TOKYO J. MATH.
VOL. 12, NO. 2, 1989

An Example of a Normal Isolated Singularity with Constant Plurigenera δ_m Greater than 1

Hiroyasu TSUCHIHASHI

Tohoku Gakuin University

(Communicated by N. Iwahori)

Introduction. The plurigenera $\delta_m(X, x)$ of normal isolated singularities (X, x) were defined by Watanabe [4], as analogies of plurigenera P_m of complex manifolds. Thus δ_m have the properties similar to P_m. For instance, if P_m are bounded, then P_m are not greater than 1. The plurigenera of two-dimensional normal isolated singularities behave in the same way [1, Corollary 3.2]. However, higher dimensional normal isolated singularities may have the plurigenera δ_m greater than 1, although δ_m are bounded. The purpose of this paper is to give an example of such a normal isolated singularity.

Let $f: (\tilde{X}, E)\to(X, x)$ be a good resolution of an isolated singularity (X, x). Namely, each irreducible component E_i of the exceptional set $E=E_1+E_2+\cdots+E_s$ is a non-singular divisor on $\tilde{X}$ and E has only normal crossings as the singularities. We denote by C_i the divisor $\sum_{j\neq i} D_{ij}$ $(=E_i\cdot(E-E_i))$ on E_i, where D_{ij} is the intersection $E_i\cdot E_j$ of E_i and E_j.

DEFINITION [4, 5].

$$\delta_m(X, x)=\dim\{H^0(X\setminus\{x\}, \mathcal{O}_X(mK_X))/H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(mK_{\tilde{X}}+(m-1)E))\} .$$

Here we note that the above definition does not depend on the choice of resolutions $(\tilde{X}, E)\to(X, x)$ by [2, Theorem 2.1].

THEOREM. *$\delta_m=s$ for each positive integer m, if*

$$\dim H^0(E_i, \mathcal{O}(mK_{E_i}+(k-m)[E_i]_{|E_i}+kC_i))=\begin{cases}0 & \text{for} \quad k>m>0\\ 1 & \text{for} \quad k=m>0 ,\end{cases}$$

for each E_i and if

Received September 30, 1988

$$H^0(D_{ij}, \mathcal{O}_{D_{ij}}(K_{D_{ij}}+[E-E_i-E_j]_{|D_{ij}}))=0$$

for each $(i, j) \in I=\{(i, j) \mid 1 \leqq i<j \leqq s,\ E_i \cap E_j \neq \emptyset\}$.

PROOF. Let $\mathcal{F}(m, k)=\mathcal{O}_{\tilde{X}}(mK_{\tilde{X}}+kE)$ and let $\mathcal{F}_Z(m, k)=\mathcal{F}(m, k)\otimes\mathcal{O}_Z$ for a subvariety Z of $\tilde{X}$. Then we have the following two exact sequences of sheaves:

$$0 \longrightarrow \mathcal{F}(m, k-1) \longrightarrow \mathcal{F}(m, k) \longrightarrow \mathcal{F}_E(m, k) \longrightarrow 0 ,$$

$$0 \longrightarrow \mathcal{F}_E(m, k) \longrightarrow \bigoplus_{1\leqq i\leqq s} \mathcal{F}_{E_i}(m, k) \longrightarrow \bigoplus_{(i,j)\in I} \mathcal{F}_{D_{ij}}(m, k) \longrightarrow \cdots .$$

Here we note that

$$\mathcal{F}_{E_i}(m, k) \cong \mathcal{O}_{E_i}(mK_{E_i}+(k-m)[E_i]_{|E_i}+kC_i)) ,$$

by the adjunction formula. Hence by the first condition of the theorem, $H^0(E, \mathcal{F}_E(m, k))=0$, if $m<k$. Therefore,

$$H^0(\tilde{X}, \mathcal{F}(m, m))=H^0(\tilde{X}, \mathcal{F}(m, m+1))=\cdots=H^0(\tilde{X}\setminus E, \mathcal{O}_{\tilde{X}}(mK_{\tilde{X}})) .$$

Thus we have

$$\begin{aligned}\delta_m &= \dim H^0(\tilde{X}, \mathcal{F}(m, m))/H^0(\tilde{X}, \mathcal{F}(m, m-1)) \\ &\leqq \dim H^0(E, \mathcal{F}_E(m, m)) \leqq \sum_{i=1}^{s} \dim H^0(E_i, \mathcal{O}_{E_i}(mK_{E_i}+mC_i))=s .\end{aligned}$$

Next, we consider the case of $m=k=1$. Note that

$$\mathcal{F}_{D_{ij}}(1, 1) \cong \mathcal{O}_{D_{ij}}(K_{D_{ij}}+[E-E_i-E_j]_{|D_{ij}}) .$$

Hence by the second condition of the theorem, we have the isomorphism

$$H^0(E, \mathcal{F}_E(1, 1)) \cong \bigoplus_{1\leqq i\leqq s} H^0(E_i, \mathcal{F}_{E_i}(1, 1)) .$$

Therefore, for each i, we can take an element s_i of $H^0(E, \mathcal{F}_E(1, 1))$ so that s_i vanishes on E_j if $j\neq i$ and that s_i does not vanish on E_i. By Grauert-Riemenschneider's vanishing theorem, $H^1(\tilde{X}, \mathcal{F}(1, 0))=0$. Hence the map $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(K_{\tilde{X}}+E)) \rightarrow H^0(E, \mathcal{F}_E(1, 1))$ is surjective. Let ω_i be an element of $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(K_{\tilde{X}}+E))$ whose image under the above map is s_i. Then ω_i has poles only along E_i. Hence the images of $\omega_1^m, \omega_2^m, \cdots$ and ω_s^m under the projection $H^0(\tilde{X}, \mathcal{F}(m, m)) \rightarrow H^0(\tilde{X}, \mathcal{F}(m, m))/H^0(\tilde{X}, \mathcal{F}(m, m-1))$ are linearly independent. Therefore $\delta_m=s$. q.e.d.

In the following, we construct an example of a normal isolated singularity with a resolution satisfying the conditions of the theorem, using torus embeddings. See [3], for the notation. Let N be a free $\boldsymbol{Z}$-module of rank 3 and let $\{n_1, n_2, n_3\}$ be a basis of N. Let $\Sigma=\{$faces of $\sigma_i \mid i=1$ through $6\}$, where

$$\begin{aligned}
\sigma_1&=\boldsymbol{R}_{\geq 0}n_1+\boldsymbol{R}_{\geq 0}n_2+\boldsymbol{R}_{\geq 0}(2n_2-n_3)\ ,\\
\sigma_2&=\boldsymbol{R}_{\geq 0}n_1+\boldsymbol{R}_{\geq 0}n_2+\boldsymbol{R}_{\geq 0}n_3\ ,\\
\sigma_3&=\boldsymbol{R}_{\geq 0}n_1+\boldsymbol{R}_{\geq 0}n_3+\boldsymbol{R}_{\geq 0}(-n_2+2n_3)\ ,\\
\sigma_4&=\boldsymbol{R}_{\geq 0}n_2+\boldsymbol{R}_{\geq 0}(2n_2-n_3)+\boldsymbol{R}_{\geq 0}(-n_1+n_2+n_3)\ ,\\
\sigma_5&=\boldsymbol{R}_{\geq 0}n_2+\boldsymbol{R}_{\geq 0}n_3+\boldsymbol{R}_{\geq 0}(-n_1+n_2+n_3)\quad\text{and}\\
\sigma_6&=\boldsymbol{R}_{\geq 0}n_3+\boldsymbol{R}_{\geq 0}(-n_1+n_2+n_3)+\boldsymbol{R}_{\geq 0}(-n_2+2n_3)\ .
\end{aligned}$$

Let B, F_1 and F_2 be the closures in $T_N\operatorname{emb}(\Sigma)$ of $\operatorname{orb}(\boldsymbol{R}_{\geq 0}n_2+\boldsymbol{R}_{\geq 0}n_3)$, $\operatorname{orb}(\boldsymbol{R}_{\geq 0}n_2)$ and $\operatorname{orb}(\boldsymbol{R}_{\geq 0}n_3)$, respectively. Then F_1 and F_2 are compact submanifolds in the complex manifold $T_N\operatorname{emb}(\Sigma)$ intersecting along B. Let

$$\begin{aligned}
\Lambda=\{\text{faces of }\lambda=\boldsymbol{R}_{\geq 0}n_1&+\boldsymbol{R}_{\geq 0}(-n_2+2n_3)\\
&+\boldsymbol{R}_{\geq 0}(-n_1+n_2+n_3)+\boldsymbol{R}_{\geq 0}(2n_2-n_3)\}\ .
\end{aligned}$$

Then $T_N\operatorname{emb}(\Lambda)$ has the isolated singularity $\operatorname{orb}(\lambda)$ and (N, Σ) is a subdivision of (N, Λ). Hence we have the holomorphic map $h\colon T_N\operatorname{emb}(\Sigma)\to T_N\operatorname{emb}(\Lambda)$. Here we note that h is a resolution of the singularity $\operatorname{orb}(\lambda)$ and $h^{-1}(\operatorname{orb}(\lambda))=F_1+F_2$. Let $L_1=T_N\operatorname{emb}(\{\text{faces of }\sigma_1, \sigma_2, \sigma_4\text{ and }\sigma_5\})$ and let $L_2=T_N\operatorname{emb}(\{\text{faces of }\sigma_2, \sigma_3, \sigma_5\text{ and }\sigma_6\})$. Then L_1 (resp. L_2) is an open set of $T_N\operatorname{emb}(\Sigma)$ and has the structure of the total space of a line bundle such that F_1 (resp. F_2) is the 0-section and that $F_2\cap L_1$ (resp. $F_1\cap L_2$) consists of fibers over B. Hence we can take open neighborhoods W_1 and W_2 of F_1 and F_2, respectively, and the holomorphic smooth projections $p_i\colon W_i\to F_i$ so that $p_{i|F_i}=\mathrm{id}$ and that $p_1(F_2\cap W_1)=p_2(F_1\cap W_2)=B$. Here we may assume that $W_1\cap W_2$ is connected and simply connected, because B is a non singular rational curve. Since F_1 and F_2 are Hirzebruch surfaces of degree 1 with $(B_{|F_i})^2=-1$, we have a birational holomorphic map $q_i\colon F_i\to \boldsymbol{P}^2$ to a projective plane, contracting B to a point z_i. Take a non-singular curve C_i of degree 6 in $\boldsymbol{P}^2$ which does not pass through z_i. Then the double covering E_i of F_i ramifying along $q_i^{-1}(C_i)$ is birational to a $K3$ surface. Here we may assume that

$$(p_1^{-1}\circ q_1^{-1})(C_1)\cap W_2=(p_2^{-1}\circ q_2^{-1})(C_2)\cap W_1=\varnothing\ ,$$

taking W_1 and W_2 small enough. Let $\tilde{X}_i=W_i\times_{F_i}E_i$. Then $\tilde{X}_i$ is a complex manifold containing the compact submanifold E_i and the projection

$f_i: \tilde{X}_i \to W_i$ is the double covering map ramifying along $(p_i^{-1} \circ q_i^{-1})(C_i)$. Moreover, $f_i^{-1}(W_1 \cap W_2)$ consists of two connected components R_i and T_i each of which is isomorphic to $W_1 \cap W_2$. We obtain a complex manifold $\tilde{X}$ patching up $\tilde{X}_1$ and $\tilde{X}_2$ as follows: We identify the points $x_1 \in R_1$ (resp. T_1) and $x_2 \in R_2$ (resp. T_2) if and only if $f_1(x_1) = f_2(x_2)$. Then we have a finite proper holomorphic map $f: \tilde{X} \to W = W_1 \cup W_2$ of degree 2 such that $f(x) = f_i(x)$, if $x \in \tilde{X}_i$. Since $F = F_1 + F_2$ is contractible to a point, $E := f^{-1}(F)$ $(= E_1 + E_2)$ is also contractible to a point. It is easy to verify that $(\tilde{X}, E)$ satisfies the condition of the theorem. Therefore, we obtain an isolated singularity (X, x) with $\delta_m = 2$, for each positive integer m. Moreover, for any positive integer r, we can obtain an isolated singularity with $\delta_m = 2r$ for each positive integer m, taking an r-sheeted unramified covering of $\tilde{X}$ and then contracting the inverse image of E.

References

[1] S. Ishii, Two dimensional singularities with bounded plurigenera δ_m are $\boldsymbol{Q}$-Gorenstein singularities, to appear in Proc. Iowa City Singularities Conference, Contemporary Mathematics Series of Amer. Math. Soc.

[2] F. Sakai, Kodaira dimensions of complements of divisors, *Complex Analysis and Algebraic Geometry, a collection of papers dedicated to K. Kodaira*, Iwanami-Shoten (1977), 240–257.

[3] T. Oda, *Convex Bodies and Algebraic Geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge · Band 15, Springer-Verlag, 1987.

[4] K. Watanabe, On plurigenera of normal isolated singularities I, Math. Ann., **250** (1980), 65–94.

[5] K. Watanabe, On plurigenera of normal isolated singularities II, *Complex Analytic Singularities* (T. Suwa and P. Wagreigh, eds.), Adv. Studies Pure Math., **8** (1987), Kinokuniya/North-Holland, 671–685.

Present Address:
Department of Liberal Arts, Tohoku Gakuin University
Sendai 981, Japan