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On Real Quadratic Fields and Periodic Expansions

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§0. Introduction.

We denote the continued fraction expansion of a real number α_0 ($\alpha_0>1$) by

$$[a_1: a_2, a_3, \cdots]$$

and define

$$\alpha_{i-1}=[a_i: a_{i+1}, a_{i+2}, \cdots] \qquad (i\geqq 1) .$$

With these tools some expansions are introduced as follows:

For any real x ($0<x<1$),

$$x=\sum_{n=1}^{\infty}\frac{b_n}{\alpha_0\alpha_1\alpha_2\cdots\alpha_{n-1}} , \tag{1}$$

where the digits b_n's can be found so that

$$0\leqq b_n\leqq a_n \quad \text{and} \quad \text{if } b_n=a_n \text{ then } b_{n+1}=0 ,$$

and similarly, for every real x ($-1/\alpha_0\leqq x\leqq 1$),

$$x=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cdot c_n}{\alpha_0\alpha_1\alpha_2\cdots\alpha_{n-1}} , \tag{1*}$$

where the digits c_n's can be found so that

$$0\leqq c_n\leqq a_n \quad \text{and} \quad \text{if } c_n\neq 0 \text{ then } c_{n+1}\neq a_{n+1} .$$

The second expansion is called the canonical form on discrepancy problem ([1] [2] [4] [5]).

The purpose of this paper is to characterize the quadratic fields

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$\boldsymbol{Q}(\alpha_0)$ for the quadratic algebraic number α_0 with respect to these representations.

THEOREM 1. *Let α_0 be a real quadratic algebraic number. Then x is a number of quadratic field $\boldsymbol{Q}(\alpha_0)$ if and only if the sequence $\{b_i\}_{i=1,2,\dots}$ in the expansion* (1) *is periodic, that is, there are some n and m such that $\{b_i\}=\{b_1, b_2, \cdots, b_n, \overline{b_{n+1}, \cdots, b_{n+m}}\}$.*

THEOREM 1*. *Let α_0 be a real quadratic algebraic number. Then x is a number of quadratic field $\boldsymbol{Q}(\alpha_0)$ if and only if the sequence $\{c_i\}_{i=1,2,\dots}$ in the expansion* (1*) *is periodic.*

The similar result is found in [3]. The main idea of the proof is to consider the natural extensions of the expansions in the sense of ergodic theory.

§ 1. Definitions and notations.

Let α_0 ($\alpha_0>1$) be a real quadratic irrational. We summalize several results for the continued fraction expansion of α_0.

DEFINITION 1. A quadratic irrational α is said to be reduced if $\alpha>1$ and $-1<\bar{\alpha}<0$, where $\bar{\alpha}$ denotes the algebraic conjugate of α.

From Definition 1, the following facts are well known ([6]):

THEOREM (Lagrange). *Let $\alpha=[a_1: a_2, a_3, \cdots]$. Then*

(1) *A quadratic irrational α is reduced if and only if the digits $\{a_i\}_{i=1,2,\dots}$ is purely periodic, that is, there is some k such that $\{a_i\}_{i=1,2,\dots}=\overline{\{a_1, a_2, \cdots, a_{k-1}, a_k\}}$.*

(2) *The number α is a quadratic irrational if and only if the digits $\{a_i\}_{i=1,2,\dots}$ is periodic.*

For the quadratic irrational α_0, we denote as usual the continued fraction expansion of α_0 by

$$\alpha_0=[a_1: a_2, a_3, \cdots, a_{N-1}, \overline{a_N, \cdots, a_{N+k-1}}]\,, \tag{2}$$

and call k the length of the period of the sequence $\{a_i\}_{i=1,2,\dots}$ and N the first reduced index.

Put p_n and q_n recursively as follows:

$$p_0=1\,,\quad q_0=0\,,\quad p_1=a_1\,,\quad q_1=1\,,$$
$$p_{n+1}=a_{n+1}p_n+p_{n-1}\,,\quad q_{n+1}=a_{n+1}q_n+q_{n-1}\qquad \text{for}\quad n\geqq 1\,.$$

Then the following properties are shown by induction.

PROPERTIES. *For any irrational* α,

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^n \qquad \text{for} \quad n \geqq 1 , \tag{3}$$

$$\alpha_0 = \frac{p_{n-1} + \alpha_n p_n}{q_{n-1} + \alpha_n q_n} \qquad \text{for} \quad n \geqq 1 , \tag{4}$$

$$\alpha_1 \alpha_2 \cdots \alpha_j = \frac{(-1)^{j+1}}{q_j \alpha_0 - p_j} \qquad \text{for} \quad j \geqq 1 . \tag{5}$$

Particularly, if α_0 *is purely periodic with period* k, *then*

$$\alpha_0 = \frac{(p_k - q_{k-1}) + \sqrt{d}}{2q_k} , \tag{6}$$

where $d = (p_k + q_{k-1})^2 - 4(-1)^k$, *and*

$$\alpha_0 \alpha_1 \cdots \alpha_{k-1} = \frac{(p_k + q_{k-1}) + \sqrt{d}}{2} . \tag{7}$$

Hence, $\alpha_0 \alpha_1 \cdots \alpha_{k-1}$ *is a root of* $t^2 - (p_k + q_{k-1})t + (-1)^k = 0$, *that is,* $\alpha_0 \alpha_1 \cdots \alpha_{k-1}$ *is a quadratic integer.*

We introduce a kind of expansion of x called a modified β-expansion associated with continued fraction expansion (see [1]). For any $\alpha_0 > 1$, let

$$I_j = \{\alpha_j\} \times [0, 1) \qquad (j \geqq 0)$$

and define the transformation T_j from I_j to I_{j+1} by

$$T_j(\alpha_j, x) = (\alpha_{j+1}, \alpha_j x - b_{j+1}) ,$$

where $b_{j+1} = [\alpha_j x]$ for any $j \geqq 0$. Then, from

$$T_j\left(\alpha_j, \left[\frac{a_j}{\alpha_j}, 1\right]\right) = \left(\alpha_{j+1}, \left[0, \frac{1}{\alpha_{j+1}}\right]\right) ,$$

we know

$$0 \leqq b_{j+1} \leqq a_{j+1} \quad \text{and} \quad \text{if } b_{j+1} = a_{j+1} \text{ then } b_{j+2} = 0$$

for $j \geqq 0$. Using the transformation T_j, we have the expansion of x:

$$x = \frac{b_1}{\alpha_0} + \frac{b_2}{\alpha_0 \alpha_1} + \cdots + \frac{b_n}{\alpha_0 \alpha_1 \cdots \alpha_{n-1}} + \frac{x_n}{\alpha_0 \alpha_1 \cdots \alpha_{n-1}} ,$$

where x_n is given by

$$T_{n-1}\circ\cdots\circ T_1\circ T_0(\alpha_0, x)=(\alpha_n, x_n)\ ,$$

i.e.

$$x_n=-b_n-\alpha_{n-1}b_{n-1}-\alpha_{n-2}\alpha_{n-1}b_{n-2}-\cdots \\ -\alpha_1\cdots\alpha_{n-1}b_1+\alpha_0\alpha_1\cdots\alpha_{n-1}x\ . \tag{8}$$

The other expansion, which is called the canonical form in discrepancy problem, is introduced analogously (see [1]). For any $j\geqq 0$, let

$$I_j^*=\{\alpha_j\}\times\left[-\frac{1}{\alpha_j}, 1\right]$$

and define the transformation T_j^* from I_j^* to I_{j+1}^* by

$$T_j^*(\alpha_j, x)=(\alpha_{j+1}, -\alpha_j x+c_{j+1})\ ,$$

where c_{j+1} is defined by

$$c_{j+1}=\max(a_{j+1}-[\alpha_j(1-x)], 0) \qquad \text{for} \quad j\geqq 0\ .$$

Then the sequence $\{c_j\}$ satisfies

$$0\leqq c_j\leqq a_j \quad \text{and} \quad \text{if } c_j\neq 0 \text{ then } a_{j+1}\neq c_{j+1} \qquad (j\geqq 1)\ .$$

Using the transformation T_j^*, for any $n\geqq 1$, x is expanded as follows:

$$x=\frac{c_1}{\alpha_0}-\frac{c_2}{\alpha_0\alpha_1}+\cdots+\frac{(-1)^{n-1}\cdot c_n}{\alpha_0\alpha_1\cdots\alpha_{n-1}}-\frac{(-1)^{n-1}\cdot x_n^*}{\alpha_0\alpha_1\cdots\alpha_{n-1}}\ ,$$

where x_n^* is given by

$$T_{n-1}^*\circ\cdots\circ T_1^*\circ T_0^*(\alpha_0, x)=(\alpha_n, x_n^*)\ ,$$

i.e.

$$x_n^*=c_n-\alpha_{n-1}c_{n-1}+\alpha_{n-2}\alpha_{n-1}c_{n-2}-\cdots \\ +(-1)^{n-1}\alpha_1\alpha_2\cdots\alpha_{n-1}c_1-(-1)^{n-1}\alpha_0\alpha_1\cdots\alpha_{n-1}x\ . \tag{9}$$

§ 2. Proof of Theorem 1.

From now on, we assume α_0 ($\alpha_0>1$) is a quadratic irrational, that is, α_0 is expanded as (2). Therefore $I_j=I_{j+k}$ and $T_j=T_{j+k}$ for $j\geqq N$, where N is the first reduced index. We define the domain $\bar{I}_j$ and the transformation $\bar{T}_j$ from $\bar{I}_j$ to $\bar{I}_{j+1}$, which is called a natural extension of T_j in ergodic theory, as follows: Put

$$\bar{I}_j=\{\alpha_j\}\times\left[0,\ \frac{1}{\alpha_j}\right)\times\left[\frac{1}{\bar{\alpha}_j},\ 1\right)$$
$$\cup\{\alpha_j\}\times\left[\frac{1}{\alpha_j},\ 1\right)\times\left[1+\frac{1}{\bar{\alpha}_j},\ 1\right)\qquad\text{for}\quad j\geqq 0\,.$$

Let us define the partition $P_j=\{X_{j,m}\mid m=0,1,\cdots,a_{j+1}\}$ of $\bar{I}_j$ by

$$X_{j,m}=\{(\alpha_j,x,y)\in\bar{I}_j\mid(\alpha_j,x)\in\langle m\rangle_j\}\,,$$

where

$$\langle m\rangle_j=\{\alpha_j\}\times\left[\frac{m}{\alpha_j},\ \frac{m+1}{\alpha_j}\right)\qquad\text{for}\quad 0\leqq m\leqq a_{j+1}-1$$

and

$$\langle a_{j+1}\rangle_j=\{\alpha_j\}\times\left[\frac{a_{j+1}}{\alpha_j},\ 1\right)\qquad\text{for}\quad j\geqq 0\,.$$

For any $j\geqq 1$, we also define another partition

$$Q_j=\{Y_{j,m}\mid m=0,1,\cdots,a_j\}$$

of $\bar{I}_j$ by

$$Y_{j,m}=\{(\alpha_j,x,y)\in\bar{I}_j\mid(\alpha_j,y)\in(m)_j\}\,,$$

where

$$(0)_j=\{\alpha_j\}\times[\overline{\alpha_{j-1}},\ 1)\,,$$
$$(m)_j=\{\alpha_j\}\times[\overline{\alpha_{j-1}}-m,\ \overline{\alpha_{j-1}}-(m-1))\qquad\text{for}\quad 1\leqq m\leqq a_j-1$$

and

$$(a_j)_j=\{\alpha_j\}\times\left[\frac{1}{\bar{\alpha}_j},\ \overline{\alpha_{j-1}}-(a_j-1)\right)\,.$$

Define the transformation $\bar{T}_j\colon\bar{I}_j\to\bar{I}_{j+1}$ by

$$\bar{T}_j(\alpha_j,x,y)=(\alpha_{j+1},\ \alpha_jx-m,\ \bar{\alpha}_jy-m)\tag{10}$$

if $(\alpha_j,x,y)\in X_{j,m}$ for $j\geqq 0$. From the definitions of the partition P_j, Q_j and the definition of $\bar{T}_j$, we see $\bar{T}_j(X_{j,m})=Y_{j+1,m}$ (see Figure (1)). Therefore we have the following Lemma 1.

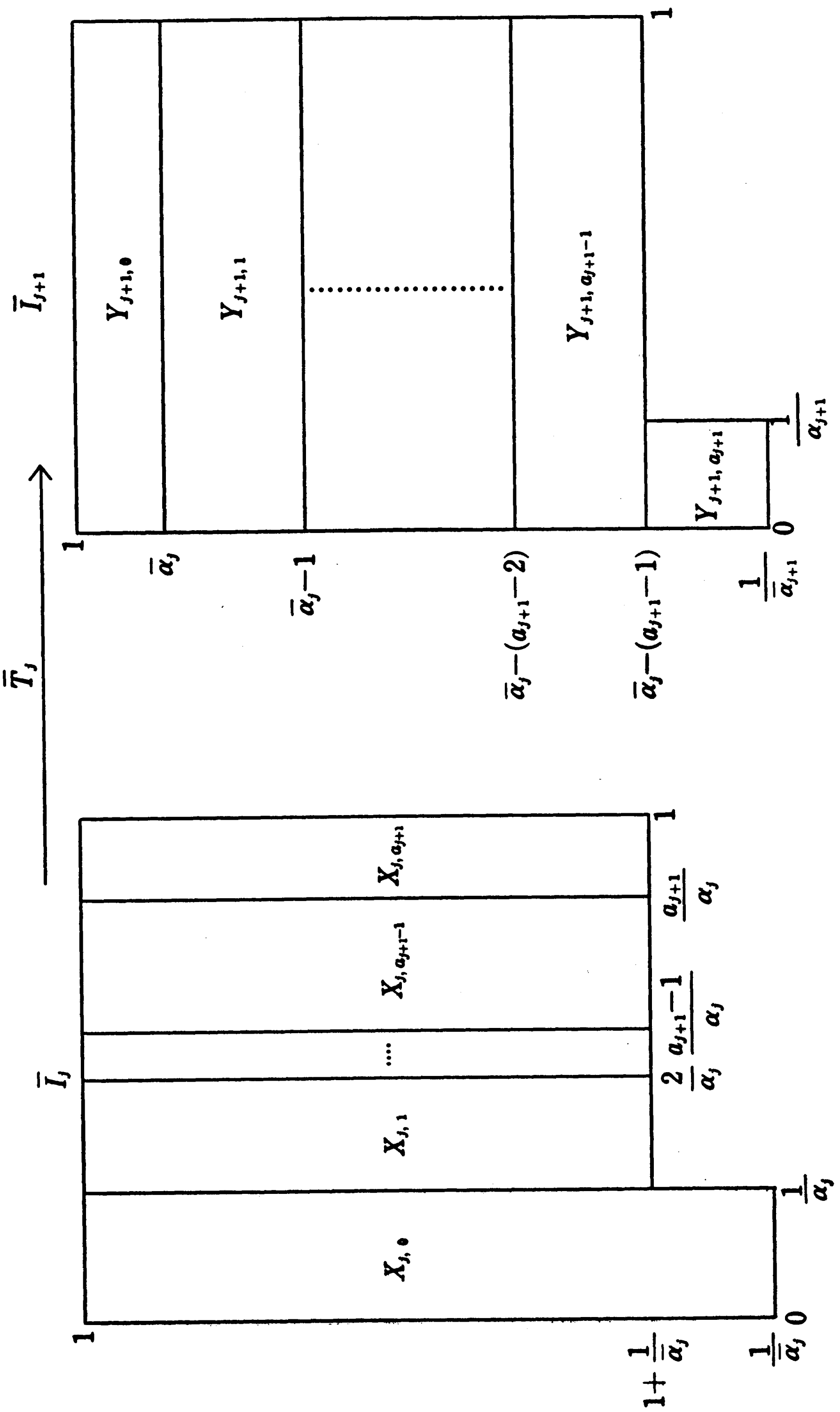
$\bar{I}_j$
$X_{j,0}$
$X_{j,1}$
$X_{j,a_{j+1}-1}$
$X_{j,a_{j+1}}$
$\bar{T}_j$
$\bar{I}_{j+1}$
$Y_{j+1,0}$
$Y_{j+1,1}$
$Y_{j+1,a_{j+1}-1}$
$Y_{j+1,a_{j+1}}$

FIGURE (1)

LEMMA 1. *The transformation* $\bar{T}_j$ *is bijective.*

DEFINITION 2. Let α be purely periodic and $x \in \boldsymbol{Q}(\alpha)$. The number x is said to be α-reduced if $(\alpha, x, \bar{x}) \in \bar{I}_0$.

Note that $\boldsymbol{Q}(\alpha_0)=\boldsymbol{Q}(\alpha_1)=\cdots$ and $\boldsymbol{Q}(\alpha_i)$ is T_i-invariant.

LEMMA 2. *Let* α_0 *be purely periodic.*

(A) *If* x *is* α_0*-reduced, then* x_1 *is* α_1*-reduced and*

(B) *if* x *is* α_0*-reduced, then there uniquely exists* (α_{k-1}, x_{-1})*, which is* $T_{k-1}(\alpha_{k-1}, x_{-1})=(\alpha_0, x)$ *and* x_{-1} *is* α_{k-1}*-reduced.*

PROOF. Assume that x is α_0-reduced, i.e. $(\alpha_0, x, \bar{x}) \in \bar{I}_0$, then

$$\begin{aligned}\bar{T}_0(\alpha_0, x, \bar{x}) &= (\alpha_1, \alpha_0 x - b_1, \bar{\alpha}_0 \bar{x} - b_1) \\ &= (\alpha, x, \overline{(x)}) ,\end{aligned}$$

that is, the following relation validates; $(\bar{x})_1=\overline{(x_1)}$. From $\bar{T}_0(\bar{I}_0)=\bar{I}_1$, we have (A). (B) is obtained by the injectivity of $\bar{T}_k$.

LEMMA 3. *We put*

$$x=\frac{s_0+t_0\sqrt{d}}{r_0} \quad \text{and} \quad x_n=\frac{s_n+t_n\sqrt{d}}{r_n}$$

where r_i*,* s_i *and* t_i *are integers for* $i=0, 1, \cdots$. *If* α_0 *is reduced and* x *is* α_0*-reduced, then* $|r_n|$*,* $|s_n|$*,* $|t_n|$ *are bounded.*

REMARK. We understand in Theorem 1 that the assumption in Lemma 3 is not necessary to obtain the boundedness of $|r_n|$, $|s_n|$ and $|t_n|$. But it is not easy to prove this lemma without the assumption.

PROOF. Because of (7), $\alpha_0\alpha_1 \cdots \alpha_{k-1}$ is a quadratic integer. Then, for any $l \geqq 1$ there exist the integers $L_1(l)$ and $L_2(l)$ such that

$$(\alpha_0\alpha_1 \cdots \alpha_{k-1})^l=\frac{L_1(l)+L_2(l)\sqrt{d}}{2} .$$

Therefore for any $n \geqq 1$, there exist the integer M such that

$$\alpha_0\alpha_1 \cdots \alpha_{n-1}=\frac{M_1(n)+M_2(n)\sqrt{d}}{M_3(n)} \quad \text{and} \quad |M_3(n)|<M .$$

From (8) it follows that $|r_n|$ is bounded.

Denote the number n in such a manner that $n=kq+i$ $(0 \leqq i<k)$,

then, x_n is α_i-reduced, and we have the inequality

$$0<x_n<1$$

and

$$\frac{1}{\bar{\alpha}_i}<\bar{x}_n<1 .$$

That is,

$$0<\frac{s_n+t_n\sqrt{d}}{r_n}<1$$

and

$$\frac{m_i}{n_i-\sqrt{d}}<\frac{s_n-t_n\sqrt{d}}{r_n}<1 ,$$

where

$$\alpha_i=\frac{n_i+\sqrt{d}}{m_i} .$$

On the other hand, from $\alpha_i>1$, $-1<\bar{\alpha}_i<0$, we know

$$0<n_i<\sqrt{d} , \qquad 0<m_i<2\sqrt{d} .$$

From the boundedness of $|r_n|$, $|m_i|$ and $|n_i|$, it follows that $|s_n|$ and $|t_n|$ are also bounded.

PROPOSITION 1. *If α_0 is purely periodic and x is α_0-reduced, then $\{b_i\}_{i=1,2,\cdots}$ is purely periodic.*

At the end of this section, we point out that the converse of this proposition also holds.

PROOF. By Lemma 2 (A) and Lemma 3, there exist positive integers i and n such that

$$x_i=x_{nk+i} .$$

We denote the partition Q_0 of $\bar{I}_0$ by $\{Y_{k,m} \mid m=0, 1, \cdots, a_k\}$. Therefore by using Lemma 2 (B) repeatedly, we have

$$x_{i-1}=x_{nk+i-1} \quad \text{and so} \quad x=x_{nk} .$$

PROOF OF THEOREM 1. Assume that α_0 is a quadratic number, and α_0 is expanded in the form (2). Let x be a number of $\boldsymbol{Q}(\alpha_0)$. Put

$$(\alpha_N, x_N, y_N)=(\alpha_N, x_N, 0)\in \bar{I}_N ,$$

where N is the first reduced index. Then we see

$$(\alpha_m, x_m, y_m)=\bar{T}_{m-1}\circ\cdots\circ\bar{T}_N(\alpha_N, x_N, y_N)\in\bar{I}_m$$

for all $m\geqq N$. By the formulae (8) and (10),

$$\begin{aligned}(x_m, y_m)=(&-b_m-\alpha_{m-1}b_{m-1}-\cdots-\alpha_1\cdots\alpha_{m-1}b_1+\alpha_0\alpha_1\cdots\alpha_{m-1}x ,\\ &-b_m-\overline{\alpha_{m-1}}b_{m-1}-\cdots-\overline{\alpha_{N+1}}\cdots\overline{\alpha_{m-1}}b_{N-1})\quad\text{for } m\geqq N .\end{aligned}$$

Therefore the distance between $(x_m, \bar{x}_m)$ and (x_m, y_m) is estimated as follows:

$$\begin{aligned}|\bar{x}_m-y_m|&\leqq|\bar{\alpha}_N|\cdots|\overline{\alpha_{m-1}}|\,|b_N|+|\overline{\alpha_{N-1}}|\cdots|\overline{\alpha_{m-1}}|\,|b_{N-1}|+\cdots\\ &\quad+|\bar{\alpha}_1|\cdots|\overline{\alpha_{m-1}}|\,|b_1|+|\bar{\alpha}_0|\,|\bar{\alpha}_1|\cdots|\overline{\alpha_{m-1}}|\,|\bar{x}|\\ &\leqq K\cdot\lambda^{m-N}\quad\text{for some positive constant } K ,\end{aligned}\tag{11}$$

where

$$\lambda=\max\{|\bar{\alpha}_N|, |\overline{\alpha_{N+1}}|, \cdots, |\overline{\alpha_{N+k-1}}|\} .$$

For any $\varepsilon>0$ and any $m\geqq N$, we define an ε-boundary $\bar{I}_{m,\varepsilon}$ of the domain $\bar{I}_m$ as follows:

$$\begin{aligned}\bar{I}_{m,\varepsilon}&=\overline{I^{(1)}_{m,\varepsilon}}\cup\overline{I^{(2)}_{m,\varepsilon}}\cup\overline{I^{(3)}_{m,\varepsilon}}\\ &=\{\alpha_m\}\times[0, 1)\times(1-\varepsilon, 1]\\ &\quad\cup\{\alpha_m\}\times\left[\frac{1}{\alpha_m}, 1\right)\times\left[1+\frac{1}{\bar{\alpha}_m}, 1+\frac{1}{\bar{\alpha}_m}+\varepsilon\right)\\ &\quad\cup\{\alpha_m\}\times\left[0, \frac{1}{\alpha_m}\right)\times\left[\frac{1}{\bar{\alpha}_m}, \frac{1}{\bar{\alpha}_m}+\varepsilon\right) .\end{aligned}$$

We discuss the following two cases:
(1) the case that there exists $m>N$ such that

$$(\alpha_m, x_m, y_m)\notin\bar{I}_{m,\varepsilon} ,$$

and
(2) the case that $(\alpha_m, x_m, y_m)\in\bar{I}_{m,\varepsilon}$ for any $m>N$.

In the case (1), by (11), we obtain

$$(\alpha_m, x_m, \bar{x}_m) \in \bar{I}_m$$

for large m. Therefore x_m is α_m-reduced. Thus, by Proposition 1, we obtain the assertion of Theorem 1.

In the case (2), by the definition of $\bar{T}_m$, we know

$$\bar{T}_m(\overline{I^{(2)}_{m,\varepsilon}}) \subset \overline{I_{m+1}} - \overline{I_{m+1,\varepsilon}} \quad \text{and} \quad \bar{T}_m(\overline{I^{(3)}_{m,\varepsilon}}) \subset \overline{I^{(1)}_{m+1,\varepsilon}} .$$

Therefore only one among the following two holds:

$$\begin{cases} (\alpha_{2p}, x_{2p}, y_{2p}) \in \{\alpha_{2p}\} \times \left[0, \dfrac{1}{\alpha_{2p}}\right) \times \left[\dfrac{1}{\bar{\alpha}_{2p}}, 1+\dfrac{1}{\bar{\alpha}_{2p}}\right) \subset \bar{I}_{2p} \quad \text{and} \\ (\alpha_{2p+1}, x_{2p+1}, y_{2p+1}) \in \{\alpha_{2p+1}\} \times [0, 1) \times (\overline{\alpha_{2p+1}}, 1) \subset \overline{I_{2p+1}} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for large } 2p , \end{cases}$$

or

$$\begin{cases} (\alpha_{2p}, x_{2p}, y_{2p}) \in \{\alpha_{2p}\} \times [0, 1) \times (\bar{\alpha}_{2p}, 1) \subset \overline{I_{2p}} \quad \text{and} \\ (\alpha_{2p+1}, x_{2p+1}, y_{2p+1}) \in \{\alpha_{2p+1}\} \times \left[0, \dfrac{1}{\alpha_{2p+1}}\right) \times \left[\dfrac{1}{\overline{\alpha_{2p+1}}}, 1+\dfrac{1}{\overline{\alpha_{2p+1}}}\right) \subset \overline{I_{2p+1}} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for large } 2p . \end{cases}$$

That is,

$$(b_{2p}, b_{2p+1}) = (0, a_{2p+1}) \text{ or } (a_{2p}, 0) .$$

This means $\{b_i\}_{i=1,2,\dots}$ is periodic. Conversely, if $\{b_i\}_{i=1,2,\dots}$ is periodic, it is easy to show that x belongs to $Q(\alpha_0)$ by (8).

REMARK. We can show the converse of Proposition 1. In fact, if the sequence $\{b_i\}_{i=1,2,\dots}$ is purely periodic, then the number x belongs to $Q(\alpha_0)$ and there is some n such that x_n is α_n-reduced by Theorem 1. For Lemma 2 (A) there is a number j such that $x_{n+j}=x$, $\alpha_{n+j}=\alpha$ and x_{n+j} is α_{n+j}-reduced. Therefore the number x is α_0-reduced.

§ 3. Proof of Theorem 1*.

The proof of Theorem 1* is obtained in analogy to § 2. Therefore we give only a sketch of the proof.

We define the domain $\bar{I}^*_j$, partition P^*_j and Q^*_j, and the transformation $\bar{T}^*_j$ from $\bar{I}^*_j$ to $\overline{I^*_{j+1}}$ as follows (see Figure (1*)): For $j \geqq 0$, put

$$\bar{I}_j^* = \{\alpha_j\} \times \left[-\frac{1}{\alpha_j}, 1\right] \times [0, 1)$$
$$\cup \{\alpha_j\} \times \left[-\frac{1}{\alpha_j}, 1-\frac{1}{\alpha_j}\right] \times \left[1, -\frac{1}{\alpha_j}\right).$$

For $j \geqq 0$, let us define the partition

$$P_j^* = \{X_{j,m}^* \mid m = 0, 1, \cdots, a_{j+1}\}$$

of $\bar{I}_j^*$ as follows: For $0 \leqq m \leqq a_{j+1} - 2$ or $m = a_{j+1}$,

$$X_{j,m}^* = \{(\alpha_j, x, y) \in \bar{I}_j^* \mid (\alpha_j, x) \in \langle m \rangle_j^*\},$$

where

$$\langle 0 \rangle_j^* = \{\alpha_j\} \times \left[-\frac{1}{\alpha_j}, 1 - \frac{a_{j+1}}{\alpha_j}\right),$$

$$\langle m \rangle_j^* = \{\alpha_j\} \times \left[1 - \frac{a_{j+1} - (m-1)}{\alpha_j}, 1 - \frac{a_{j+1} - m}{\alpha_j}\right) \quad \text{for } 1 \leqq m \leqq a_{j+1} - 2$$

and

$$\langle a_{j+1} \rangle_j^* = \{\alpha_j\} \times \left[1 - \frac{1}{\alpha_j}, 1\right].$$

Moreover, we define

$$X_{j,a_{j+1}-1}^* = \left\{(\alpha_j, x, y) \in \bar{I}_j^* \;\middle|\; x \in \left[1 - \frac{2}{\alpha_j}, 1 - \frac{1}{\alpha_j}\right]\right\}$$
$$- \{\alpha_j\} \times \left\{1 - \frac{1}{\alpha_j}\right\} \times [0, 1).$$

For $j \geqq 1$, we also define another partition

$$Q_j^* = \{Y_{j,m}^* \mid m = 0, 1, \cdots, a_j\} \quad \text{of } \bar{I}_j^*$$

by

$$Y_{j,m}^* = \{(\alpha_j, x, y) \in \bar{I}_j^* \mid (\alpha_j, y) \in (m)_j^*\}$$

where

$$(m)_j^* = [m, m+1) \quad \text{for} \quad 0 \leqq m \leqq a_j - 1$$

and

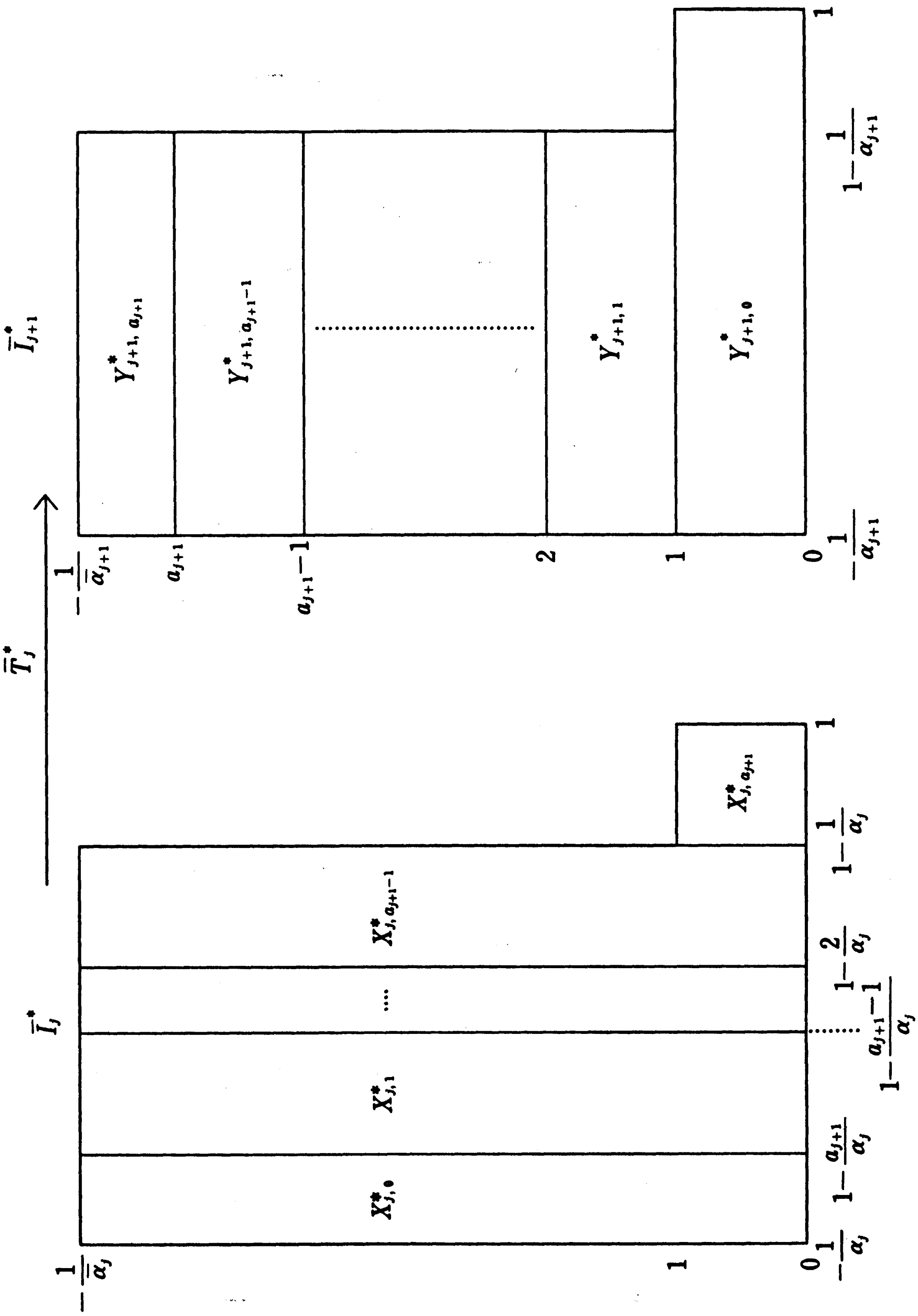
$\bar{I}_j^*$
$\bar{T}_j^*$
$\bar{I}_{j+1}^*$
$X^*_{j,0}$
$X^*_{j,1}$
$X^*_{j,a_{j+1}-1}$
$X^*_{j,a_{j+1}}$
$Y^*_{j+1,a_{j+1}}$
$Y^*_{j+1,a_{j+1}-1}$
$Y^*_{j+1,1}$
$Y^*_{j+1,0}$

FIGURE (1*)

$$(\alpha_j)_j^* = \left[\alpha_j,\ -\frac{1}{\bar{\alpha}_j}\right).$$

For $j \geqq 0$, define the transformation $\bar{T}_j^*: \bar{I}_j^* \to \bar{I}_{j+1}^*$ by

$$\bar{T}_j^*(\alpha_j, x, y) = (\alpha_{j+1}, -\alpha_j x + m, -\bar{\alpha}_j y + m) \quad \text{if } (\alpha_j, x, y) \in X_{j,m}^*. \tag{12}$$

Under the definitions, we see $\bar{T}_j^*(X_{j,m}^*) \subset Y_{j+1,m}^*$, then

LEMMA 1*. *The transformation $\bar{T}_j^*$ is bijective except for the boundary.*

DEFINITION 2*. Let α be purely periodic and $x \in \boldsymbol{Q}(\alpha)$. The number x is said to be α^*-reduced if $(\alpha, x, \bar{x}) \in \bar{I}_0^*$.

REMARK. We see that $(\alpha, \beta, \bar{\beta})$ belongs to (the interior of $\bar{I}_0^*) \cup \{\alpha\} \times \{0\} \times \{0\}$ if β is α^*-reduced. It is easy to show this fact in the following way: let us assume $(\alpha, \beta, \bar{\beta}) \in \{\alpha\} \times \{-1/\alpha\} \times [0, -1/\bar{\alpha})$ and we get $\bar{\beta} = -1/\bar{\alpha}$ since $\beta = -1/\alpha$. This is a contradiction. Let us assume $(\alpha, \beta, \bar{\beta}) \in \{\alpha\} \times [-1/\alpha, 1] \times \{0\}$ and we get $\beta = \bar{\beta} = 0$. Let us assume $(\alpha, \beta, \bar{\beta}) \in \{\alpha\} \times \{1-1/\alpha\} \times [1, -1/\bar{\alpha}]$ and we get $\bar{\beta} = 1 - 1/\bar{\alpha}$ and $(\alpha, \beta, \bar{\beta}) \notin \bar{I}_0^*$. This is a contradiction. Let us assume $(\alpha, \beta, \bar{\beta}) \in \{\alpha\} \times \{1\} \times [0, 1)$ and we get $\bar{\beta} = 1$. This is also a contradiction.

In analogy to § 2, we have the following statements:

LEMMA 2*. *Let α_0 be purely periodic.*

(A) *If x is α_0^*-reduced, then x_1^* is α_1^*-reduced, and*

(B) *if x is α_0^*-reduced, then there uniquely exists (α_{k-1}, x_{-1}^*), which is $T_{k-1}^*(\alpha_{k-1}, x_{-1}^*) = (\alpha_0, x)$ and x_{-1}^* is α_{k-1}^*-reduced.*

PROOF. Noting the above Remark, it is easy to prove Lemma 2* from Lemma 1*.

LEMMA 3*. *We put*

$$x = \frac{s_0 + t_0\sqrt{d}}{r_0} \quad \textit{and} \quad x_n^* = \frac{s_n + t_n\sqrt{d}}{r_n}$$

where r_i, s_i and t_i are integers for $i = 0, 1, \cdots$. If α_0^* is reduced and x is α_0^*-reduced, then $|r_n|$, $|s_n|$, $|t_n|$ are bounded.

PROPOSITION 1*. *If α_0 is purely periodic and x is α_0^*-reduced, then $\{c_i\}_{i=1,2,\cdots}$ is purely periodic.*

PROOF OF THEOREM 1*. For any $\varepsilon>0$ and m $(m\geqq N)$, we define an ε-boundary $\bar{I}^*_{m,\varepsilon}$ of the domain $\bar{I}^*_m$ as follows:

$$\begin{aligned}\bar{I}^*_{m,\varepsilon}&=\overline{I}^{(1)*}_{m,\varepsilon}\cup\overline{I}^{(2)*}_{m,\varepsilon}\cup\overline{I}^{(3)*}_{m,\varepsilon}\\&=\{\alpha_m\}\times\left[-\frac{1}{\alpha_j},1-\frac{1}{\alpha_j}\right]\times\left(-\frac{1}{\bar{\alpha}_j}-\varepsilon,-\frac{1}{\bar{\alpha}_j}\right]\\&\quad\cup\{\alpha_m\}\times\left(1-\frac{1}{\alpha_j},1\right]\times(1-\varepsilon,1)\\&\quad\cup\{\alpha_m\}\times\left[-\frac{1}{\alpha_j},1\right]\times[0,\varepsilon)\,.\end{aligned}$$

Under the same scheme as in §2, the case that $(x^*_m, y^*_m)\notin\bar{I}^*_{m,\varepsilon}$ for some $m\geqq N$ is reduced to Proposition 1*, that is,

$$(x^*_m, \bar{x}^*_m)\in\bar{I}^*_m\,.$$

In the case that $(x^*_m, y^*_m)\in\bar{I}^*_{m,\varepsilon}$ for any $m\geqq N$, we obtain

$$(c_{2p}, c_{2p+1})=(0, a_{2p+1})\ \text{or}\ (a_{2p}, 0)\ \text{or}\ (0, 0)$$

for large $2p$ by the relation:

$$\bar{T}^*_m(\overline{I}^{(2)*}_{m,\varepsilon})\subset\overline{I}^{(1)*}_{m+1,\varepsilon}\,.$$

Therefore $\{c_i\}_{i=1,2,\cdots}$ is periodic.

REMARK. By using the same method as in proof of Theorem 1, we can also show the converse of Proposition 1*.

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