c-A)^{k-1} V_{k-1}(s)x ds \in D(A) \)
for \( x \in X \) and \( t \geq 0 \);

(3.12) \( \langle c-A \rangle V_k(t) \in B(X), \quad ||(c-A)^k V_k(t)|| \leq K_k e^{b_k t} \) and
\[
||\langle c-A \rangle V_k(t+h) - \langle c-A \rangle V_k(t)|| \leq M_k e^{a_k(t+h)}h \quad \text{for } t, h \geq 0,
\]
where \( K_k, M_k, a_k \) and \( b_k \) are nonnegative constants, and hence \( \langle c-A \rangle V_k(\cdot)x: [0, \infty) \rightarrow X \) is continuous for \( x \in X \);

(3.13) \( \langle c-A \rangle V_k(t) = \langle c-A \rangle^{k-1} V_k(t) - \langle c-A \rangle^{k-1} V_{k-1}(t) \)
\[
+ \frac{t^{k-1}}{(k-1)!} \langle c-A \rangle^{k-1} C \quad \text{for } t \geq 0.
\]

Now, define \( U(t), t \geq 0 \), by \( U(t)x = \langle c-A \rangle^{n+1} V_{n+1}(t)x \) for \( x \in X \). Then, by (3.12), \( U(t) \in B(X) \) and \( ||U(t+h) - U(t)|| \leq K e^{\omega(t+h)}h \) for \( t, h \geq 0 \), where \( K = M_{n+1} \) and \( \omega = a_{n+1} \). Clearly, (0.5), (0.7) and (0.8) are satisfied. Similarly as in the proof of [10, (ii)⇒(i) in Theorem 1], we obtain that \( (\alpha, \infty) \subseteq \rho(A) \) and

\[
R(\lambda: A)x = \int_0^\infty \lambda^{n+1} e^{-\lambda t} U(t)x dt \quad \text{for } x \in X \text{ and } \lambda > \alpha,
\]

where \( ||S(t)|| \leq M e^{\alpha t} \) for \( t \geq 0 \) and \( \alpha > 0 \). It follows from [1, Theorem 3.1] that \( U(t), t \geq 0 \), satisfy (0.6) with \( n \) replaced by \( n+1 \). Thus \( \{ U(t): t \geq 0 \} \) is an \( (n+1) \)-times integrated semigroup on \( X \) with generator \( A \). (We note here that

\[
U(t)x = \int_0^t \cdots \int_0^{t_{n+1}} S(t_{n+1})(c-A)^{n+1}x dt_{n+1} \cdots dt_2 dt_1
\]
for \( x \in D(A^{n+1}) \) by (1.6).)

Finally, we show "(i)⇒(iii)". Let \( A \) be the generator of an \( (n+1) \)-times integrated semigroup \( \{ U(t): t \geq 0 \} \) on \( X \) satisfying \( ||U(t+h) - U(t)|| \leq K e^{\omega(t+h)}h \) for \( t, h \geq 0 \). We first note that for \( x \in X \) and \( x^* \in X^* \), \( \langle U(t)x, x^* \rangle \)
is differentiable a.e. $ t \in [0, \infty) $ and

\begin{equation}
\frac{d}{dt} \langle U(t)x, x^* \rangle \leq K' e^{\omega t} ||x|| ||x^*|| \quad \text{for a.e. } t \in [0, \infty).
\end{equation}

Now, by the definition of the generator, $(\omega', \infty) \subset \rho(A)$ and

\[ R(\lambda; A)x = \int_0^\infty \lambda^{n+1} e^{-\lambda t} U(t)x dt \quad \text{for } x \in X \text{ and } \lambda > \omega'. \]

Since

\[ (\lambda - A) \left( \int_0^\infty e^{-\lambda t} \sum_{k=0}^{n-1} \frac{t^k}{k!} A^k x dt + \int_0^\infty \lambda e^{-\lambda t} U(t)A^n x dt \right) = x \]

for $ x \in D(A^n) $ and $ \lambda > |\omega'| $, we obtain

\[ R(\lambda; A)x = \int_0^\infty e^{-\lambda t} \sum_{k=0}^{n-1} \frac{t^k}{k!} A^k x dt + \int_0^\infty \lambda e^{-\lambda t} U(t)A^n x dt \]

for $ x \in D(A^n) $ and $ \lambda > |\omega'| $. Therefore for $ x \in D(A^n) $, $ x^* \in X^* $ and $ \lambda > |\omega'| $

\[ \langle R(\lambda; A)x, x^* \rangle = \int_0^\infty e^{-\lambda t} \sum_{k=0}^{n-1} \frac{t^k}{k!} \langle A^k x, x^* \rangle dt + \int_0^\infty e^{-\lambda t} \frac{d}{dt} \langle U(t)A^n x, x^* \rangle dt . \]

Differentiating this $ m-1 $ times with respect to $ \lambda $ and using (3.14),

\[ (m-1)!(\langle R(\lambda; A)^n R(\lambda; A)^{\ast} x, x^* \rangle \]

\[ \leq \int_0^\infty t^{m-1} e^{-\lambda t} \left\{ \sum_{k=0}^{n-1} \frac{t^k}{k!} ||A^k R(\lambda; A)^{\ast}|| + K' e^{\omega t} ||A^n R(\lambda; A)^{\ast}|| \right\} dt \cdot ||x|| ||x^*|| \]

\[ \leq \frac{(m-1)! M ||x|| ||x^*||}{(\lambda - \alpha)^m} \quad \text{for } x \in X, x^* \in X^*, \lambda > \alpha \text{ and } m \geq 1 , \]

where $ M = \max(||A^k R(\lambda; A)^{\ast}||, k = 0, 1, \cdots, (n-1), K' ||A^n R(\lambda; A)^{\ast}||) $ and $ \alpha = \max(1, |\omega'|) $. Thus (iii) holds good.

Now, we shall prove (c$_1$)-(c$_8$). We first note that $ A_i $ is a closed linear operator in the Banach space $ \overline{D(A)} $ and

\begin{equation}
\begin{cases}
(\alpha, \infty) \subset \rho(A_i) \equiv (\lambda; (\lambda - A_i)^{-1} \in B(\overline{D(A)}) \\
(\lambda - A_i)^{-1} = R(\lambda; A_i)|_{\overline{D(A)}} \quad \text{for } \lambda > \alpha .
\end{cases}
\end{equation}

Let $ \{S_i(t): t \geq 0\} $ be the $ C_t $-semigroup on $ \overline{D(A)} $ defined by (3.3). Since $ C_i^{-1} A_i C_i $ is the generator of $ \{S_i(t): t \geq 0\} $ by Theorem 2.1 and $ A_i \subset C_i^{-1} A_i C_i $, (3.15) implies $ A_i = C_i^{-1} A_i C_i $. This proves (c$_1$). (c$_2$) and (c$_3$) can be proved by the same way as in the proof of [10, (ii) $ \Rightarrow $ (i) in Theorem 1]. Q.E.D.
Addendum.

After this paper was submitted for publication, the authors received the following due to R. deLaubenfels:


Theorem 2.4 (b) and Lemma 2.8 in [d1] show that (1.6) and (1.7) hold true even if \( \{S(t): t \geq 0\} \) does not satisfy (0.3). Proposition 1.1, (1.5) and Proposition 1.4 (a) are also obtained in [d2]. It should be noted that Proposition 1.4 (a) does not hold if (0.3) is not assumed. (See [d2, Example 6.1].)

Let \( A, c \) and \( n \) be as in Theorem 3.1. It is shown in [d2, Theorem 2.4] that \( A \) is the generator of an \((n+1)\)-times integrated semigroup on \( X \) if and only if \( A \) is the generator of a C-semigroup on \( X \) with \( C = R(c; A)^{n+1} \).

References


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