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On the Convergence of Series of Fourier Coefficients of Vector Valued Functions

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§ 1. Introduction.

Let X be a Banach space with norm $\|\cdot\|$ and let $x(t)$ be an X valued function on $-\infty<t<\infty$ which is 2π periodic: $\|x(t+2\pi)-x(t)\|=0$ for every t, and integrable (in Bochner sense [2]) on $T\equiv(-\pi, \pi)$. One may define the Fourier coefficients of $x(t)$:

$$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi} x(t)e^{-int}dt\ , \qquad n=0, \pm 1, \cdots . \tag{1}$$

Kandil [6] has studied the unconditional convergence of $\sum_{n=-\infty}^{\infty} c_n$ in X and the convergence of $\sum_{n=-\infty}^{\infty} \|c_n\|$ when X is a Hilbert space, and he gave sufficient conditions for those sorts of convergence, which are analogous to the known criteria for the absolute convergence of ordinary Fourier series of complex valued functions.

The purpose of this paper is to generalize Kandil's results as well as the author's theorem on the convergence of $\sum \|c_n\|$ when X is the space of random variables [7], [8].

We introduce some notations. Let $L^p(T)$ be the class of complex valued functions $f(t)$ with $\int_T |f(t)|^p dt<\infty$, $p>0$, as usual. The class of $x(t)$ with $\|x(t)\|\in L^p(T)$ is denoted by $\boldsymbol{L}^p(T)$, $p\geqq 1$. We remark that such function $x(t)$ is integrable. Write

$$|||x(\cdot)|||_p=\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \|x(t)\|^p dt\right]^{1/p}\ , \qquad p\geqq 1\ .$$

Letting

$$\Delta_h^{(r)}x(t)=\sum_{k=0}^{r} (-1)^{r-k}\binom{r}{k}x(t+kh)\ ,$$

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the difference of the r-th order, we describe the r-th integrated modulus of continuity of $x(t)$ by

$$\sup_{|h|\leq\delta} |||\Delta_h^{(r)}x(\cdot)|||_p = M_p^{(r)}(\delta)\ , \qquad \delta>0$$

and the continuity modulus of $x(t)$ by

$$\sup_{|h|\leq\delta}\sup_{t\in T} \|\Delta_h^{(r)}x(t)\| = \sup_{|h|\leq\delta} |||\Delta_h^r x(\cdot)|||_\infty = M^{(r)}(\delta)\ , \qquad \delta>0\ .$$

§ 2. Unconditional convergence of $\sum_{n=-\infty}^{\infty} c_n$.

The unconditional convergence in X is usually defined for unilateral series $\sum_{n=1}^{\infty} x_n$, $x_n \in X$, $n=1, 2, \cdots$, but the definition of unconditional convergence for bilateral series $\sum_{n=-\infty}^{\infty} x_n$ is also naturally made, and the equivalence of unconditional convergence in X and the convergence of all subseries, which is due to Orlicz [10, I], is also true for bilateral series, and the unconditional convergence of $\sum_{n=-\infty}^{\infty} x_n$ is equivalent to the unconditional convergence of both series $\sum_{n=-\infty}^{0} x_n$ and $\sum_{n=0}^{\infty} x_n$.

We shall study the conditional convergence of the Fourier series $\sum_{n=-\infty}^{\infty} c_n$ of $x(t)$, c_n being given by (1). We actually do this along the line of Kandil [6]. Let X^* be the conjugate space of a Banach space X. Suppose $x(t) \in L^p(T)$ for some $p \geqq 1$ and let $x^* \in X^*$. The norm of a linear functional x^* is denoted by $\|x^*\|$. Write the Fourier coefficients of the complex valued function $x^*x(t)$ by

$$c_n^* = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^*x(t)e^{-int}dt = x^*c_n\ , \qquad n=0, \pm 1, \pm 2, \cdots\ . \tag{2}$$

We see

$$|c_n^*| \leqq \|x^*\|\cdot\|c_n\|\ . \tag{3}$$

Since $|x^*x(t)|^p \leqq \|x^*\|^p \cdot \|x(t)\|^p$,

$$x^*x(t) \in L^p(T)\ , \tag{4}$$

and the ordinary Hausdorff-Young inequality when $1\leqq p\leqq 2$ applies to have

$$\Big(\sum_{k=-\infty}^{\infty} |c_k^*|^{p'}\Big)^{1/p'} \leqq C\Big[\frac{1}{2\pi}\int_{-\pi}^{\pi} |x^*x(t)|^p dt\Big]^{1/p} \leqq C\|x^*\|\cdot|||x(\cdot)|||_p\ , \tag{5}$$

where, $1/p+1/p'=1$, C is a constant depending only on p.

LEMMA 1. *If* $1\leqq p\leqq 2$, $x(t)\in L^p(T)$, *then*

$$(\sum_{|k|\geqq n}|x^*c_k|^{p'})^{1/p'}\leqq C\|x^*\|M_p^{(r)}(1/n)\ , \tag{6}$$

for every positive integer n, where r is any positive integer and C is a constant depending only on p and r.

PROOF. The Fourier coefficients of $\Delta_h^{(r)}x(t)$ are $c_k(1-e^{ikh})^r$, $k=0, \pm 1,\cdots$ and we have from (5) with $x^*c_k(1-e^{ikh})^r$ in place of c_k^*,

$$\left[\sum_{k=-\infty}^{\infty}|x^*c_k(1-e^{ikh})^r|^{p'}\right]^{1/p'}\leqq C\|x^*\|\cdot|||\Delta_h^{(r)}x(\cdot)|||_p\ ,$$

C being a constant depending only on p. In what follows C's are constants depending only on p and r and may be different on each occurrence.

Integrating both sides of the last inequality with respect to h on $(0, 1/n)$, multiplying by n and using (5), we have

$$\begin{aligned} n\int_0^{1/n}\left(\sum_{k=-\infty}^{\infty}|x^*c_k|^{p'}\left|2\sin\frac{kh}{2}\right|^{rp'}\right)^{1/p'}dh &\leqq C\|x^*\|n\int_0^{1/n}|||\Delta_h^{(r)}x(\cdot)|||_p dh \\ &\leqq C\|x^*\|\sup_{|h|\leqq 1/n}|||\Delta_h^{(r)}x(\cdot)|||_p \\ &=C\|x^*\|M_p^{(r)}(1/n)\ . \end{aligned} \tag{7}$$

On the other hand, the left hand side of (7) is not less than

$$2^r n\int_0^{1/n}\left(\sum_{|k|\geqq n}|x^*c_k|^{p'}\left|\sin\frac{kh}{2}\right|^{rp'}\right)^{1/p'}dh$$

which is, because of the Minkowski inequality, not less than

$$2^r\left[\sum_{|k|\geqq n}|x^*c_k|^{p'}\left(n\int_0^{1/n}\left|\sin\frac{kh}{2}\right|^r dh\right)^{p'}\right]^{1/p'}\ .$$

Since, for $|k|\geqq n$,

$$n\int_0^{1/n}\left|\sin\frac{kh}{2}\right|^r dh=\frac{n}{k}\int_0^{k/n}\left|\sin\frac{u}{2}\right|^r du\geqq C>0\ ,$$

the last one is not less than

$$C(\sum_{|k|\geqq n}|x^*c_k|^{p'})^{1/p'}\ .$$

This shows Lemma 1.

THEOREM 1. *Let $1\leqq p\leqq 2$, $x(t)\in L^p(T)$. If*

$$\sum_{n=1}^{\infty}n^{-1/p'}M_p^{(r)}(1/n)<\infty\ , \tag{8}$$

for some positive integer r, then

$$\sum_{n=-\infty}^{\infty} |x^* c_n| < \infty . \tag{9}$$

PROOF. The proof is carried out by the well known method of Riemann's condensation as was done in the author's papers [7], [8].

We have

$$\sum_{n=3}^{\infty} |x^* c_n| = \sum_{n=1}^{\infty} \sum_{k=2^n+1}^{2^{n+1}} |x^* c_n|$$

which is, by means of the Hölder inequality, not greater than

$$\sum_{n=1}^{\infty} 2^{n/p} \Big(\sum_{k=2^n+1}^{2^{n+1}} |x^* c_n|^{p'} \Big)^{1/p'} \leqq \sum_{n=1}^{\infty} 2^{n/p} \Big(\sum_{k=2^n+1}^{\infty} |x^* c_n|^{p'} \Big)^{1/p'} .$$

Because of Lemma 1, this is not less than

$$\begin{aligned} & C\|x^*\| \sum_{n=1}^{\infty} 2^{n/p} M_p^{(r)}(1/2^n) \\ \leqq & C\|x^*\| \sum_{n=1}^{\infty} 2^{-n/p'} \Big(\sum_{m=2^{n-1}+1}^{2^n} 1 \Big) M_p^{(r)}(1/2^n) \\ \leqq & C\|x^*\| \sum_{n=1}^{\infty} 2^{-n/p'} \sum_{m=2^{n-1}+1}^{2^n} M_p^{(r)}(1/m) \\ \leqq & C\|x^*\| \sum_{m=1}^{\infty} m^{-1/p'} M_p^{(r)}(1/m) < \infty . \end{aligned}$$

The same is true for $\sum_{n=-\infty}^{-3} |x^* c_n|$ and the proof is complete.

The combination of Theorem 1 and Orlicz-Banach's theorem ([4] p. 93, [9]) that if the Banach space X is weakly complete, then the series $\{x_n,\ n=1, 2, \cdots\}$, $x_n \in X$ is unconditionally convergent if and only if $\sum_{n=1}^{\infty} |x^* x_n| < \infty$, gives us:

THEOREM 2. *Let* $1 \leqq p \leqq 2$, $x(t) \in L^p(T)$. *If X is weakly complete and the condition* (8) *is satisfied, then* $\sum_{n=-\infty}^{\infty} c_n$ *is unconditionally convergent.*

We note that the conclusion of this theorem cannot be replaced by the convergence of $\sum_{n=-\infty}^{\infty} \|c_n\|$. This was shown by Kandil [6].

§ 3. Series of Fourier coefficients of functions of strongly bounded variation.

Let 2π-periodic X valued function $x(t)$ be of strongly bounded variation, namely

$$V=\sup_{D}\sum_{j=1}^{n}\|x(t_j)-x(t_{j-1})\|<\infty \tag{10}$$

for all divisions D of T, $-\pi\leqq t_0<\cdots<t_n\leqq\pi$. We easily see that for $h>0$

$$\int_{-\pi}^{\pi}\|x(t+h)-x(t)\|dt\leqq 2hV\,. \tag{11}$$

From this and since $\Delta_h^{(r)}x(t)=\Delta_h^{(r-1)}[x(t+h)-x(t)]$ we have

$$\int_{-\pi}^{\pi}\|\Delta_h^{(r)}x(t)\|dt\leqq 2^r hV\,. \tag{12}$$

LEMMA 2. *If $x(t)\in L^p(T)$, $p\geqq 1$ and is of strongly bounded variation, then for every positive integer r and for $\delta>0$, $1/p+1/p'=1$,*

$$M_p^{(r)}(\delta)\leqq C[M^{(r)}(\delta)]^{1/p'}\delta^{1/p}\,, \tag{13}$$

where C is a constant depending only on r and p.

This is substantially known and in fact the proof is just the same as that of Lemma 2.1 of [8]. From Lemma 2 and Theorem 1 we readily have

THEOREM 3. *If $1\leqq p\leqq 2$, X is weakly complete and $x(t)$ is of strongly bounded variation, and*

$$\sum_{n=1}^{\infty}n^{-1}[M^{(r)}(1/n)]^{1/p'}<\infty \tag{14}$$

is satisfied, $1/p+1/p'=1$, then $\sum_{n=-\infty}^{\infty}c_n$ is unconditionally convergent.

§ 4. Series of Fourier coefficients of Hilbert valued functions.

We study the convergence of $\sum\|c_n\|$ as Kandil [6] did when X is a Hilbert space.

This case is very much similar to that already considered in [8] and actually much simpler, because, for Hilbert valued periodic functions, the Parseval relation holds true: *if X is a Hilbert space and $x(t)\in L^2(T)$, then*

$$|||x(\cdot)|||_2^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}\|x(t)\|^2dt=\sum_{n=-\infty}^{\infty}\|c_n\|^2\,. \tag{15}$$

For a general Banach valued function of $L^2(T)$, (15) does not necessarily hold [2].

(15) is a particular case of the fact that if $x(t)$ and $y(t)$ are Hilbert valued function of $L^2(T)$, then the Fourier series of

$$h(t)=\frac{1}{2\pi}\int_{-\pi}^{\pi}(x(t-u),\ y(u))du$$

is absolutely convergent and

$$h(t)=\sum_{n=-\infty}^{\infty}(c_n,\ \bar{d}_n)e^{int}\ , \tag{16}$$

where $(\cdot,\cdot)$ is the inner product and c_n, d_n are Fourier coefficients of $x(t)$ and $y(t)$ respectively.

Now Lemma 1 with $p=2$ keeps being true if $\|c_k\|$ is taken for x^*c_k, and we have

$$\Big(\sum_{|k|\geqq n}\|c_k\|^2\Big)^{1/2}\leqq CM_2^{(r)}(1/n)\ , \tag{17}$$

C being a constant depending only on r. Furthermore the proof of Theorem 1 goes through, x^*c_k being replaced by $\|c_k\|$ and $p=2$. Thus we have the following

THEOREM 4. *If $x(t)$ is a Hilbert valued function of $L^2(T)$ and the condition*

$$\sum_{n=1}^{\infty}n^{-1/2}M_2^{(r)}(1/n)<\infty \tag{18}$$

is satisfied for some positive integer r, then

$$\sum_{n=-\infty}^{\infty}\|c_n\|<\infty\ . \tag{19}$$

We also have

THEOREM 5. *If $x(t)\in L^2(T)$ is a Hilbert valued function of strongly bounded variation and the condition*

$$\sum_{n=1}^{\infty}n^{-1}[M_2^{(r)}(1/n)]^{1/2}<\infty \tag{20}$$

is satisfied for some positive integer r, then (19) *holds.*

These are slight generalizations of Kandil's results [6].

§ 5. Series of Fourier coefficients of $L^{\alpha}(0, 1)$ valued functions.

Let $\{f_n(u),\ n=1, 2, \cdots\}$ be a sequence of real valued functions on

$(0, 1)$. Suppose $f_n(u) \in L^\alpha(0, 1)$, $n=1, 2, \cdots$, $\alpha \geqq 1$. Write, as usual, $\|f_n\|_\alpha = \left[\int_0^1 |f_n(u)|^\alpha du\right]^{1/\alpha}$.

Orlicz [10] I, II, has given the theorem:

A. *If a series* $\sum_{n=1}^\infty f_n(u)$, $f_n(u) \in L^\alpha(0, 1)$, *is unconditionally convergent (in* $L^\alpha(0, 1)$*), then*

(i) *for* $1 \leqq \alpha \leqq 2$

$$\sum_{n=1}^\infty \|f_n\|_\alpha^2 < \infty , \tag{21}$$

(ii) *for* $2 \leqq \alpha < \infty$

$$\sum_{n=1}^\infty \|f_n\|_\alpha^\alpha < \infty . \tag{22}$$

Now consider an $L^\alpha(0, 1)$ valued, 2π-periodic function of t. We may write such function $x(t, u)$, $t \in T$, $u \in (0, 1)$. Suppose throughout $x(t, u)$ is measurable in rectangle $T \times (0, 1)$.

Write for $1 \leqq \alpha < \infty$, $1 \leqq p \leqq \infty$,

$$|||x(\cdot, \cdot)|||_{p,\alpha} = \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \|x(t, \cdot)\|_\alpha^p dt\right]^{1/p} , \tag{23}$$

$$M_{p,\alpha}^{(r)}(\delta) = \sup_{|h| \leqq \delta} |||\Delta_h^{(r)} x(\cdot, \cdot)|||_{p,\alpha} \tag{24}$$

for any positive integer r. The class of $x(t, u)$ with $|||x(\cdot, \cdot)|||_{p,\alpha} < \infty$ is denoted by $\boldsymbol{L}^{p,\alpha}(T \times (0, 1))$. Obviously $|||x(\cdot, \cdot)|||_{p,\alpha} \leqq |||x(\cdot, \cdot)|||_{q,\alpha}$ and $\boldsymbol{L}^{p,\alpha}(T \times (0, 1)) \supset \boldsymbol{L}^{q,\alpha}(T \times (0, 1))$ for $1 \leqq p \leqq q$.

For $x(t, u) \in \boldsymbol{L}^{1,\alpha}(T \times (0, 1))$, the Fourier coefficients are

$$c_n(u) = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(t, u) e^{-int} dt , \qquad n = 0, \pm 1, \cdots \tag{25}$$

and we see $c_n(u) \in L^\alpha(0, 1)$, for the Minkowski inequality gives, for $1 \leqq \alpha$,

$$\|c_n(\cdot)\|_\alpha \leqq \left\{\int_0^1 \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} |x(t, u)| dt\right]^\alpha du\right\}^{1/\alpha} \leqq \frac{1}{2\pi}\int_{-\pi}^{\pi} \|x(t, \cdot)\|_\alpha dt = |||x(\cdot, \cdot)|||_{1,\alpha} < \infty .$$

Let $1 \leqq \alpha < \infty$, $1 \leqq p \leqq 2$. Since $L^\alpha(0, 1)$ is weakly complete ([4] p. 289, 290), we see, from Theorem 2, that for $x(t, u) \in \boldsymbol{L}^{p,\alpha}(T \times (0, 1))$, $\sum_{n=-\infty}^\infty c_n(u)$ is unconditionally convergent in $L^\alpha(0, 1)$, if

$$\sum_{n=1}^\infty n^{-1/p'} M_{p,\alpha}^{(r)}(1/n) < \infty \tag{26}$$

is satisfied for some positive integer r.

Now applying A with $\mathrm{Re}\, c_{\pm n}(u)$, $\mathrm{Im}\, c_{\pm n}(u)$, for $f_n(u)$ and combining results obtained, we have

THEOREM 6. *Let* $1 \leqq \alpha < \infty$, $1 \leqq p \leqq 2$ *and* $x(t, u) \in L^{p,\alpha}(T \times (0, 1))$. *If* (26) *is satisfied for some positive integer* r, *then*

(i) *for* $1 \leqq \alpha \leqq 2$,

$$\sum_{n=-\infty}^{\infty} \|c_n(\cdot)\|_\alpha^2 < \infty , \tag{27}$$

(ii) *for* $2 \leqq \alpha < \infty$,

$$\sum_{n=-\infty}^{\infty} \|c_n(\cdot)\|_\alpha^\alpha < \infty . \tag{28}$$

This result will be partly generalized in the corollary of Theorem 7 later on.

§6. Series of Fourier coefficients of $L^\alpha(S)$ valued functions.

We shall now consider the more general case of $L^\alpha(S)$ valued functions where S is a general measure space, a σ-field Σ of measurable sets and (positive) measure μ being given.

$L^\alpha(S)$ denotes the class of functions $f(u)$ defined on S such that $\int_S |f(u)|^\alpha d\mu < \infty$. Let $x(t)$ be 2π-periodic $L^\alpha(S)$ valued functions. As before we write $x(t)$ by $x(t, u)$ so that for each t, $\int_S |x(t, u)|^\alpha d\mu < \infty$. Throughout in what follows we assume that S is σ-finite and $x(t, u)$, $t \in T$, $u \in S$ is measurable on the product space $T \times S$, product measure being introduced.

We write

$$\|x(t, \cdot)\|_\alpha = \left[\int_S |x(t, u)|^\alpha d\mu\right]^{1/\alpha} \tag{29}$$

$$|||x(\cdot, \cdot)|||_{p,\alpha} = \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \|x(t, \cdot)\|_\alpha^p dt\right]^{1/p} , \tag{30}$$

for $1 \leqq \alpha, p$. (30) is a generalization of (23). The same notations (23) and (30) will give no confusion. The class of $x(t, u)$ with $|||x(\cdot, \cdot)|||_{p,\alpha} < \infty$ is denoted by $L^{p,\alpha}(T \times S)$. Since we assume S σ-finite and $x(t, u)$ is integrable on $(T \times S)$, Fubini-Tonelli theorem is true and the Minkowski inequality and the interchange of the order of integrals are allowed to apply in what follows. Write for any positive integer r, without any confusion with (24),

$$M_{p,\alpha}^{(r)}(\delta)=\sup_{|h|\leqq\delta}|||\Delta_h^{(r)}x(\cdot,\cdot)|||_{p,\alpha}\ . \tag{31}$$

Consider the Fourier coefficients of $x(t, u)$ on T:

$$c_n(u)=\frac{1}{2\pi}\int_{-\pi}^{\pi}x(t, u)e^{-int}dt\ ,\qquad n=0,\ \pm1,\cdots .$$

The author [7] [8] discussed the convergence of $\sum|c_n(u)|$ in the case in which S is a probability space. The convergence of $\sum\|c_n(\cdot)\|^{\beta}$ for some $\beta>0$ was also studied implicitly in there. Let $1/p+1/p'=1$.

LEMMA 3. *If $1\leqq p\leqq\alpha\leqq p'$, $1\leqq\alpha<\infty$ and $x(t, u)\in L^{p,\alpha}(T\times S)$, then*

$$\left[\sum_{n=-\infty}^{\infty}\|c_n(\cdot)\|_{\alpha}^{p'}\right]^{1/p'}\leqq C|||x(\cdot,\cdot)|||_{p,\alpha} \tag{32}$$

where C is a constant depending only on α and p.

This is an immediate consequence of the Minkowski inequality and the ordinary Hausdorff-Young inequality, because, noting $1\leqq p\leqq2$, $\alpha\leqq p'$, we see that

$$\begin{aligned}\left[\sum_{n=-\infty}^{\infty}\|c_n(\cdot)\|_{\alpha}^{p'}\right]^{1/p'}&=\left\{\sum_{n=-\infty}^{\infty}\left[\int_S|c_n(u)|^{\alpha}d\mu\right]^{p'/\alpha}\right\}^{1/p'}\\ &\leqq\left\{\int_S\left[\sum_{n=-\infty}^{\infty}|c_n(u)|^{p'}\right]^{\alpha/p'}d\mu\right\}^{1/\alpha}\\ &\leqq C\left\{\int_S\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}|x(t, u)|^{p}dt\right]^{\alpha/p}d\mu\right\}^{1/\alpha}\\ &\leqq C\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\int_S|x(t, u)|^{\alpha}d\mu\right]^{p/\alpha}dt\right\}^{1/p}\\ &=C|||x(\cdot,\cdot)|||_{p,\alpha}\ ,\end{aligned}$$

C being a constant depending only on α and p.

Using Lemma 3, we also have

LEMMA 4. *Let $x(t, u)\in L^{p,\alpha}(T\times S)$.*
(i) *If $1\leqq p\leqq2$, $1\leqq p<\alpha<\infty$, $1/\alpha+1/\alpha'=1$, then*

$$\left[\sum_{|k|\geqq n}\|c_k(\cdot)\|_{\alpha}^{p'}\right]^{1/p'}\leqq CM_{p,\alpha}^{(r)}(1/n)\ , \tag{33}$$

(ii) *if $1\leqq\alpha\leqq p\leqq2$, then*

$$\left[\sum_{|k|\geqq n}\|c_k(\cdot)\|_{\alpha}^{\alpha'}\right]^{1/\alpha'}\leqq CM_{p,\alpha}^{(r)}(1/n)\ , \tag{34}$$

where r is any positive integer and C's are constants depending only on p, α and r.

The proof of Lemma 4 is carried out just in a similar way as in the proof of Lemma 1 with $\|c_k(\cdot)\|_\alpha$ in place of $|x^*c_k|$ and as a matter of fact, the lemma was shown in Theorem 5.2 (5.4) of [8] (in which the left hand side of (5.4) should be read as $[\sum_{|k|\geq n}\|C_k(\omega)\|_r^{\theta'}]^{1/\theta'}$ and a constant C is missing on the top of the right hand side of (5.5)), when S is the probability space. The proof does not require any change.

THEOREM 7. *Let* $1\leqq p\leqq 2$, $1\leqq\alpha<\infty$, $0<\beta$ *and suppose* $x(t, u)\in L^{p,\alpha}(T\times S)$.

(i) *If* $1\leqq\alpha\leqq p\leqq 2$, $0<\beta\leqq\alpha'$ *and*

$$\sum_{n=1}^{\infty} n^{-\beta/\alpha'}[M_{p,\alpha}^{(r)}(1/n)]^\beta<\infty \tag{35}$$

is satisfied for some positive integer r, then

$$\sum_{n=-\infty}^{\infty}\|c_n(\cdot)\|_\alpha^\beta<\infty\ , \tag{36}$$

(ii) *if* $1\leqq p\leqq\alpha<\infty$, $1\leqq p\leqq 2$, $0<\beta\leqq p'$ *and*

$$\sum_{n=1}^{\infty} n^{-\beta/p'}[M_{p,\alpha}^{(r)}(1/n)]^\beta<\infty \tag{37}$$

is satisfied for some positive integer r, then (36) *holds.*

The proof is carried out by the known argument in the theory of absolute convergence of ordinary Fourier series ([11], Chap VI, 3. Also see [8]). Just for completeness, we give the proof.

PROOF. In order to prove (i) and (ii) simultaneously, we write $\theta=\min(p, \alpha)$ and $1/\theta+1/\theta'=1$. Note that $\theta\geqq 1$. $1\leqq p\leqq 2$ and $0<\beta\leqq\theta'$ in both cases. (35) and (37) reduce to

$$\sum_{n=1}^{\infty} n^{-\beta/p'}[M_{p,\alpha}^{(r)}(1/n)]^\beta<\infty\ . \tag{38}$$

Now using the Hölder inequality, we have

$$\sum_{n=3}^{\infty}\|c_n(\cdot)\|_\alpha^\beta=\sum_{n=1}^{\infty}\sum_{j=2^n+1}^{2^{n+1}}\|c_n(\cdot)\|_\alpha^\beta$$
$$\leqq\sum_{n=1}^{\infty}\left[\sum_{j=2^n+1}^{2^{n+1}}\|c_j(\cdot)\|_\alpha^{\theta'}\right]^{\beta/\theta'}2^{n(1-\beta/\theta')}$$

$$\leqq \sum_{n=1}^{\infty}\left[\sum_{j=2^n+1}^{2^{n+1}} \|c_j(\cdot)\|_\alpha^{\theta'}\right]^{\beta/\theta'} 2^{n(1-\beta/\theta')}$$

$$\leqq \sum_{n=1}^{\infty} 2^{-n} \sum_{m=2^{n-1}+1}^{2^n} m^{1-\beta/\theta'}\left[\sum_{j=m}^{\infty} \|c_j(\cdot)\|_\alpha^{\theta'}\right]^{\beta/\theta'}$$

$$\leqq \sum_{n=1}^{\infty} \sum_{m=2^{n-1}+1}^{2^n} m^{-\beta/\theta'}\left[\sum_{j=m}^{\infty} \|c_j(\cdot)\|_\alpha^{\theta'}\right]^{\beta/\theta'}$$

which is, because of Lemma 4, not greater than

$$C\sum_{n=1}^{\infty} \sum_{m=2^{n-1}+1}^{2^n} m^{-\beta/\theta'}[M_{p,\alpha}^{(r)}(1/m)]^\beta$$

$$=C\sum_{n=1}^{\infty} m^{-\beta/\theta'}[M_{p,\alpha}^{(r)}(1/m)]^\beta \,.$$

C's are constants depending only on p, α and r.

The same thing is true for $\sum_{n=-\infty}^{-3} \|c_n(\cdot)\|_\alpha^\beta$ and the proof is complete.

§ 7. Some remarks.

First we mention that, for $x(t, u) \in L^{1,\alpha}(T\times S)$, $1\leqq \alpha<\infty$,

$$\|c_n(\cdot)\|_\alpha \to 0 \,, \qquad n\to \pm\infty \,. \tag{39}$$

By the ordinary Riemann-Lebesgue lemma, $c_n(u)\to 0$ as $n\to\pm\infty$, and

$$|c_n(u)|\leqq K(u)$$

where $K(u)=\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}|x(t, u)|dt$ and $\displaystyle\int_S K^\alpha(u)d\mu$ is, by the Minkowski inequality, not greater than

$$\int_S\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}|x(t, u)|dt\right]^\alpha d\mu\leqq\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\int_S|x(t, u)|^\alpha d\mu\right]^{1/\alpha}dt\right\}^\alpha$$

$$\equiv |||x(\cdot,\cdot)|||_\alpha^\alpha<\infty \,.$$

Hence the Lebesgue convergence theorem applies to have (39).

Now we give some remarks in connection with Theorem 7.

REMARK 1. If $1\leqq p\leqq\alpha$, $1\leqq p\leqq 2$, then taking $\beta=p'$ in Theorem 7 (ii) we have the covergence of $\sum_{n=-\infty}^{\infty}\|c_n(\cdot)\|_\alpha^{p'}$ under the condition (37) with p' for β. However from Lemma 3 if $1\leqq p\leqq\alpha\leqq p'$, $\sum_{n=-\infty}^{\infty}\|c_n(\cdot)\|_\alpha^{p'}<\infty$ without the condition (37).

In this case $\sum_{n=-\infty}^{\infty}\|c_n(\cdot)\|_\alpha^\beta<\infty$ for $\beta\geqq p'$ since $\|c_n(\cdot)\|_\alpha\to 0$. Note for $\beta>p'$, (37) is automatically satisfied since as we easily see $M_{p,\alpha}^{(r)}(1/n)=O(1)$.

Theorem 7 says that if $0<\beta<p'$, still $\sum_{n=-\infty}^{\infty}\|c_n(\cdot)\|_\alpha^\beta<\infty$ as far as

(37) is satisfied for some positive integer r.

REMARK 2. We shall give some other special cases of Theorem 7 particularly in relationship to Theorem 6.

We first take $\beta=1$. Then Theorem 7 (ii) becomes:

Let $1\leqq p\leqq 2$, $1\leqq p\leqq \alpha<\infty$. *Then the condition that*

$$\sum_{n=1}^{\infty} n^{-1/p'} M_{p,\alpha}^{(r)}(1/n)<\infty \tag{40}$$

for some positive integer r, *implies*

$$\sum_{n=-\infty}^{\infty} \|c_n(\cdot)\|_\alpha<\infty\ . \tag{41}$$

When $S=(0, 1)$, Σ is the σ-field of Lebesgue measurable sets and μ is the Lebesgue measure, the condition (40) is no more than (26) and the condition (41) is much better than (27) in Theorem 6, since $\|c_n(\cdot)\|_\alpha\to 0$, $n\to\pm\infty$, so that even if S is the above special case (0, 1), (41) holds true whether α is less than 2 or not, as far as $1\leqq p\leqq\alpha$. Hence this partly generalizes Theorem 6 (i).

REMARK 3. Taking $\beta=2$ or α, we have the following special case of Theorem 7 which we state as a corollary.

COROLLARY (of Theorem 7). *Let* $1\leqq p\leqq 2$, $1\leqq\alpha<\infty$ *and* $x(t, u)\in L^{p,\alpha}(T\times S)$.

(i) $1\leqq p\leqq\alpha<\infty$ *and*

$$\sum_{n=1}^{\infty} n^{-2/p'}[M_{p,\alpha}^{(r)}(1/n)]^2<\infty \tag{42}$$

is satisfied for some positive integer r, *then*

$$\sum_{n=-\infty}^{\infty} \|c_n(\cdot)\|_\alpha^2<\infty\ . \tag{43}$$

(ii) $1\leqq p\leqq\alpha\leqq p'$ *and*

$$\sum_{n=1}^{\infty} n^{-\alpha/p'}[M_{p,\alpha}^{(r)}(1/n)]^2<\infty \tag{44}$$

is satisfied for some positive integer r, *then* (43) *holds.*

(iii) $1\leqq\alpha\leqq p\leqq 2$ *and*

$$\sum_{n=1}^{\infty} n^{-2/\alpha'}[M_{p,\alpha}^{(r)}(1/n)]^2<\infty \tag{45}$$

is satisfied for some positive integer r, then

$$\sum_{n=-\infty}^{\infty} \|c_n(\cdot)\|_\alpha^\alpha < \infty . \tag{46}$$

(iv) $1 \leqq \alpha \leqq p \leqq 2$ *and*

$$\sum_{n=1}^{\infty} n^{-\alpha/\alpha'} [M_{p,\alpha}^{(r)}(1/n)]^\alpha < \infty \tag{47}$$

is satisfied for some positive integer r, then (46) *holds.*

In order to get (27) in Theorem 6, the condition (26) and $1 \leqq \alpha \leqq 2$, were needed, while in the above Corollary (i) with $S=(0, 1)$ and Lebesgue measure, the condition (42) is much weaker and the conclusion (43) is just the same as (27). Further $1 \leqq p \leqq \alpha$ does not exclude $\alpha > 2$.

In Corollary (ii), the condition (44) with $S=(0, 1)$ is weaker than (26) but besides $1 \leqq p \leqq \alpha$, the restriction $\alpha \leqq p'$ was made, though this does not force $\alpha \leqq 2$.

§ 8. Almost everywhere convergence of $\sum |c_n(u)|$.

We shall consider the almost everywhere convergence of $\sum_{n=-\infty}^{\infty} |c_n(u)|$ in S, where $c_n(u)$ is the Fourier coefficients of $x(t, u) \in L^{p,\alpha}(T \times S)$. The author studied the same problem where S is a probability space and applied the results to the discussion of sample continuity of a stochastic process.

Since we have been assuming S σ-finite, there are $\{S_j, j=1, 2, \cdots\}$ such that $S=\bigcup_{j=1}^\infty S_j$, $\mu(S_j) < \infty$. Therefore in order to have $\sum_{n=-\infty}^{\infty} |c_n(u)| < \infty$ for almost all $u \in S$, it is sufficient to show it for almost all $u \in S_j$ for each j, and hence it is sufficient to prove

$$\sum_{n=-\infty}^{\infty} \int_{S_j} |c_n(u)| d\mu < \infty . \tag{48}$$

Now for $\alpha \geqq 1$

$$\int_{S_j} |c_n(u)| d\mu \leqq [\mu(S_j)]^{1/\alpha'} \left[\int_{S_j} |c_n(u)|^\alpha d\mu \right]^{1/\alpha} \leqq [\mu(S_j)]^{1/\alpha'} \|c_n(\cdot)\|_\alpha .$$

Thus the sufficient condition for $\sum_{n=-\infty}^{\infty} \|c_n(\cdot)\|_\alpha < \infty$ ensures (48), so that Theorem 7 with $\beta=1$ gives us the following theorem.

THEOREM 8. *Let* $x(t, u) \in L^{p,\alpha}(T \times S)$. *Either if* $1 \leqq \alpha \leqq p \leqq 2$ *and*

$$\sum_{n=1}^{\infty} n^{-1/\alpha'} M_{p,\alpha}^{(r)}(1/n) < \infty \ , \tag{49}$$

for some positive integer r, or if $1 \leqq p \leqq \alpha \leqq \infty$, $1 \leqq p \leqq 2$ and

$$\sum_{n=1}^{\infty} n^{-1/p'} M_{p,\alpha}^{(r)}(1/n) < \infty \ , \tag{50}$$

for some positive integer r, then $\sum_{n=-\infty}^{\infty} |c_n(u)| < \infty$ almost everywhere in S.

References

[1] S. BANACH, *Théorie des Opérations Linéaires*, Monografje Mat., Warsaw (1932).

[2] S. BOCHNER, Integration von Funktionen deren Werte die Elemente eines Vektorraums sind, Fund. Math., **20** (1933), 262–276.

[3] N. DUNFORD, Uniformity in linear spaces, Trans. Amer. Math. Soc., **44** (1938), 305–356.

[4] N. DUNFORD and J. T. SCHWARZ, *Linear Operators, Part I*, Intersci. Publ. Inc. (1967).

[5] E. HILL and R. S. PHILLIPS, *Functional Analysis and Semi-Groups*, Amer. Math. Soc. Publ., **31** (1974).

[6] M. A. KANDIL, Convergence of Fourier coefficients series for vector valued functions, Bull. Cal. Math. Soc., **75** (1983), 77–81.

[7] T. KAWATA, Absolute convergence of Fourier series of periodic stochastic processes and its applications, Tohoku Math. J., **35** (1983), 459–475.

[8] ———, Lipschitz classes of periodic stochastic processes and Fourier series, Tokyo J. Math., **8** (1985), 277–301.

[9] W. ORLICZ, Beiträge zur Theorie der Orthogonalentwicklungen. II, Studia Math., **1** (1929), 241–255.

[10] ———, Uber unbedingte Konvergenz in Funktionenräumen (I), (II), Studia Math., **4** (1933), 33–37, 41–47.

[11] A. ZYGMUND, *Trigonometric Series, I*, Cambridge Univ. Press (1968).

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