TOKYO J. MATH.
VOL. 13, NO. 1, 1990

Construction of Vector Bundles and Reflexive Sheaves

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(Communicated by S. Suzuki)

§0. Introduction.

Let X be a smooth algebraic variety defined over a (not necessarily algebraically closed) field k. Let E be a vector bundle on X of rank $r-1$ $(r\geqq 2)$. Given a vector bundle F of rank r on X and an injection $\sigma: E\to F$, we can consider the closed subscheme $D(\sigma)=\{x\in X \mid \operatorname{rank}\sigma(x)< r-1\}$ of X. In §1, we discuss the relation between vector bundles and these closed subschemes associated with them. Our result is summarized as follows:

THEOREM (1.7). *Fix a vector bundle E as above and a line bundle L on X, and set $M=\det E$. Let $\mathscr{F}$ be the set of pairs (F, σ_F), where F is a vector bundle on X of rank r with $\det F=L$, and $\sigma_F: E\to F$ is an injection with $D(\sigma_F)$ of pure codimension 2. Let $\mathscr{G}$ be the set of pairs (Y, τ_Y), where Y is a Cohen-Macaulay closed subscheme of X of pure codimension 2, and $\tau_Y: E^\vee\to\omega_Y(-K_X+M-L)$ is a surjection. Then there exists a map $f: \mathscr{F}\to\mathscr{G}$ which is surjective in case $h^2(E(M-L))=0$.* (See (1.5), (1.6) and (1.7) for the precise statements.)

This theorem includes a result of Vogelaar [V] as a special case in which the following conditions are satisfied:
(1) X is a projective variety over an algebraically closed field,
(2) $E=\mathscr{O}_X^{\oplus r-1}$,
(3) Y is a locally complete intersection.
So our result is a generalization of that of Vogelaar's. We note that the above theorem also provides a way for constructing vector bundles. As an application, in §2, we will construct an indecomposable vector bundle of rank 3 on $\boldsymbol{P}^3$ which can never be obtained by Vogelaar's method.

In §3, we describe a method for constructing reflexive sheaves from

Received July 3, 1989

line bundles and closed subschemes of codimension 2. The precise statement of our result is as follows:

THEOREM (3.2). *Let X be a locally factorial Gorenstein projective variety of dimension $n \geqq 3$ defined over a (not necessarily algebraically closed) field k and L a line bundle on X. Let Y be a closed subscheme of X of codimension 2 and $\mathscr{I}_Y$ the ideal defining Y. Assume that for any ideal $\mathscr{I}_{Y'} \supsetneqq \mathscr{I}_Y$, $h^{n-1}(\mathscr{I}_Y(K_X+L)) > h^{n-1}(\mathscr{I}_{Y'}(K_X+L))$. Then $H^{n-1}(\mathscr{I}_Y(K_X+L))$ induces the exact sequence*

$$0 \longrightarrow H^{n-1}(\mathscr{I}_Y(K_X+L)) \otimes \mathscr{O}_X \longrightarrow F \longrightarrow \mathscr{I}_Y(L) \longrightarrow 0$$

with F reflexive.

From this theorem we can show the following: *Let X be a smooth projective variety of dimension $n \geqq 3$ over an algebraically closed field. Given a line bundle L on X with $h^2(\mathscr{O}_X(-L))=0$, and a codimension two closed subvariety Y of X with $h^{n-2}(\mathscr{O}_Y(K_X+L))>0$, we can construct a reflexive sheaf F on X with $c_1(F)=L$ and $c_2(F)=Y$.* (See (3.3).)

Basically we use the standard notation from algebraic geometry. The dualizing sheaf of a Cohen-Macaulay scheme X of pure dimension is denoted by ω_X. We denote by K_X the canonical bundle of a Gorenstein variety X. The words "vector bundles" and "locally free sheaves" are used interchangeably. The tensor products of line bundles are denoted additively. Thus, for example, if E is a coherent sheaf and if L and M are two line bundles, $E(L+M)$ means $E \otimes \mathscr{L} \otimes \mathscr{M}$, where $\mathscr{L}$ and $\mathscr{M}$ are invertible sheaves corresponding to L and M, respectively.

§ 1. The connection between vector bundles and closed subschemes of pure codimension 2.

(1.1) Throughout this section, X will stand for a smooth algebraic variety defined over a (not necessarily algebraically closed) field k. A *vector bundle* on X will mean a locally free sheaf on X of finite rank. Our aim is to explain the connection between vector bundles on X and closed subschemes of X of pure codimension 2. This generalizes the well-known connection by Vogelaar. This also provides a method for constructing vector bundles.

(1.2) Let E and F be two vector bundles on X of rank $r-1$ and r $(r \geqq 2)$, respectively. Given an injection $\sigma: E \to F$, set $Z := \{x \in X \mid \text{rank } \sigma(x) < r-1\}$. If Z has pure codimension 2, then the cokernel G of σ is a torsion free sheaf of rank 1. Therefore there exists a line bundle N on X of which G is a subsheaf, such that $\operatorname{codim}_X(\operatorname{Supp} N/G) \geqq 2$. This implies that $\mathscr{I} := G(-N)$ is a sheaf of ideals in $\mathscr{O}_X$. The closed subscheme of X defined by $\mathscr{I}$ is called the *dependency locus* of σ, and is denoted by $D(\sigma)$. Then $D(\sigma)=Z$ as sets. Note $N=\det F-\det E$. Before showing the relation between vector bundles and closed subschemes of pure codimension 2, we quote two algebraic results as needed.

(1.3) LEMMA. *Let A be a regular local ring of dimension s and B a quotient of A of dimension $s-t$. Then B is Cohen-Macaulay if and only if* $\operatorname{Ext}^q_A(B, A)=0$ *for all* $q>t$.

For a proof, we refer to [AK], Corollary 3.5.22.

(1.4) LEMMA. *Let A be a Cohen-Macaulay local ring of dimension s and B a quotient of A of dimension $s-t$. Then* $\operatorname{Ext}^q_A(B, A)=0$ *for all* $q<t$.

For a proof, we refer to [AK], Lemma 4.5.1.

(1.5) Let L be a line bundle on X and E a vector bundle on X of rank $r-1$ ($r \geqq 2$) with $\det E=M$. In the rest of this section we are always in the following situation:

$\mathscr{F}$: the set of pairs (F, σ_F), where F is a vector bundle on X of rank r with $\det F=L$, and $\sigma_F: E \to F$ is an injection whose dependency locus $D(\sigma_F)$ has pure codimension 2,

$\mathscr{G}$: the set of pairs (Y, τ_Y), where Y is a Cohen-Macaulay closed subscheme of X of pure codimension 2, and $\tau_Y: E^\vee \to \omega_Y(-K_X+M-L)$ is a surjection.

(1.6) Given $(F, \sigma_F) \in \mathscr{F}$, put $Y := D(\sigma_F)$. Then we obtain from (1.2) an exact sequence

$$0 \longrightarrow E \xrightarrow{\sigma_F} F \longrightarrow \mathscr{I}_Y(L-M) \longrightarrow 0 . \tag{1.6.1}$$

On the other hand, taking the long exact sequence of $\mathscr{E}xt$ induced by the short exact sequence

$$0 \longrightarrow \mathscr{I}_Y(L-M) \longrightarrow \mathscr{O}_X(L-M) \longrightarrow \mathscr{O}_Y(L-M) \longrightarrow 0 \tag{1.6.2}$$

and using (1.4), we have

$$\mathscr{H}om(\mathscr{I}_Y(L-M), \mathscr{O}_X) \cong \mathscr{O}_X(M-L) ,$$
$$\mathscr{E}xt^1(\mathscr{I}_Y(L-M), \mathscr{O}_X) \cong \mathscr{E}xt^2(\mathscr{O}_Y(L-M), \mathscr{O}_X) = \omega_Y(-K_X+M-L) ,$$

where $\omega_Y = \mathcal{E}xt^2(\mathcal{O}_Y, K_X)$. Thus the exact sequence of $\mathcal{E}xt$ applied to (1.6.1) gives

$$0 \longrightarrow \mathcal{O}_X(M-L) \longrightarrow F^\vee \longrightarrow E^\vee \longrightarrow \omega_Y(-K_X+M-L) \longrightarrow 0 .$$

We denote by τ_Y the last surjection and set $f(F, \sigma_F)=(Y, \tau_Y)$.

(1.7) **Theorem.** (A) *The correspondence* $f:(F, \sigma_F) \mapsto (Y, \tau_Y)$ *is a map from* $\mathcal{F}$ *into* $\mathcal{G}$.

(B) *Assume* $h^2(E(M-L))=0$. *Then* f *is surjective. Furthermore, if* $h^1(E(M-L))=0$, *then* f *is bijective.*

Proof. (A) It is sufficient to prove that Y is Cohen-Macaulay. The long exact sequences of $\mathcal{E}xt$ derived from (1.6.1) and (1.6.2) yield $\mathcal{E}xt^q(\mathcal{O}_Y(L-M), \mathcal{O}_X)=0$ for all $q>2$. Our desired result thus follows from (1.3).

(B) We take $(Y, \tau_Y) \in \mathcal{G}$ and investigate $\mathrm{Ext}^1(\mathcal{J}_Y(L-M), E)$. Combining the spectral sequence

$$E_2^{pq}=H^p(\mathcal{E}xt^q(\mathcal{J}_Y(L-M), E)) \Longrightarrow E^{p+q}=\mathrm{Ext}^{p+q}(\mathcal{J}_Y(L-M), E)$$

relating local and global Ext with the discussion in (1.6), we have the exact sequence

$$\begin{aligned} 0 &\longrightarrow E_2^{10}=H^1(\mathcal{H}om(\mathcal{J}_Y(L-M), E)) \cong H^1(E(M-L)) \\ &\longrightarrow E^1=\mathrm{Ext}^1(\mathcal{J}_Y(L-M), E) \\ &\longrightarrow E_2^{01}=H^0(\mathcal{E}xt^1(\mathcal{J}_Y(L-M), E)) \cong H^0(\omega_Y(-K_X+M-L)\otimes E) \\ &\longrightarrow E_2^{20}=H^2(\mathcal{H}om(\mathcal{J}_Y(L-M), E)) \cong H^2(E(M-L)) . \end{aligned}$$

The morphism τ_Y can be interpreted as giving an element $\tau \in H^0(\omega_Y(-K_X+M-L)\otimes E)$. Assume $h^2(E(M-L))=0$. Then we can lift τ to an element $\xi \in \mathrm{Ext}^1(\mathcal{J}_Y(L-M), E)$, so it determines an extension

$$0 \longrightarrow E \longrightarrow F \longrightarrow \mathcal{J}_Y(L-M) \longrightarrow 0 . \tag{1.7.1}$$

We denote by σ_F the first injection. Applying (1.3) to the long exact sequence of $\mathcal{E}xt$ derived from $0 \to \mathcal{J}_Y(L-M) \to \mathcal{O}_X(L-M) \to \mathcal{O}_Y(L-M) \to 0$ gives $\mathcal{E}xt^q(\mathcal{J}_Y(L-M), \mathcal{O}_X)=0$ for all $q \geqq 2$. We combine this with (1.7.1) to find $\mathcal{E}xt^q(F, \mathcal{O}_X)=0$ for all $q \geqq 2$. Applying the functor $\mathcal{H}om(\cdot, \mathcal{O}_X)$ to the sequence (1.7.1) yields an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_X(M-L) \longrightarrow F^\vee \longrightarrow E^\vee \longrightarrow \omega_Y(-K_X+M-L) \\ \longrightarrow \mathcal{E}xt^1(F, \mathcal{O}_X) \longrightarrow 0 , \end{aligned}$$

in which the connecting morphism $E^\vee \to \omega_Y(-K_X+M-L)$ is τ_Y. Since

τ_Y is surjective, $\mathscr{E}xt^1(F, \mathscr{O}_X)=0$. Thus $(F, \sigma_F)\in \mathscr{F}$ and $f(F, \sigma_F)=(Y, \tau_Y)$, which implies that f is surjective. Furthermore, if $h^1(E(M-L))=0$, then f is clearly bijective. Q.E.D.

§2. Example.

(2.1) In this section, we will construct an indecomposable vector bundle of rank 3 on $\boldsymbol{P}^3$. We note here that this bundle cannot be obtained by Vogelaar's method. In fact, we use a curve in $\boldsymbol{P}^3$ which is *not* a locally complete intersection.

(2.2) Throughout this section, the ground field k is assumed to be algebraically closed. Let C be a complete algebraic curve with $h^1(\mathscr{O}_C)=g$ and $\mathscr{F}$ a torsion free sheaf of rank 1 on C. Put

$$\deg \mathscr{F} := \chi(\mathscr{F})-\chi(\mathscr{O}_C) ,$$
$$\Delta(\mathscr{F}) := 1+\deg \mathscr{F}-h^0(\mathscr{F})=g-h^1(\mathscr{F}) .$$

Then we have the following result due to Fujita.

PROPOSITION (Fujita). *If* $\deg \mathscr{F}\geqq 2\Delta(\mathscr{F})\geqq 0$, *then* $\mathscr{F}$ *is generated by its global sections.*

For a proof we refer to [F], Proposition 1.6.

(2.3) The dualizing sheaf ω_C on C is torsion free of rank 1 and $\deg \omega_C=2\Delta(\omega_C)=2g-2$. Thus ω_C is generated by its global sections for $g\geqq 1$ by (2.2).

(2.4) Let X be a smooth quasi-projective algebraic variety and Y a closed subvariety of X of codimension i. Let $A^i(X)$ be the group of cycles of codimension i on X modulo rational equivalence. We also denote by Y the class of Y in $A^i(X)$ by abuse of notation. Grothendieck ([G]. p. 151, (16)) proved the

FORMULA. $c_j(\mathscr{O}_Y)=0 \ (0<j<i)$,
$c_i(\mathscr{O}_Y)=(-1)^{i-1}(i-1)!\, Y$.

For $i=2$, $c_2(\mathscr{O}_Y)=-Y$.

(2.5) THEOREM. *Let* X *be a 3-dimensional smooth projective variety with* $h^1(\mathscr{O}_X)=0$ *and* C *a curve in* X *with* $g\geqq 1$. *Let* t *be the number of global sections generating* ω_C. *Then there exists a vector bundle* F *on*

X of rank $t+1$ with $c_1(F)=c_1(X)$ and $c_2(F)=C$.

PROOF. Take $L=-K_X$ and $E=\mathcal{O}_X^{\oplus t}$, and apply (1.7) and (2.4). Q.E.D.

(2.6) We shall apply (2.5) to the simplest case $X=\boldsymbol{P}^3:=P$. Let F and C be as in (2.5). Since the Chow ring is isomorphic to $\boldsymbol{Z}[h]/h^4$, we may consider the Chern classes $c_1(F)$, $c_2(F)$, $c_3(F)$ as integers. So $c_1(F)=4$ and $c_2(F)=\deg C:=d$.

(2.7) In order to calculate $c_3(F)$, we need the

RIEMANN-ROCH THEOREM. *Let $\mathcal{F}$ be a coherent sheaf of rank r on $\boldsymbol{P}^3$, with Chern classes c_1, c_2, c_3. Then*

$$\chi(\mathcal{F})=r+\binom{c_1+3}{3}-2c_2+\frac{1}{2}(c_3-c_1c_2)-1\ .$$

For a proof, we refer to [H2], Theorem 2.3.

(2.8) Now we go back to the situation (2.6). The exact sequence $0\rightarrow\mathcal{O}_P^{\oplus t}\rightarrow F\rightarrow\mathcal{I}_C(4)\rightarrow 0$ gives rise to $c_i(F)=c_i(\mathcal{I}_C(4))$, hence by (2.7)

$$\chi(\mathcal{I}_C(4))=\frac{1}{2}c_3(F)-4d+\binom{7}{3}\ .$$

On the other hand, by the exact sequence $0\rightarrow\mathcal{I}_C(4)\rightarrow\mathcal{O}_P(4)\rightarrow\mathcal{O}_C(4)\rightarrow 0$, we see that

$$\chi(\mathcal{I}_C(4))=\chi(\mathcal{O}_P(4))-\chi(\mathcal{O}_C(4))=g-1-4d+\binom{7}{3}\ .$$

So $c_3(F)=2g-2=\deg\omega_C$.

(2.9) In the rest of this section we assume char $k\neq 2, 3$. Let s, t be the homogeneous coordinates on $\boldsymbol{P}^1$ and w, x, y, z on $\boldsymbol{P}^3$. Consider the rational curve C of degree 6 in $\boldsymbol{P}^3$ which is the image of the map $f:\boldsymbol{P}^1\rightarrow\boldsymbol{P}^3$ defined by $f(s:t)=(w:x:y:z):=(st^2(s+t)^3:s^2t^3(s+t):s^3t(s+t)^2:(s-t)^6)$. We set

$$M(s,t):=\begin{bmatrix}\partial w/\partial s & \partial w/\partial t\\ \partial x/\partial s & \partial x/\partial t\\ \partial y/\partial s & \partial y/\partial t\\ \partial z/\partial s & \partial z/\partial t\end{bmatrix}=\begin{bmatrix}t^2(s+t)^2(4s+t) & st(s+t)^2(2s+5t)\\ st^3(3s+2t) & s^2t^2(3s+4t)\\ s^2t(s+t)(5s+3t) & s^3(s+t)(s+3t)\\ 6(s-t)^5 & -6(s-t)^5\end{bmatrix}.$$

An easy calculation shows that the rank of $M(s,t)$ is 2 for any $(s:t)\in\boldsymbol{P}^1$.

Let p be the point $(0:0:0:1)$. Then $f^{-1}(p)$ consists of three distinct points $(0:1)$, $(1:0)$, $(1:-1)$. Put $V:=\boldsymbol{P}^1-\{(0:1), (1:0), (1:-1)\}$ and assume $(s:t)\in V$. Since $t\neq 0$, we can set $t=1$, and use s as an affine parameter. Then we have

$$\begin{aligned} f(s:1) &= (s(s+1)^3 : s^2(s+1) : s^3(s+1)^2 : (s-1)^6) \\ &= \left(\frac{(s+1)^2}{s} : 1 : s(s+1) : \frac{(s-1)^6}{s^2(s+1)}\right). \end{aligned}$$

From this it is easy to see that f is injective on V. Thus, in sum, C has exactly one singular point p.

Let U be the open set $\{z\neq 0\}$. Then p is the origin in U. We use $M(s,t)$ to see that the tangent directions in U at $(0:1)$, $(1:0)$ and $(1:-1)$ are $(1,0,0)$, $(0,0,1)$ and $(0,-1,0)$, respectively. Therefore C is *not* a locally complete intersection and blowing up C at p desingularizes C in one step. Of course the multiplicity of p on C is 3. Let $\delta_p=\text{length}(\tilde{\mathcal{O}}_p/\mathcal{O}_p)$, where $\tilde{\mathcal{O}}_p$ is the integral closure of $\mathcal{O}_p$. Then the arithmetic genus g of C is equal to δ_p.

(2.10) In order to calculate δ_p, we quote the following

LEMMA. *Let C be a complete algebraic curve with only one singular point p. Assume that blowing up C at p desingularizes C in one step. Let ρ be the multiplicity of p on C. Then $\rho-1\leqq\delta_p\leqq\rho(\rho-1)/2$. Furthermore $\delta_p=\rho(\rho-1)/2$ if and only if* length $\mathfrak{m}/\mathfrak{m}^2=2$, *where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_p$.*

For a proof, see for example [K].

(2.11) We now return to our case (2.9). Applying (2.10) yields $g=\delta_p=2$, so by (2.3), ω_C is generated by two global sections. Combining this with (2.5), (2.6) and (2.8), we obtain a 3-bundle F on $\boldsymbol{P}^3$ with $c_0(F)=1$, $c_1(F)=4$, $c_2(F)=6$ and $c_3(F)=2$. Since the polynomial X^3+4X^2+6X+2 is irreducible by Eisenstein's criterion, F is indecomposable.

§ 3. Construction of reflexive sheaves.

(3.1) The aim of this section is to describe a way to construct reflexive sheaves from line bundles and closed subschemes of codimension 2.

(3.2) THEOREM. *Let X be a locally factorial Gorenstein projective variety of dimension $n\geqq 3$ defined over a (not necessarily algebraically* *closed) field k and L a line bundle on X. Let Y be a closed subscheme of X of codimension 2 and $\mathscr{I}_Y$ the ideal defining Y. Assume that for any ideal $\mathscr{I}_{Y'} \supsetneqq \mathscr{I}_Y$, $h^{n-1}(\mathscr{I}_Y(K_X+L)) > h^{n-1}(\mathscr{I}_{Y'}(K_X+L))$. Then $H^{n-1}(\mathscr{I}_Y(K_X+L))$ induces the exact sequence*

$$0 \longrightarrow H^{n-1}(\mathscr{I}_Y(K_X+L)) \otimes \mathscr{O}_X \longrightarrow F \longrightarrow \mathscr{I}_Y(L) \longrightarrow 0$$

with F reflexive.

PROOF. By Serre duality we have an isomorphism

$$\begin{aligned}\varphi:\ &\mathrm{Ext}^1(\mathscr{I}_Y(L), H^{n-1}(\mathscr{I}_Y(K_X+L))\otimes\mathscr{O}_X)\\ &\xrightarrow{\ \sim\ } \mathrm{Hom}(H^{n-1}(\mathscr{I}_Y(K_X+L)), H^{n-1}(\mathscr{I}_Y(K_X+L)))\ .\end{aligned}$$

Let $\varphi(\xi)=\mathrm{id}$. Then ξ defines a global extension

$$0 \longrightarrow H^{n-1}(\mathscr{I}_Y(K_X+L)) \otimes \mathscr{O}_X \longrightarrow F \longrightarrow \mathscr{I}_Y(L) \longrightarrow 0 \qquad (\xi)$$

over X. We show that F is reflexive. Since F is torsion free, the natural map $\mu: F \to F^{\vee\vee}$ is injective. We consider the commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & H^{n-1}(\mathscr{I}_Y(K_X+L))\otimes\mathscr{O}_X & \longrightarrow & F & \longrightarrow & \mathscr{I}_Y(L) & \longrightarrow 0 & & (\xi)\\ & \| & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\nu} & & & \\ 0 \longrightarrow & H^{n-1}(\mathscr{I}_Y(K_X+L))\otimes\mathscr{O}_X & \longrightarrow & F^{\vee\vee} & \longrightarrow & S & \longrightarrow 0 & & (\xi') \end{array}$$

where ξ' is an element of $\mathrm{Ext}^1(S, H^{n-1}(\mathscr{I}_Y(K_X+L))\otimes\mathscr{O}_X)$ given by the second extension. We note that ν^*: $\mathrm{Ext}^1(S, H^{n-1}(\mathscr{I}_Y(K_X+L))\otimes\mathscr{O}_X) \to \mathrm{Ext}^1(\mathscr{I}_Y(L), H^{n-1}(\mathscr{I}_Y(K_X+L))\otimes\mathscr{O}_X)$ satisfies $\nu^*(\xi')=\xi$. We claim that S is torsion free of rank 1. Suppose to the contrary that $S_{\mathrm{Tor}}\neq 0$. Let $\xi''=i^*(\xi')$, where $i: S_{\mathrm{Tor}} \to S$ is the inclusion map. Then we have the commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & H^{n-1}(\mathscr{I}_Y(K_X+L))\otimes\mathscr{O}_X & \longrightarrow & E & \longrightarrow & S_{\mathrm{Tor}} & \longrightarrow 0 & & (\xi'')\\ & \| & & \downarrow & & \downarrow{\scriptstyle i} & & & \\ 0 \longrightarrow & H^{n-1}(\mathscr{I}_Y(K_X+L))\otimes\mathscr{O}_X & \longrightarrow & F^{\vee\vee} & \longrightarrow & S & \longrightarrow 0 & & (\xi')\ . \end{array}$$

On the other hand, since $\mathrm{Supp}(S_{\mathrm{Tor}})\subset Y$, $\mathrm{Ext}^1(S_{\mathrm{Tor}}, H^{n-1}(\mathscr{I}_Y(K_X+L))\otimes\mathscr{O}_X)\cong H^{n-1}(S_{\mathrm{Tor}}\otimes K_X)^\vee\otimes H^{n-1}(\mathscr{I}_Y(K_X+L))=0$. Hence $F^{\vee\vee}_{\mathrm{Tor}}\neq 0$, which is a contradiction. Since X is locally factorial and $\det F=\det(F^{\vee\vee})$, we can write $S=\mathscr{I}_{Y'}(L)$ for some closed subscheme Y' of codimension $\geqq 2$. The Serre duality theorem says that

$$\psi:\ \mathrm{Ext}^1(\mathscr{I}_{Y'}(L),\ H^{n-1}(\mathscr{I}_Y(K_X+L))\otimes\mathcal{O}_X)$$
$$\xrightarrow{\ \sim\ }\mathrm{Hom}(H^{n-1}(\mathscr{I}_{Y'}(K_X+L)),\ H^{n-1}(\mathscr{I}_Y(K_X+L)))\ .$$

Let $\eta=\psi(\xi')$. Then, by the functoriality of Serre duality, we obtain the commutative diagram

$$\begin{array}{ccc} H^{n-1}(\mathscr{I}_Y(K_X+L)) & \xrightarrow{\ \mathrm{id}\ } & H^{n-1}(\mathscr{I}_Y(K_X+L)) \\ & {}_{f}\searrow \qquad \nearrow_{\eta} & \\ & H^{n-1}(\mathscr{I}_{Y'}(K_X+L)) & \end{array}$$

where f is the natural map induced by $\nu\otimes K_X$. So $h^{n-1}(\mathscr{I}_Y(K_X+L))\leqq h^{n-1}(\mathscr{I}_{Y'}(K_X+L))$. Combining this with the hypothesis gives $\mathscr{I}_Y=\mathscr{I}_{Y'}$. Therefore μ is an isomorphism and F is reflexive. Q.E.D.

(3.3) COROLLARY. *Let X be a smooth projective variety of dimension $n\geqq 3$ defined over an algebraically closed field k and L a line bundle on X such that $h^2(\mathcal{O}_X(-L))=0$. Let Y be a closed subvariety of X of codimension 2. Assume $h^{n-2}(\mathcal{O}_Y(K_X+L))>0$. Then there exists a reflexive sheaf F of rank r on X with $c_1(F)=L$ and $c_2(F)=Y$, where $r=h^{n-1}(\mathscr{I}_Y(K_X+L))+1$.*

PROOF. Given any ideal $\mathscr{I}_{Y'}\supsetneqq\mathscr{I}_Y$, we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{I}_Y(K_X+L) & \longrightarrow & \mathcal{O}_X(K_X+L) & \longrightarrow & \mathcal{O}_Y(K_X+L) & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{I}_{Y'}(K_X+L) & \longrightarrow & \mathcal{O}_X(K_X+L) & \longrightarrow & \mathcal{O}_{Y'}(K_X+L) & \longrightarrow & 0\ . \end{array}$$

Since $h^{n-1}(\mathcal{O}_Y(K_X+L))=h^{n-1}(\mathcal{O}_{Y'}(K_X+L))=h^{n-2}(\mathcal{O}_{Y'}(K_X+L))=0$,

$$h^{n-1}(\mathscr{I}_Y(K_X+L))>h^{n-1}(\mathcal{O}_X(K_X+L))=h^{n-1}(\mathscr{I}_{Y'}(K_X+L))\ ;$$

the assertion now follows from (3.2). Q.E.D.

ACKNOWLEDGEMENT. I would like to express my sincere thanks to Professors H. Kaji and M. Tomari: Kaji communicated a curve in (2.9) to me; Tomari told me Kirby's result [K]. I would also like to express my hearty thanks to the referee for his or her valuable comments.

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