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Classifying 3-Dimensional Lens Spaces by Eta-Invariants

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Dedicated to Professor Akio Hattori on his sixtieth birthday

Let C^2 be the space of pairs (z_0, z_1) of complex numbers with the standard flat Kähler metric. Let p be a positive integer and q_0, q_1 be integers relatively prime to p. Put $z=\exp\frac{2\pi\sqrt{-1}}{p}$ and define an isometry g of C^2 by

$$g: (z_0, z_1) \longrightarrow (z^{q_0}z_0, z^{q_1}z_1) .$$

Then g generates a cyclic subgroup $G=\{g^k\}_{k=0,1,\cdots,p-1}$ of the unitary group $U(2)$ and the elements g^k act on the unit sphere

$$S^3=\{(z_0, z_1)\in C^2; z_0\bar{z}_0+z_1\bar{z}_1=1\}$$

without fixed point. The differentiable manifold S^3/G has a unique riemannian metric so that the covering projection $\varphi: S^3\to S^3/G$ gives a local isometry of S^3 onto S^3/G. This riemannian manifold S^3/G is called a lens space and is denoted by $L(p; q_0, q_1)$.

The following theorem on lens spaces is well known. (See Cohen [3].)

THEOREM. *The following assertions are equivalent*:

(1) $L(p; q_0, q_1)$ *is isometric to* $L(p; \bar{q}_0, \bar{q}_1)$.

(2) $L(p; q_0, q_1)$ *is diffeomorphic to* $L(p; \bar{q}_0, \bar{q}_1)$.

(3) $L(p; q_0, q_1)$ *is homeomorphic to* $L(p; \bar{q}_0, \bar{q}_1)$.

(4) *There are integers* l *and* $e_i\in\{-1, 1\}$ $(i=0, 1)$ *such that* (q_0, q_1) *is a permutation of* $(e_0l\bar{q}_0, e_1l\bar{q}_1)$.

Since g^k is also a generator of G, the lens space $L(p; kq_0, kq_1)$ is identical to $L(p; q_0, q_1)$. Hence, choosing a suitable generator for G, we

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may assume that $q_0=1$ and we denote $L(p; 1, q)$ simply by $L(p; q)$. The above theorem is now restated as

COROLLARY. *Two 3-dimensional lens spaces $L(p; q)$ and $L(p; \bar{q})$ are isometric if and only if*

$$q \equiv \pm \bar{q} \text{ modulo } p \qquad \text{or} \qquad q\bar{q} \equiv \pm 1 \text{ modulo } p\,.$$

Note that the positive (resp. negative) signs on the above equations correspond to the orientation preserving (resp. reversing) isometry. Also note that we have only to consider the case $q \leqq \left[\frac{p}{2}\right]$ by this corollary.

A. Ikeda and Y. Yamamoto [5] studied the eigenvalues and their multiplicities, *i.e.*, the spectrum of the Laplace operator Δ acting on the space of smooth functions on the lens space and showed that if two 3-dimensional lens spaces are isospectral in the sense of Δ then they are isometric. That is to say, the spectrum of Δ determines the geometric structure of 3-dimensional lens space. In the preceding papers ([6], [7] and [8]) we studied the spectrum of the operator $A=\pm(*d-d*)$, *i.e.*, square root of Δ, acting on the space $\Lambda^{ev}(S^3)$ of even forms on S^3 and the eta-invariant

$$\eta(p; q) = -\frac{1}{p}\sum_{i=1}^{p-1} \cot\frac{i}{p}\pi \cot\frac{iq}{p}\pi = \frac{1}{3p}(p-1)(2pq-3p-q+3) - \frac{2}{p}\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2$$

for the 3-dimensional lens space $L(p; q)$ which is a spectral invariant of the operator A. Since the isospectrality in the sense of A implies that of Δ, the aim at that time was to study to what extent the eta-invariants explain the isometric structure of 3-dimensional lens spaces. We showed that if $\eta(p; q)=\eta(p; \bar{q})$ then the equation

$$(q-\bar{q})(q\bar{q}-1) \equiv 0 \qquad \text{modulo } p$$

holds so that we get $L(p; q) \simeq L(p; \bar{q})$ ($\simeq$ denotes the orientation preserving isometry) when we restrict p to be of the form kp', where p' is a prime and $k=1$ or 2 or 3. When p is a general composite number, we first evaluated the Dedekind sum $\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2$ (as stated in Theorem 1.1) taking into account the length of the sequence of remainders in euclidean algorithm for calculating the greatest common divisor of p and q. Using this evaluation, we obtained Theorem 1.6 which states that the eta-invariant is an "almost complete" (orientation preserving) isometric invariant. That is, some exceptional cases were left for classifying lens spaces by these eta-invariants.

In this paper, we study the eta-invariant

$$\eta_\alpha(p; q) = -\frac{1}{p}\sum_{i=1}^{p-1} \cot\frac{i}{p}\pi \cot\frac{iq}{p}\pi \cos\frac{2\alpha i}{p}\pi ,$$

which corresponds to the irreducible unitary representation α of $\pi_1(L(p; q)) \cong \boldsymbol{Z}/p\boldsymbol{Z}$. First, we represent this cotangent sum by a generalized Dedekind sum $\sum_{k=0}^{q-1}\left[\frac{kp+p-\alpha}{q}\right]^2$ in Lemma 2.2 and then we evaluate this generalized Dedekind sum as a polynomial of p, q, $s=p-\alpha$ and some recursively defined terms r_i and s_i. The result is

$$\sum_{k=0}^{q-1}\left[\frac{kp+s}{q}\right]^2 = \frac{1}{3}(p-1)^2(q-1) - \frac{1}{6}p(p-q) - \frac{1}{6} + \frac{p}{12}(1-3\cdot(-1)^n) + s(p-1) + \rho_s$$
$$+\frac{p}{6}\sum_{i=0}^{n}(-1)^i\left[\frac{r_{i-1}}{r_i}\right] + \frac{p}{6}\sum_{i=0}^{n}\frac{(-1)^i}{r_{i-1}r_i} + p\sum_{i=0}^{n}(-1)^i\left(\frac{s_i^2}{r_{i-1}r_i} + \left[-\frac{s_i}{r_i}\right]\right),$$

which is stated in Theorem 3.3.

Using the exact value of $\eta_\alpha(p; q)$, we obtain the following theorem which is an improvement of Donnelly's proposition for 3-dimensional case (cf. Donnelly [4], Prop. 4.3).

THEOREM 4.3. *Let q and $\bar{q}$ $\left(\leqq\left[\frac{p}{2}\right]\right)$ be positive integers relatively prime to p. Then two lens spaces $L(p; q)$ and $L(p; \bar{q})$ are (orientation preservingly) isometric to each other, i.e., the equation $q\bar{q}\equiv 1$ modulo p holds if and only if their eta-invariants satisfy the equations:*

(1) $$\eta(p; q) = \eta(p; \bar{q}) \quad \textit{and} \quad \eta_1(p; q) = \eta_{\bar{q}}(p; \bar{q})$$

or

(2) $$\eta(p; q) = \eta(p; \bar{q}) \quad \textit{and} \quad \eta_1(p; q) = \eta_{q^*}(p; \bar{q}) ,$$

where q^* is a positive integer less than $p/2$ defined by the equation $qq^*\equiv 1$ modulo p.

§1. Eta-invariants for $L(p; q)$.

In this section, we rearrange the result and the argument of the preceding paper [8]. However, notations are slightly different.

Let p be an integer greater than 2, and q $(<p)$ be a positive integer relatively prime to p and $[x]$ denote the greatest integer less than or equal to the real number x. The eta-invariant for $L(p; q)$ is given by the explicit formula

$$\eta(p;q)=-\frac{1}{p}\sum_{i=1}^{p-1}\cot\frac{i}{p}\pi\cot\frac{iq}{p}\pi\ ,$$

and this cotangent sum is represented as

$$\eta(p;q)=\frac{1}{3p}(p-1)(2pq-3p-q+3)-\frac{2}{p}\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2$$

(see, for example, Donnelly [4] Prop. 4.1 and Katase [8] Th. 4). Hence we know that $3p\eta(p;q)$ is an integer and we are able to treat in integral category for given integers p and q. By definition, the eta-invariant $\eta(p;q)$ is an orientation preservingly isometric invariant. In fact, if $q\bar{q}\equiv 1$ modulo p, for example, then we get

$$\eta(p;q)=-\frac{1}{p}\sum_{j=1}^{p-1}\cot\frac{j\bar{q}}{p}\pi\cot\frac{jq\bar{q}}{p}\pi=-\frac{1}{p}\sum_{j=1}^{p-1}\cot\frac{j\bar{q}}{p}\pi\cot\frac{j}{p}\pi=\eta(p;\bar{q})\ .$$

On the other hand, to study whether the eta-invariant $\eta(p;q)$ is a complete invariant for the isometric class of 3-dimensional lens spaces or not, we evaluate the Dedekind sum $\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2$ by p, q, and some recursively defined terms. Let $r_{-1}=p$, $r_0=q$, and r_i $(i=1, 2, \cdots)$ be positive integers defined by

$$r_i=r_{i-2}-\left[\frac{r_{i-2}}{r_{i-1}}\right]r_{i-1}\ .$$

Since p and q are relatively prime, $r_n=1$ for some integer n $(\geqq 1)$. Also let $a_{-1}=1$, $a_0=\left[\frac{p}{q}\right]$, and a_j $(j=1, 2, \cdots, n-1)$ be positive integers defined inductively by the relation

$$p=a_jr_j+a_{j-1}r_{j+1}\ .$$

Since the integer a_{n-1} satisfies the equation

$$\begin{aligned}\frac{(-1)^n a_{n-1}}{p}&=\sum_{i=0}^{n}\frac{(-1)^i}{r_{i-1}r_i}\\&=\frac{1}{pq}+\sum_{i=1}^{n}\frac{(-1)^i}{r_{i-1}r_i}\\&=\frac{1}{pq}+\frac{(-1)^n a_{n-1}^{(1)}}{q}\end{aligned}$$

for some positive integer $a_{n-1}^{(1)}$, we get the equation $qa_{n-1}\equiv(-1)^n$ modulo p. Note that the integer a_{n-1} $(1\leqq a_{n-1}<p/2)$ is uniquely obtained by this

property for given p and q.

The following theorem is a result from the reciprocity formula for Dedekind sums.

THEOREM 1.1 ([8] Th. 3). $\sum_{k=1}^{q-1}\left[\frac{kq}{q}\right]^2$ *is evaluated as a polynomial of fewer terms*:

$$\sum_{k=1}^{q-1}\left[\frac{kp}{q}\right]^2=\frac{1}{3}(p-1)^2(q-1)-\frac{p}{6}(p-q)+\frac{p}{12}(1-3\cdot(-1)^n)-\frac{1}{6}$$
$$+\frac{p}{6}\sum_{i=0}^{n}(-1)^i\left[\frac{r_{i-1}}{r_i}\right]+\frac{1}{6}(-1)^n a_{n-1}\,.$$

COROLLARY 1.2.

$$3p\eta(p;q)=\frac{3}{2}(1+(-1)^n)p-q-p\sum_{i=0}^{n}(-1)^i\left[\frac{r_{i-1}}{r_i}\right]-(-1)^n a_{n-1}\,.$$

Hence, if we assume $\eta(p;q)=\eta(p;\bar{q})$, then

$$3p\eta(p;q)-3p\eta(p;\bar{q})=\bar{q}-q+(-1)^{\bar{n}}\bar{a}_{\bar{n}-1}-(-1)^n a_{n-1}$$
$$+p\left(\frac{3}{2}\{(-1)^{\bar{n}-1}-1-((-1)^{n-1}-1)\}+\sum_{i=0}^{\bar{n}}(-1)^i\left[\frac{\bar{r}_{i-1}}{\bar{r}_i}\right]-\sum_{j=0}^{n}(-1)^j\left[\frac{r_{j-1}}{r_j}\right]\right)=0$$

and we get the following simultaneous equations:

$$\bar{q}-q+(-1)^{\bar{n}}\bar{a}_{\bar{n}-1}-(-1)^n a_{n-1}=sp \tag{1.1}$$

and

$$\frac{3}{2}\{(-1)^{\bar{n}-1}-1-((-1)^{n-1}-1)\}+\sum_{i=0}^{\bar{n}}(-1)^i\left[\frac{\bar{r}_{i-1}}{\bar{r}_i}\right]-\sum_{j=0}^{n}(-1)^j\left[\frac{r_{j-1}}{r_j}\right]=-s\,, \tag{1.2}$$

where $s=0$ or ± 1.

Multiplying $q\bar{q}$ to both sides of the equation (1.1), and using the relations $qa_{n-1}\equiv(-1)^n$ modulo p and $\bar{q}\bar{a}_{\bar{n}-1}\equiv(-1)^{\bar{n}}$ modulo p, we obtain the equation

$$(\bar{q}-q)(q\bar{q}-1)\equiv 0 \quad \text{modulo } p\,. \tag{1.3}$$

In general, we can not conclude from this equation that

$$q=\bar{q} \qquad \text{or} \qquad q\bar{q}\equiv 1 \text{ modulo } p\,.$$

However, we obtain the following

THEOREM 1.3 ([8] Th. 5). *Let p be a prime number or $p=kp'$ where p' is a prime number and $k=2$ or 3. Then two lens spaces $L(p;q)$ and $L(p;\bar{q})$ are (orientation preservingly) isometric to each other (i.e., $L(p;q)\simeq L(p;\bar{q})$) if and only if $\eta(p;q)=\eta(p;\bar{q})$.*

REMARK 1.4. This theorem does not hold when $p=5p'$ where p' is a prime number greater than or equal to 5. In fact, we know that $L(65;8)$ is not isometric to $L(65;18)$ although $\eta(65;8)=\eta(65;18)$. However, we can prove that Theorem 1.3 holds if $p=5p'$ ($p'\geqq 7$) and

$$q\equiv\bar{q}\equiv\pm 1 \quad \text{modulo } 5$$

when $\eta(p;q)=\eta(p;\bar{q})$. (In case $p'=5$, there is a counter-example: $L(25;4)$ is not isometric to $L(25;9)$ although $\eta(25;4)=\eta(25;9)$.)

When p is a composite number $p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$, where p_i's are prime and e_i's are non-negative integers, it follows, in general, from the equation (1.3) that

$$q-\bar{q}=la \quad \text{and} \quad q\bar{q}-1=mb$$

for some factors a and b of p and for some integers l and m. Hence we need further study on the exact value of $\eta(p;q)$ or the simultaneous equations (1.1) and (1.2). By analyzing the properties of the integers a_i, we obtain the following lemma concerning the isometric equivalence condition for two lens spaces and the length of the sequence of remainders in the euclidean algorithm for calculating the greatest common divisor.

LEMMA 1.5 ([8] Th. 8).
(1) *If $q+\bar{q}=p$ and $q<p/2$, then $\bar{n}=n+1$.*
(2) *If $q\bar{q}\equiv(-1)^n$ modulo p for q and $\bar{q}<p/2$, then $\bar{n}=n$.*

Note that, taking this lemma and the equation $\eta(p;q)=-\eta(p;p-q)$ into account, we have only to consider the case: $1\leqq q$, $\bar{q}\leqq\left[\frac{p}{2}\right]$. The main result obtained in the preceding paper is the following

THEOREM 1.6 ([8] Th. 12). *Let q and $\bar{q}$ ($q\neq\bar{q}$) be positive integers relatively prime to p satisfying the equations $r_n=\bar{r}_{\bar{n}}=1$ and $\eta(p;q)=\eta(p;\bar{q})$.*

(1) *If $n=\bar{n}$ are even, then, for given q, $L(p;q)\simeq L(p:\bar{q})$ holds for $\bar{q}=a_{n-1}$ only; $L(p;q)\not\simeq L(p;\bar{q})$ for any integer $\bar{q}$ other than a_{n-1}. (If $n=\bar{n}=2$, there exists no such an integer $\bar{q}$ other than a_1 that satisfies the assumption.)*

(2) *If* $n\neq\bar{n}$ *or* $n=\bar{n}$ *are odd, then* $L(p;q)\neq L(p;\bar{q})$ *for any* $\bar{q}$.

§2. Eta-invariants with unitary representations.

For any unitary representation α of $\pi_1(L(p;q))\cong \boldsymbol{Z}/p\boldsymbol{Z}$, the eta-invariant $\eta_\alpha(p;q)$ associated to α is given by the explicit formula

$$\eta_\alpha(p;q)=-\frac{1}{p}\sum_{i=1}^{p-1}\cot\frac{i}{p}\pi\cot\frac{iq}{p}\pi\cos\frac{2i\alpha}{p}\pi$$

(see Donnelly [4], Prop. 4.1). Note that the usual eta-invariant $\eta(p;q)$ corresponds to the trivial representation $\alpha=0$. One of the result concerning the classification of lens spaces by this eta-invariant is the following theorem obtained by H. Donnelly ([4], Prop. 4.3).

THEOREM 2.1 (Donnelly). *Let* p *be a positive integer,* q *and* $\bar{q}$ $\left(1\leqq q,\bar{q}<\frac{p}{2}\right)$ *be integers relatively prime to* p, *and suppose that, for each irreducible unitary representation* α *of* $G=\boldsymbol{Z}/p\boldsymbol{Z}$, *there exists a unitary representation* β ($=l\alpha$ *modulo* p *for some positive integer* l *less than* p) *such that the equation* $\eta_\alpha(p;q)=\eta_\beta(p;\bar{q})$ *holds. Then* $L(p;q)$ *and* $L(p;\bar{q})$ *are* (*orientation preservingly*) *isometric to each other.*

PROOF. Here we give an elementary proof of this theorem which is essentially the same as Donnelly's. Multiplying $\cos\frac{2k\alpha}{p}\pi$ $(1\leqq k\leqq p-1)$ to $\eta_\alpha(p;q)$ and summing them from $\alpha=0$ to $p-1$, we have

$$\begin{aligned}\sum_{\alpha=0}^{p-1}\eta_\alpha(p;q)\cos\frac{2k\alpha}{p}\pi&=-\frac{1}{p}\sum_{i=1}^{p-1}\cot\frac{i}{p}\pi\cot\frac{iq}{p}\pi\Big(\sum_{\alpha=0}^{p-1}\cos\frac{2i\alpha}{p}\pi\cos\frac{2k\alpha}{p}\pi\Big)\\&=-\frac{1}{2p}\sum_{i=1}^{p-1}\cot\frac{i}{p}\pi\cot\frac{iq}{p}\pi\Big(\sum_{\alpha=0}^{p-1}\Big(\cos\frac{2(i+k)\alpha}{p}\pi+\cos\frac{2(i-k)\alpha}{p}\pi\Big)\Big).\end{aligned}$$

Since

$$\sum_{\alpha=0}^{p-1}\cos\frac{2m\alpha}{p}\pi=\begin{cases}p & \text{if } p\mid m\\ 0 & \text{if } p\nmid m,\end{cases}$$

we have

$$\begin{aligned}\sum_{\alpha=0}^{p-1}\eta_\alpha(p;q)\cos\frac{2k\alpha}{p}\pi&=-\frac{1}{2}\Big(\cot\frac{k}{p}\pi\cot\frac{kq}{p}\pi+\cot\frac{p-k}{p}\pi\cot\frac{(p-k)q}{p}\pi\Big)\\&=-\cot\frac{k}{p}\pi\cot\frac{kq}{p}\pi.\end{aligned}$$

On the other hand, we have

$$\sum_{\alpha=0}^{p-1}\eta_\beta(p;\bar q)\cos\frac{2k\alpha}{p}\pi=-\frac{1}{p}\sum_{j=1}^{p-1}\cot\frac{j}{p}\pi\cot\frac{j\bar q}{p}\pi\Big(\sum_{\alpha=0}^{p-1}\cos\frac{2jl\alpha}{p}\pi\cos\frac{2k\alpha}{p}\pi\Big)$$
$$=-\frac{1}{2p}\sum_{j=1}^{p-1}\cot\frac{j}{p}\pi\cot\frac{j\bar q}{p}\pi\Big(\sum_{\alpha=0}^{p-1}\Big(\cos\frac{2(jl+k)\alpha}{p}\pi+\cos\frac{2(jl-k)\alpha}{p}\pi\Big)\Big)$$
$$=-\frac{1}{2}\Big(\cot\frac{j_k}{p}\pi\cot\frac{j_k\bar q}{p}\pi+\cot\frac{j_{p-k}}{p}\pi\cot\frac{j_{p-k}\bar q}{p}\pi\Big)$$
$$=-\cot\frac{j_k}{p}\pi\cot\frac{j_k\bar q}{p}\pi\,,$$

where the integer j_k $(1\leqq j_k\leqq p-1)$ is defined by the equation $j_kl\equiv k$ modulo p.

Thus the G eta-invariants $\cot\frac{k}{p}\pi\cot\frac{kq}{p}\pi$ and $\cot\frac{j_k}{p}\pi\cot\frac{j_k\bar q}{p}\pi$ of the cover S^3 of $L(p;q)$ and $L(p;\bar q)$ coincide and the conclusion follows. (See, Atiyah and Bott [2] or Wall [16], p. 215. Also see Katase [11], Th. 1.1, where this property is proved elementarily). □

As we have proved by using a proposition concerning the G eta-invariant, the assumption in Theorem 2.1 seems to be too strong. To weaken this assumption, we reduce $\eta_\alpha(p;q)$ given by the trigonometric sum to a sum using Gauss symbols.

Put $z=\exp\frac{2\pi\sqrt{-1}}{p}$. Then we get $\sqrt{-1}\cot\frac{k}{p}\pi=\frac{1+z^k}{1-z^k}$ and hence

$$\eta_\alpha(p;q)=\frac{1}{p}\operatorname{Re}\Big(\sum_{i=1}^{p-1}\frac{1+z^i}{1-z^i}\frac{1+z^{qi}}{1-z^{qi}}z^{\alpha i}\Big)$$
$$=\frac{1}{p}\operatorname{Re}\Big(\sum_{i,h,k=1}^{p-1}\Big(1-\frac{2h}{p}\Big)\Big(1-\frac{2k}{p}\Big)z^{(h+qk+\alpha)i}\Big)$$
$$=\frac{1}{p}\operatorname{Re}\Big(\sum_{i=1}^{p-1}\Big(\sum_{\substack{h,k\\ p\mid(h+qk+\alpha)}}\Big(1-\frac{2h}{p}\Big)\Big(1-\frac{2k}{p}\Big)+\sum_{\substack{h,k\\ p\nmid(h+qk+\alpha)}}\Big(1-\frac{2h}{p}\Big)\Big(1-\frac{2k}{p}\Big)z^{(h+qk+\alpha)i}\Big)\Big)$$
$$=\frac{1}{p}\operatorname{Re}\Big(\sum_{i=1}^{p-1}\sum_{\substack{h,k\\ p\mid(h+qk+\alpha)}}\Big(1-\frac{2h}{p}\Big)\Big(1-\frac{2k}{p}\Big)+\sum_{\substack{h,k\\ p\nmid(h+qk+\alpha)}}\Big(1-\frac{2h}{p}\Big)\Big(1-\frac{2k}{p}\Big)\sum_{i=1}^{p-1}z^{(h+qk+\alpha)i}\Big)$$
$$=\frac{1}{p}\operatorname{Re}\Big((p-1)\sum_{\substack{h,k\\ p\mid(h+qk+\alpha)}}\Big(1-\frac{2h}{p}\Big)\Big(1-\frac{2k}{p}\Big)-\sum_{\substack{h,k\\ p\nmid(h+qk+\alpha)}}\Big(1-\frac{2h}{p}\Big)\Big(1-\frac{2k}{p}\Big)\Big)$$

$$= \sum_{\substack{h,k \\ p \mid (h+qk+\alpha)}} \left(1-\frac{2h}{p}\right)\left(1-\frac{2k}{p}\right) - \frac{1}{p}\sum_{h,k}\left(1-\frac{2h}{p}\right)\left(1-\frac{2k}{p}\right)$$
$$= \sum_{\substack{h,k \\ p \mid (h+qk+\alpha)}} \left(1-\frac{2h}{p}\right)\left(1-\frac{2k}{p}\right).$$

Now assume $\alpha \neq 0$ throughout this section. Let $s=p-\alpha$, and ρ_s $(0<\rho_s<p)$ be the integer defined by the equation $q\rho_s \equiv s$ modulo p, and consider the sum

$$\sum_{\substack{h+kq\equiv s \\ h\neq s}} \left(1-\frac{2}{p}h-\frac{2}{p}k+\frac{4}{p^2}hk\right).$$

Since we must exclude the points $(h, k)=(0, \rho_s)$ and $(s, 0)$, we get

$$\sum_{\substack{h+kq\equiv s \\ h\neq s}} 1 = p-2\,, \quad \sum_{\substack{h+kq\equiv s \\ h\neq s}} h = \frac{1}{2}p(p-1)-s\,, \quad \text{and} \quad \sum_{\substack{h+kq\equiv s \\ h\neq s}} k = \frac{1}{2}p(p-1)-\rho_s\,.$$

As for the sum containing hk, consider the lattice points (h, k) satisfying $h+kq=s+np$ $(n=0, 1, \cdots, q)$ and $1\leqq h,\ k\leqq p-1$. The resulting pairs (h, k) for the case $q\leqq s$ (also effective for $q>s$) are as follows.

$$n=0: \qquad (s-q,\ 1),\ (s-2q,\ 2), \cdots\cdots, \left(s-\left[\frac{s}{q}\right]q,\ \left[\frac{s}{q}\right]\right),$$

$$n=1: \qquad \left(p+s-\left(\left[\frac{s}{q}\right]+1\right)q,\ \left[\frac{s}{q}\right]+1\right), \cdots, \left(p+s-\left[\frac{p+s}{q}\right]q,\ \left[\frac{p+s}{q}\right]\right),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$n=q-1: \qquad \cdots\cdots\cdots, \left((q-1)p+s-\left[\frac{(q-1)p+s}{q}\right]q,\ \left[\frac{(q-1)p+s}{q}\right]\right),$$

$$n=q: \qquad \left(qp+s-\left(\left[\frac{(q-1)p+s}{q}\right]+1\right)q,\ \left[\frac{(q-1)p+s}{q}\right]+1\right), \cdots\cdots, (qp+s-(p-1)q,\ p-1)\,.$$

Note that h appears up to the last pair on the row $n=q-1$ if $q+s\geqq p$, because $\left[\frac{(q-1)p+s}{q}\right]=\left[p-\frac{p-s}{q}\right]=p-\left[\frac{p-s}{q}\right]-1$ when $\frac{p-s}{q}$ is not an integer.

If $q+s\geqq p$, for example, we get from the above result that

$$\sum_{\substack{h+kq\equiv s \\ h\neq s}} hk = p\left(1\times\left(\left[\frac{s}{q}\right]+1\right)+\cdots+1\times\left[\frac{p+s}{q}\right]+2\times\left(\left[\frac{p+s}{q}\right]+1\right)\right.$$
$$\left.+\cdots+2\times\left[\frac{2p+s}{q}\right]+\cdots\cdots+(q-1)\times\left(\left[\frac{(q-2)p+s}{q}\right]+1\right)\right.$$

$$+\cdots+(q-1)\times\left[\frac{(q-1)p+s}{q}\right]\Big)-\frac{1}{6}qp(p-1)(2p-1)+\frac{1}{2}sp(p-1)$$

$$=p\sum_{k=1}^{q-1}k\Big(\frac{1}{2}\left[\frac{kp+s}{q}\right]\Big(\left[\frac{kp+s}{q}\right]+1\Big)-\frac{1}{2}\left[\frac{(k-1)p+s}{q}\right]\Big(\left[\frac{(k-1)p+s}{q}\right]+1\Big)\Big)$$

$$-\frac{1}{6}qp(p-1)(2p-1)+\frac{1}{2}sp(p-1)$$

$$=\frac{p}{2}\Big(\sum_{k=1}^{q-1}\Big(k\left[\frac{kp+s}{q}\right]^2-(k-1)\left[\frac{(k-1)p+s}{q}\right]^2-\left[\frac{(k-1)p+s}{q}\right]^2\Big)$$

$$+\sum_{k=1}^{q-1}\Big(k\left[\frac{kp+s}{q}\right]-(k-1)\left[\frac{(k-1)p+s}{q}\right]-\left[\frac{(k-1)p+s}{q}\right]\Big)\Big)$$

$$-\frac{1}{6}qp(p-1)(2p-1)+\frac{1}{2}sp(p-1)$$

$$=\frac{p}{2}\Big((q-1)\left[\frac{(q-1)p+s}{q}\right]^2-\sum_{k=1}^{q-1}\left[\frac{(k-1)p+s}{q}\right]^2$$

$$+(q-1)\left[\frac{(q-1)p+s}{q}\right]-\sum_{k=1}^{q-1}\left[\frac{(k-1)p+s}{q}\right]\Big)$$

$$-\frac{1}{6}qp(p-1)(2p-1)+\frac{1}{2}sp(p-1)$$

$$=\frac{p}{2}\Big((p-1)^2(q-1)-\sum_{k=0}^{q-1}\left[\frac{kp+s}{q}\right]^2+\left[\frac{(q-1)p+s}{q}\right]^2$$

$$+(p-1)(q-1)-\sum_{k=0}^{q-1}\left[\frac{kp+s}{q}\right]+\left[\frac{(q-1)p+s}{q}\right]\Big)$$

$$-\frac{1}{6}qp(p-1)(2p-1)+\frac{1}{2}sp(p-1)$$

$$=\frac{1}{12}p(p-1)(2pq-q+3)+\frac{1}{2}sp(p-2)-\frac{p}{2}\sum_{k=0}^{q-1}\left[\frac{kp+s}{q}\right]^2.$$

Here we have used the equation $\sum_{k=0}^{q-1}\left[\frac{kp+s}{q}\right]=\frac{1}{2}(p-1)(q-1)+s$. Note that the above equation holds whether $q+s\geqq p$ or not.

Thus we have obtained the following

LEMMA 2.2. *The eta-invariant associated to the unitary representation* α ($\neq 0$) *is reduced as follows*:

$$\eta_\alpha(p;q)=\frac{1}{3p}(2p^2q-3p^2-3pq+3p+q-3)+\frac{2}{p}(p-\alpha)(p-1)$$

$$+\frac{2}{p}\rho_{p-\alpha}-\frac{2}{p}\sum_{i=0}^{q-1}\left[\frac{ip+p-\alpha}{q}\right]^2.$$

Note that, considering the procedure of the case $\alpha=0$ (*i.e.*, $s=p$), we must add 1 on the right-hand side of the above equation and the resulting invariant $\eta_0(p; q)$ coincides with $\eta(p; q)$. Also note that $3p\eta_\alpha(p; q)$ is an integer for any α $(0\leqq\alpha<p)$ and we may argue in integral category.

§ 3. Generalized Dedekind sums.

At first, we show a reciprocity formula for generalized Dedekind sums (cf. Knuth [13]).

LEMMA 3.1. *Let p and q be relatively prime positive integers and s be a positive integer less than p. Then we have the reciprocity relations:*

$$(3.1)\quad p\sum_{h=0}^{p-1}\left[\frac{hq-s}{p}\right]^2+q\sum_{k=0}^{q-1}\left[\frac{kp+s}{q}\right]^2=\frac{1}{6}(p-1)(2p-1)(q-1)(2q-1)$$
$$+s(p-q-1+s)+2q\rho_s$$

and

$$(3.2)\quad p\sum_{h=0}^{p-1}\left[\frac{hq+s}{p}\right]^2+q\sum_{k=0}^{q-1}\left[\frac{kp-s}{q}\right]^2=\frac{1}{6}(p-1)(2p-1)(q-1)(2q-1)$$
$$-s(p-q-1-s)-2pq+2p\rho_{-s}-2pq\left[-\frac{s}{q}\right].$$

PROOF. Since the numbers $kp+s-\left[\frac{kp+s}{q}\right]q$ for $k=0, 1, \cdots, q-1$ are simply the numbers $0, 1, \cdots, q-1$ in some order, we have

$$\sum_{k=0}^{q-1}\left(kp+s-\left[\frac{kp+s}{q}\right]q\right)^2=\frac{1}{6}q(q-1)(2q-1).$$

The left-hand side of this equation is equal to

$$\frac{1}{6}p^2q(q-1)(2q-1)+spq(q-1)+s^2q-2pq\sum_{k=0}^{q-1}k\left[\frac{kp+s}{q}\right]$$
$$-2sq\sum_{k=0}^{q-1}\left[\frac{kp+s}{q}\right]+q^2\sum_{k=0}^{q-1}\left[\frac{kp+s}{q}\right]^2$$
$$=\frac{1}{6}p^2q(q-1)(2q-1)+sq(q-1-s)-2pq\sum_{k=0}^{q-1}k\left[\frac{kp+s}{q}\right]+q^2\sum_{k=0}^{q-1}\left[\frac{kp+s}{q}\right]^2.$$

Now using the simple trick of expanding $[x]$ as $\sum_{0<k\leqq x}1$ (cf. Zagier [17]. Another method will be found in Katase [12]), we have

$$\sum_{k=0}^{q-1} k\left[\frac{kp+s}{q}\right]=\sum_{k=0}^{q-1} k\Big(\sum_{0\leq h\leq \frac{kp+s}{q}} 1-1\Big)$$

$$=\sum_{-s\leq hq-s\leq kp\leq p(q-1)} k-\frac{1}{2}q(q-1)$$

$$=\sum_{h=0}^{[p-\frac{p-s}{q}]}\sum_{\frac{hq-s}{p}\leq k\leq q-1} k-\frac{1}{2}q(q-1)$$

$$=\sum_{h=0}^{[p-\frac{p-s}{q}]}\Big(\sum_{k=0}^{q-1}k-\sum_{k=0}^{[\frac{hq-s}{p}]}k\Big)+\left[\frac{q\rho_s-s}{p}\right]-\frac{1}{2}q(q-1)$$

$$=\Big(\left[p-\frac{p-s}{q}\right]+1\Big)\cdot\frac{1}{2}q(q-1)-\sum_{h=0}^{[p-\frac{p-s}{q}]}\frac{1}{2}\left[\frac{hq-s}{p}\right]\Big(\left[\frac{hq-s}{p}\right]+1\Big)$$

$$+\left[\frac{q\rho_s-s}{p}\right]-\frac{1}{2}q(q-1)\,.$$

(Note that the inequality $\left[p-\frac{p-s}{q}\right]\geqq\rho_s$ always holds, for if we assume the inequality $\left[p-\frac{p-s}{q}\right]<\rho_s$ then the integer $\tau_s=\frac{q\rho_s-s}{p}$ should be equal to $q-1$ and hence we have the equation $\rho_s=p-\frac{p-s}{q}$, a contradiction.)

If $p-s\leqq q$, the above equation reduces to

$$\frac{1}{2}pq(q-1)-\frac{1}{2}\sum_{h=0}^{p-1}\left[\frac{hq-s}{p}\right]^2-\frac{1}{2}\Big(\frac{1}{2}(p-1)(q-1)-s\Big)+\left[\frac{q\rho_s-s}{p}\right]-\frac{1}{2}q(q-1)$$

$$=\frac{1}{4}(p-1)(q-1)(2q-1)-\frac{1}{2}\sum_{h=0}^{p-1}\left[\frac{hq-s}{p}\right]^2+\frac{1}{2}s+\left[\frac{q\rho_s-s}{p}\right].$$

If $p-s>q$, the equation also reduces to

$$\frac{1}{2}pq(q-1)-\frac{1}{2}\left[\frac{p-s}{q}\right]q(q-1)-\frac{1}{2}\Big(\sum_{h=0}^{p-1}\left[\frac{hq-s}{p}\right]\Big(\left[\frac{hq-s}{p}\right]+1\Big)$$

$$-\sum_{h=p-[\frac{p-s}{q}]}^{p-1}\left[\frac{hq-s}{p}\right]\Big(\left[\frac{hq-s}{p}\right]+1\Big)\Big)+\left[\frac{q\rho_s-s}{p}\right]-\frac{1}{2}q(q-1)$$

$$=\frac{1}{2}pq(q-1)-\frac{1}{2}\left[\frac{p-s}{q}\right]q(q-1)-\frac{1}{2}\sum_{h=0}^{p-1}\left[\frac{hq-s}{p}\right]^2-\frac{1}{2}\Big(\frac{1}{2}(p-1)(q-1)-s\Big)$$

$$+\frac{1}{2}\Big(p-1-\Big(p-\left[\frac{p-s}{q}\right]\Big)+1\Big)(q-1)q+\left[\frac{q\rho_s-s}{p}\right]-\frac{1}{2}q(q-1)$$

$$=\frac{1}{4}(p-1)(q-1)(2q-1)-\frac{1}{2}\sum_{h=0}^{p-1}\left[\frac{hq-s}{p}\right]^2+\frac{1}{2}s+\left[\frac{q\rho_s-s}{p}\right].$$

Substituting

$$\sum_{k=0}^{q-1} k\left[\frac{kp+s}{q}\right]=\frac{1}{4}(p-1)(q-1)(2q-1)-\frac{1}{2}\sum_{h=0}^{p-1}\left[\frac{hq-s}{p}\right]^2+\frac{1}{2}s+\left[\frac{q\rho_s-s}{p}\right],$$

we obtain the reciprocity formula (3.1).

The formula (3.2) follows similarly. □

We next show a recursive formula for generalized Dedekind sums. Substituting $p=\left[\frac{p}{q}\right]q+r_1$ to the second sum term on the left-hand side of the equation (3.1), we have

$$\begin{aligned}
6pr_1\sum_{h=0}^{p-1}\left[\frac{hq-s}{p}\right]^2&+6pq\sum_{k=0}^{q-1}\left[\frac{kr_1+s}{q}\right]^2\\
&=(p-1)(2p-1)(q-1)(2q-1)r_1-\left[\frac{p}{q}\right]q(q-1)(2q-1)pr_1+\left[\frac{p}{q}\right]q(q-1)(2q-1)\\
&\quad-6ps(q-1-s)+6psr_1-12sr_1+12q\rho_s r_1\\
&=p(q-1)(2q-1)(r_1-1)(2r_1-1)+pqr_1(q-1)(2q-1)\left[\frac{p}{q}\right]\\
&\quad-6ps(q-r_1-1-s)+12r_1(q\rho_s-s)\ ,
\end{aligned}$$

so that

$$\begin{aligned}
\frac{1}{q}\sum_{h=0}^{p-1}\left[\frac{hq-s}{p}\right]^2+\frac{1}{r_1}\sum_{k=0}^{q-1}\left[\frac{kr_1+s}{q}\right]^2&=\frac{1}{6qr_1}(q-1)(2q-1)(r_1-1)(2r_1-1)\\
&\quad+\frac{1}{6}(q-1)(2q-1)\left[\frac{p}{q}\right]-\frac{1}{qr_1}s(q-r_1-1-s)+\frac{2}{pq}(q\rho_s-s)\ .
\end{aligned}$$

Similarly, we get from the equation (3.2) that

$$\begin{aligned}
\frac{1}{q}\sum_{h=0}^{p-1}\left[\frac{hq+s}{p}\right]^2+\frac{1}{r_1}\sum_{k=0}^{q-1}\left[\frac{kr_1-s}{q}\right]^2&=\frac{1}{6qr_1}(q-1)(2q-1)(r_1-1)(2r_1-1)\\
+\frac{1}{6}(q-1)(2q-1)\left[\frac{p}{q}\right]+\frac{1}{qr_1}s(q-r_1-1+s)&-2+\frac{2}{pq}(q\rho_{-s}+s)-2\left[-\frac{s}{q}\right].
\end{aligned}$$

Put $s_1=s-\left[\frac{s}{q}\right]q$. Then we get

$$\sum_{k=0}^{q-1}\left[\frac{kr_1\pm s}{q}\right]^2=\sum_{k=0}^{q-1}\left[\frac{kr_1\pm s_1}{q}\right]^2\pm\left[\frac{s}{q}\right]((q-1)(r_1-1)(2r_1-1)\pm 2s_1)+\left[\frac{s}{q}\right]^2 q\ .$$

Hence we get

$$\frac{1}{q}\left(\sum_{h=0}^{p-1}\left[\frac{hq+s}{p}\right]^2+\sum_{h=0}^{p-1}\left[\frac{hq-s}{p}\right]^2\right)+\frac{1}{r_1}\left(\sum_{k=0}^{q-1}\left[\frac{kr_1+s_1}{q}\right]^2+\sum_{k=0}^{q-1}\left[\frac{kr_1-s_1}{q}\right]^2\right)$$

$$=\frac{1}{3qr_1}(q-1)(2q-1)(r_1-1)(2r_1-1)+\frac{1}{3}(q-1)(2q-1)\left[\frac{p}{q}\right]$$
$$+\frac{2s^2}{qr_1}-\frac{4}{r_1}\left[\frac{s}{q}\right]s_1-\frac{2q}{r_1}\left[\frac{s}{q}\right]^2-2+\frac{2}{p}(\rho_s+\rho_{-s})-2\left[-\frac{s}{q}\right]$$
$$=\frac{1}{3qr_1}(q-1)(2q-1)(r_1-1)(2r_1-1)+\frac{1}{3}(q-1)(2q-1)\left[\frac{p}{q}\right]+\frac{2s_1^2}{qr_1}-2\left[-\frac{s}{q}\right].$$

Here, of course, $\rho_s+\rho_{-s}=p$. From this equation we obtain the following.

Lemma 3.2. *Let* $s_0=s$ *and* $s_i=s_{i-1}-\left[\frac{s_{i-1}}{r_{i-1}}\right]r_{i-1}$ $(i=1,\cdots,n)$ *be non-negative integers. Then there exist recursive relations:*

$$\frac{1}{r_i}\sum_{h=0}^{r_{i-1}-1}\left(\left[\frac{hr_i+s_i}{r_{i-1}}\right]^2+\left[\frac{hr_i-s_i}{r_{i-1}}\right]^2\right)+\frac{1}{r_{i+1}}\sum_{k=0}^{r_i-1}\left(\left[\frac{kr_{i+1}+s_{i+1}}{r_i}\right]^2+\left[\frac{kr_{i+1}-s_{i+1}}{r_i}\right]^2\right)$$
$$=\frac{1}{3r_ir_{i+1}}(r_i-1)(2r_i-1)(r_{i+1}-1)(2r_{i+1}-1)+\frac{1}{3}(r_i-1)(2r_i-1)\left[\frac{r_{i-1}}{r_i}\right]$$
$$+\frac{2s_{i+1}^2}{r_ir_{i+1}}-2\left[-\frac{s_i}{r_i}\right]$$

for $i=0,1,\cdots,n-1$.

Note that the last two terms should be omitted for $i\geqq m$ if $s_m=0$ for some m $(\leqq n-1)$.

Multiplying $(-1)^i$ to both sides of this equation and then summing from $i=0$ to $n-1$, we get

$$\frac{1}{q}\sum_{h=0}^{p-1}\left(\left[\frac{hq+s}{p}\right]^2+\left[\frac{hq-s}{p}\right]^2\right)+(-1)^{n-1}\frac{1}{r_n}\sum_{k=0}^{r_{n-1}-1}\left(\left[\frac{kr_n+s_n}{r_{n-1}}\right]^2+\left[\frac{kr_n-s_n}{r_{n-1}}\right]^2\right)$$
$$=\sum_{i=0}^{n-1}(-1)^i\left(\frac{1}{3r_ir_{i+1}}(r_i-1)(2r_i-1)(r_{i+1}-1)(2r_{i+1}-1)+\frac{1}{3}(r_i-1)(2r_i-1)\left[\frac{r_{i-1}}{r_i}\right]\right)$$
$$+2\sum_{i=0}^{n-1}(-1)^i\left(\frac{s_{i+1}^2}{r_ir_{i+1}}-\left[-\frac{s_i}{r_i}\right]\right)$$
$$=\sum_{i=0}^{n-1}(-1)^i\left(\frac{1}{3}(r_i-1)(r_{i+1}-1)-\frac{1}{3r_i}(r_i-1)(r_{i+1}-1)-\frac{1}{3r_{i+1}}(r_i-1)(r_{i+1}-1)\right.$$
$$+\frac{1}{3r_ir_{i+1}}(r_i-1)(r_{i+1}-1)+\frac{1}{3r_i}(r_i-1)^2(r_{i+1}-1)+\frac{1}{3r_{i+1}}(r_i-1)(r_{i+1}-1)^2$$
$$\left.+\frac{1}{3}(r_i-1)(r_{i+1}-1)+\frac{1}{3r_i}(r_i-1)^2(r_{i-1}-r_{i+1})+\frac{1}{3}(r_i-1)(r_{i-1}-r_{i+1})\right)$$
$$+2\sum_{i=0}^{n-1}(-1)^i\left(\frac{s_{i+1}^2}{r_ir_{i+1}}-\left[-\frac{s_i}{r_i}\right]\right)$$

$$=\frac{1}{3}(p-1)(q-1)-\frac{1}{3q}(q-1)(r_1-1)+\frac{1}{3q}(q-1)^2(r_1-1)+\frac{1}{3q}(q-1)^2(p-r_1)$$
$$-\frac{1}{3}(q-r_1-(-1)^{n-2}r_n)+\frac{1}{3}\sum_{i=0}^{n-1}(-1)^i\left[\frac{r_i}{r_{i+1}}\right]+\frac{1}{3}\cdot\frac{1+(-1)^{n-1}}{2}$$
$$-\frac{1}{3}\left(\frac{1}{q}+\frac{(-1)^{n-1}}{r_n}\right)+\frac{1}{3}\sum_{i=0}^{n-1}\frac{(-1)^i}{r_ir_{i+1}}+2\sum_{i=0}^{n-1}(-1)^i\left(\frac{s_{i+1}^2}{r_ir_{i+1}}-\left[-\frac{s_i}{r_i}\right]\right)$$
$$=\frac{1}{3q}(p-1)(q-1)(2q-1)-\frac{1}{3q}(q-1)(r_1-1)-\frac{1}{3}(q-r_1)+\frac{1}{3}\cdot\frac{1+(-1)^n}{2}$$
$$-\frac{1}{3}\left(\frac{1}{q}+(-1)^{n-1}\right)+\frac{1}{3}\sum_{i=0}^{n-1}(-1)^i\left[\frac{r_i}{r_{i+1}}\right]+\frac{1}{3}\sum_{i=0}^{n-1}\frac{(-1)^i}{r_ir_{i+1}}$$
$$+2\sum_{i=0}^{n-1}(-1)^i\left(\frac{s_{i+1}^2}{r_ir_{i+1}}-\left[-\frac{s_i}{r_i}\right]\right).$$

Since we have

$$\left[\frac{k-s_n}{r_{n-1}}\right]=\begin{cases}-1 & \text{if } k=0,\cdots,s_n-1\\ 0 & \text{otherwise}\end{cases}$$

and

$$\left[\frac{k+s_n}{r_{n-1}}\right]=\begin{cases}1 & \text{if } k=r_{n-1}-s_n,\cdots,r_{n-1}-1\\ 0 & \text{otherwise},\end{cases}$$

we obtain the equation:

$$(3.3)\quad \sum_{h=0}^{p-1}\left(\left[\frac{hq+s}{p}\right]^2+\left[\frac{hq-s}{p}\right]^2\right)=\frac{1}{3}(p-1)(q-1)(2q-1)-\frac{1}{3}(q-1)(r_1-1)$$
$$-\frac{q}{3}(q-r_1)+\frac{q}{3}\cdot\frac{1+(-1)^n}{2}-\frac{1}{3}(1+(-1)^{n-1}q)+(-1)^n2qs_n$$
$$+\frac{q}{3}\sum_{i=0}^{n-1}(-1)^i\left[\frac{r_i}{r_{i+1}}\right]+\frac{q}{3}\sum_{i=0}^{n-1}\frac{(-1)^i}{r_ir_{i+1}}+2q\sum_{i=0}^{n-1}(-1)^i\left(\frac{s_{i+1}^2}{r_ir_{i+1}}-\left[-\frac{s_i}{r_i}\right]\right).$$

On the other hand, we have

$$\sum_{h=0}^{p-1}\left[\frac{hq+s}{p}\right]^2=\sum_{h=1}^{p}\left[\frac{(p-h)q+s}{p}\right]^2=\sum_{h=0}^{p-1}\left[\frac{(p-h)q+s}{p}\right]^2+\left[\frac{s}{p}\right]^2-\left[q+\frac{s}{p}\right]^2$$
$$=\sum_{h=0}^{p-1}\left(q-1-\left[\frac{hq-s}{p}\right]\right)^2-\left(q-1-\left[\frac{\rho_sq-s}{p}\right]\right)^2$$
$$+\left(q-\left[\frac{\rho_sq-s}{p}\right]\right)^2+\left[\frac{s}{p}\right]^2-\left[q+\frac{s}{p}\right]^2$$
$$=(p-1)(q-1)^2-2(q-1)\sum_{h=0}^{p-1}\left[\frac{hq-s}{p}\right]+\sum_{h=0}^{p-1}\left[\frac{hq-s}{p}\right]^2-2\left[\frac{\rho_sq-s}{p}\right]$$

so that we have

$$\sum_{h=0}^{p-1}\left(\left[\frac{hq+s}{p}\right]^2-\left[\frac{hq-s}{p}\right]^2\right)=2((q-1)s-\tau_s)\ .$$

Here we have used the relations $\rho_s q=\tau_s p+s$. It follows from this result and the equation (3.3) that

$$\begin{aligned}\sum_{h=0}^{p-1}\left[\frac{hq-s}{p}\right]^2&=\frac{1}{6}(p-1)(q-1)(2q-1)-\frac{1}{6}(q-1)(r_1-1)-\frac{q}{6}(q-r_1)\\&\quad+\frac{q}{6}\cdot\frac{1+(-1)^n}{2}-\frac{1}{6}(1+(-1)^{n-1}q)+(-1)^n qs_n+\frac{q}{6}\sum_{i=0}^{n-1}(-1)^i\left[\frac{r_i}{r_{i+1}}\right]\\&\quad+\frac{q}{6}\sum_{i=0}^{n-1}\frac{(-1)^i}{r_ir_{i+1}}+q\sum_{i=0}^{n-1}(-1)^i\left(\frac{s_{i+1}^2}{r_ir_{i+1}}-\left[-\frac{s_i}{r_i}\right]\right)-s(q-1)+\tau_s\\&=\frac{1}{6}(p-1)(q-1)(2q-1)+\frac{1}{6}(q-1)-\frac{1}{6}q^2+\frac{q}{12}(1+3\cdot(-1)^n)\\&\quad-\frac{1}{6}+\frac{p}{6}+\frac{1}{6p}+(-1)^n qs_n-\frac{q}{6}\sum_{i=0}^{n}(-1)^i\left[\frac{r_{i-1}}{r_i}\right]\\&\quad-\frac{q}{6}\sum_{i=0}^{n}\frac{(-1)^i}{r_{i-1}r_i}+q\sum_{i=0}^{n-1}(-1)^i\left(\frac{s_{i+1}^2}{r_ir_{i+1}}-\left[-\frac{s_i}{r_i}\right]\right)-s(q-1)+\tau_s\ .\end{aligned}$$

Hence we get

$$\begin{aligned}\sum_{k=0}^{q-1}\left[\frac{kp+s}{q}\right]^2&=\frac{1}{q}\left(\frac{1}{6}(p-1)(2p-1)(q-1)(2q-1)+s(p-q-1+s)\right.\\&\qquad\left.+2q\rho_s-p\sum_{h=0}^{p-1}\left[\frac{hq-s}{p}\right]^2\right)\\&=\frac{1}{q}\left(\frac{1}{6}(p-1)^2(q-1)(2q-1)-\frac{1}{6}p(q-1)+\frac{1}{6}pq^2+\frac{1}{6}p-\frac{1}{6}p^2-\frac{1}{6}\right.\\&\qquad\left.-\frac{pq}{12}(1+3\cdot(-1)^n)+s(p-q-1+s)+sp(q-1)-\tau_s p+2q\rho_s\right)\\&\quad-(-1)^n ps_n+\frac{p}{6}\sum_{i=0}^{n}(-1)^i\left[\frac{r_{i-1}}{r_i}\right]+\frac{p}{6}\sum_{i=0}^{n}\frac{(-1)^i}{r_{i-1}r_i}\\&\quad-p\sum_{i=0}^{n-1}(-1)^i\left(\frac{s_{i+1}^2}{r_ir_{i+1}}-\left[-\frac{s_i}{r_i}\right]\right)\\&=\frac{1}{3}(p-1)^2(q-1)-\frac{1}{6}p(p-q)-\frac{1}{6}+\frac{p}{12}(1-3\cdot(-1)^n)\\&\quad+\frac{1}{q}(s(p-q+s)+sp(q-1)+q\rho_s)+\frac{p}{6}\sum_{i=0}^{n}(-1)^i\left[\frac{r_{i-1}}{r_i}\right]\\&\quad+\frac{p}{6}\sum_{i=0}^{n}\frac{(-1)^i}{r_{i-1}r_i}-p\sum_{i=0}^{n-1}(-1)^i\left(\frac{s_{i+1}^2}{r_ir_{i+1}}-\left[-\frac{s_i}{r_i}\right]\right)-(-1)^n ps_n\ .\end{aligned}$$

Thus we obtain the following

THEOREM 3.3. *The generalized Dedekind sum* $\sum_{k=0}^{q-1}\left[\frac{kp+s}{q}\right]^2$ *is evaluated as follows*:

$$\sum_{k=0}^{q-1}\left[\frac{kp+s}{q}\right]^2=\frac{1}{3}(p-1)^2(q-1)-\frac{1}{6}p(p-q)-\frac{1}{6}+\frac{p}{12}(1-3\cdot(-1)^n)+s(p-1)+\rho_s$$
$$+\frac{p}{6}\sum_{i=0}^{n}(-1)^i\left[\frac{r_{i-1}}{r_i}\right]+\frac{p}{6}\sum_{i=0}^{n}\frac{(-1)^i}{r_{i-1}r_i}+p\sum_{i=0}^{n}(-1)^i\left(\frac{s_i^2}{r_{i-1}r_i}+\left[-\frac{s_i}{r_i}\right]\right).$$

Note that the values of the generalized Dedekind sums $\sum_{h=0}^{p-1}\left[\frac{hq\pm s}{p}\right]^2$ and $\sum_{k=0}^{q-1}\left[\frac{kp-s}{q}\right]^2$ are obtained by Lemma 3.1 and the equation (3.3).

§ 4. Main theorem.

As we have obtained the exact value of the generalized Dedekind sum, the next theorem follows from Lemma 2.2 and Theorem 3.3.

THEOREM 4.1. *The eta-invariant with the unitary representation* α *is reduced as follows*:

$$3p\eta_\alpha(p;q)=-q-\frac{3}{2}p(1-(-1)^n)-p\sum_{i=0}^{n}(-1)^i\left[\frac{r_{i-1}}{r_i}\right]-(-1)^n a_{n-1}$$
$$-6p\sum_{i=0}^{n}(-1)^i\left(\frac{s_i^2}{r_{i-1}r_i}+\left[-\frac{s_i}{r_i}\right]\right)$$
$$=3p\eta(p;q)-3p-6p\sum_{i=0}^{n}(-1)^i\left(\frac{s_i^2}{r_{i-1}r_i}+\left[-\frac{s_i}{r_i}\right]\right),$$

where $s=p-\alpha<p$.

Since $\eta_{p-\alpha}(p;q)=\eta_\alpha(p;q)$, we may write the right-hand side of this equation as $3p\eta_s(p;q)$.

Now let us study the sum term on the above equation. Put

$$\sum_{i=0}^{k}(-1)^i\frac{s_i^2}{r_{i-1}r_i}=\frac{(-1)^k\xi_k}{pr_k}\qquad\text{for}\quad k=0,1,\cdots,n.$$

Then the integers ξ_k's satisfy the equations

$$\xi_0=s^2,\quad \xi_1=s^2\left[\frac{p}{q}\right]-p(s+s_1)\left[\frac{s}{q}\right],\quad\cdots,\quad\text{and}\quad \xi_k r_{k-1}+\xi_{k-1}r_k=ps_k^2.$$

Calculating further, we obtain the following lemma which will be proved inductively.

LEMMA 4.2. *Let the integers* $a_{n-1}^{(k)}$ *be defined by the equations*

$$\sum_{i=k}^{n} \frac{(-1)^i}{r_{i-1}r_i} = \frac{(-1)^n a_{n-1}^{(k)}}{r_{k-1}} \qquad \text{for} \quad k=0, 1, \cdots, n .$$

Then ξ_n *is represented as follows:*

$$\xi_n = s^2 a_{n-1} - p\left((s+s_1)\left[\frac{s}{q}\right]a_{n-1}^{(1)} + (s_1+s_2)\left[\frac{s_1}{r_1}\right]a_{n-1}^{(2)} + \cdots + (s_{n-1}+s_n)\left[\frac{s_{n-1}}{r_{n-1}}\right]a_{n-1}^{(n)}\right) .$$

Note that $a_{n-1}^{(0)}$ coincides with a_{n-1} which was introduced in Theorem 1.1 and the positive integer $a_{n-1}^{(k)}\left(<\frac{r_{k-1}}{2}\right)$ is uniquely determined by the equation $r_k a_{n-1}^{(k)} \equiv (-1)^{n+k}$ modulo r_{k-1}.

As a corollary to this lemma, we obtain the following

THEOREM 4.3. *Let* q *and* $\bar{q}$ $\left(<\frac{p}{2}\right)$ *be positive integers relatively prime to* p. *Then two lens spaces* $L(p; q)$ *and* $L(p; \bar{q})$ *are (orientation preservingly) isometric to each other, i.e., the equation* $q\bar{q} \equiv 1$ *modulo* p *holds if and only if their eta-invariants satisfy the equations:*

$$(1) \qquad \eta(p; q) = \eta(p; \bar{q}) \quad \text{and} \quad \eta_1(p; q) = \eta_{\bar{q}}(p; \bar{q})$$

or

$$(2) \qquad \eta(p; q) = \eta(p; \bar{q}) \quad \text{and} \quad \eta_1(p; q) = \eta_{q^*}(p; \bar{q}) ,$$

where q^* *is a positive integer less than* $p/2$ *defined by the equation* $qq^* \equiv 1$ *modulo* p.

PROOF. The "only if" part is trivial; we have only to substitute $j\bar{q}$ modulo p or jq^* modulo p for i in the defining cotangent sums of eta-invariants. So we prove "if" part.

(1) Let n (respectively $\bar{n}$) be the length of the remainders for calculating the $\gcd(p, q)$ (respectively $\gcd(p, \bar{q})$). By assumption, the equation

$$\sum_{i=0}^{n} (-1)^i \left(\frac{s_i^2}{r_{i-1}r_i} + \left[-\frac{s_i}{r_i}\right]\right) = \sum_{i=0}^{\bar{n}} (-1)^i \left(\frac{\bar{s}_i^2}{\bar{r}_{i-1}\bar{r}_i} + \left[-\frac{\bar{s}_i}{\bar{r}_i}\right]\right)$$

holds for $s=1$ and $\bar{s}=\bar{q}$. In particular, since $\bar{s}_1 = \cdots = \bar{s}_{\bar{n}} = 0$, we have

$$\frac{1}{p}(-1)^n a_{n-1} - \frac{1+(-1)^n}{2} = \frac{\bar{q}}{p} - 1$$

so that

$$(-1)^n a_{n-1} - \bar{q} = \frac{(-1)^n - 1}{2} p .$$

However, since a_{n-1} and $\bar{q}$ are positive integers less than $p/2$, the above equation holds only when n is an even integer and therefore we have $\bar{q} = a_{n-1} = q^*$, *i.e.*, $q\bar{q} \equiv 1$ modulo p.

(2) Similarly, we have

$$\frac{1}{p}(-1)^n a_{n-1} - \frac{1+(-1)^n}{2} = \frac{1}{p}(-1)^{\bar{n}} \bar{\xi}_{\bar{n}} + \sum_{i=0}^{\bar{n}} (-1)^i \left[-\frac{q_i^*}{\bar{r}_i} \right],$$

where q_i^*'s are defined from q^* and $\bar{r}_j$'s, so that

$$(-1)^n a_{n-1} \equiv (-1)^{\bar{n}} \bar{\xi}_{\bar{n}} \qquad \text{modulo } p .$$

Since $\bar{\xi}_{\bar{n}} \equiv q^{*2} \bar{a}_{\bar{n}-1}$ modulo p, we have

$$\begin{aligned} (-1)^{\bar{n}} q^* \bar{a}_{\bar{n}-1} &\equiv (-1)^n q a_{n-1} && \text{modulo } p \\ &\equiv 1 && \text{modulo } p \end{aligned}$$

so that $q^* = \bar{q}$, *i.e.*, $q\bar{q} \equiv 1$ modulo p. □

Note that even though the integers n and $\bar{n}$ do not appear in the statement of Theorem 4.3, they are necessarily even and coincide by Lemma 1.5. (Compare Theorem 1.6.)

REMARK 4.4. (1) The condition $\eta(p; q) = \eta(p; \bar{q})$ is necessary. In fact, when $p = 68$, $q = 11$, and $\bar{q} = 21$, we have $\eta_1(68; 11) = \eta_{21}(68; 21)$ and $\eta(68; 11) \neq \eta(68; 21)$. Of course, $L(68; 11) \neq L(68; 21)$.

(2) The equations $\eta(p; q) = \eta(p; \bar{q})$ and $\eta_1(p; q) = \eta_{\bar{s}}(p; \bar{q})$ may hold for some $\bar{s}$ $(1 \leqq \bar{s} < p)$ different from $\bar{q}$ even though $L(p; q) \neq L(p; \bar{q})$; for example, two non-isometric lens spaces $L(161; 37)$ and $L(161; 51)$ satisfy the equations $\eta(161; 37) = \eta(161; 51)$ and $\eta_1(161; 37) = \eta_{64}(161; 51)$. However, $\eta_2(161; 37) \neq \eta_{128}(161; 51) = \eta_{33}(161; 51)$ in this example. Note that, in these cases, q and $\bar{q}$ satisfy the equations

$$q - \bar{q} = ia \qquad \text{for some integer } i \ (0 < |i| < b)$$

and

$$q\bar{q} - 1 \equiv jb \ \text{ modulo } p \qquad \text{for some integer } j \ (0 < j < a) ,$$

where $p=ab$ (see [8] p. 347), and $\bar{s}$ is of the form $\bar{q}\pm ka$ for some integer k. Also note that the integer l appeared in Theorem 2.1 should be $\bar{q}$.

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