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Vector Bundles of Grassmann Type and Configuration Type of Rank 2 on an Algebraic Surface

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§0. Introduction.

The problem of constructing a holomorphic vector bundle of rank 2 on S having the given Chern classes, where S is a smooth projective surface over the complex number field $\boldsymbol{C}$, was first considered by Schwarzenberger ([17]) and solved by Maruyama ([11]). When $S=\boldsymbol{P}^2$, the structure of the moduli spaces are known (cf. [2], [7], [10], [13]). Many other important results are known (cf. [3], [6], [9], [12]). See [14] for general theory on $\boldsymbol{P}^n$. In the papers [15] and [16], Sasakura gave a method to construct vector bundles or reflexive sheaves of arbitrary ranks on complex spaces by giving explicit transition matrices and mentioned their general properties. They are called *of configuration type* or *of Grassmann type*. In this paper, we construct bundles of the above types of rank 2 on an algebraic surface and calculate their Chern classes by investigating the local structures.

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NOTATION AND TERMINOLOGIES. A surface means a complex manifold of dimension 2 embedded in a projective space. $\mathcal{O}$ denotes the structure sheaf of the surface which we concern. A vector bundle, or simply a bundle, means a holomorphic vector bundle of rank 2. A sheaf is simple if the endomorphisms of it are the homotheties. For a section s of a line bundle, $(s)_0$ denotes its zero divisor. For a meromorphic function t, $(t)_\infty$ denotes its pole divisor. Sometimes, we abbreviate the symbol $\otimes$ denoting the tensor product of sections: for example, $s_1 \cdots s_m := s_1 \otimes \cdots \otimes s_m$.

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§ 1. Preliminaries and results.

In this section, we define the vector bundles of configuration type and Grassmann type and describe some of their properties. We start with the definition of the vector bundles of Grassmann type.

Let X be a connected complex manifold of dimension n and $\mathscr{L}$ a line bundle on X. We also assume that $s_0, s_1, \cdots, s_r$ are non-zero global sections of $\mathscr{L}$ (not necessarily linearly independent) such that $\operatorname{codim}(s_0)_0 \cap (s_r)_0 = 2$ and that $(s_0)_0$ is both reduced and irreducible. Set $Z=(s_0)_0 \cap (s_r)_0$. Put $H=\begin{bmatrix} I_{r-1} & \begin{matrix} f_1 \\ \vdots \\ \vdots \end{matrix} \\ 0 & f_r \end{bmatrix}$, where I_m denotes the unit matrix of $m \times m$, $f_i = s_i/s_0$.

Let $N_0 := X-(s_0)_0$ and $N_1 := X-(s_r)_0$. Then we can define a vector bundle $\mathring{\mathscr{E}}$ on $N_0 \cup N_1 = X-Z$ with the transition matrix H on $N_0 \cap N_1$. To be more precise, let e^i be a frame on N_i ($i=0, 1$), and $e^0 = e^1 H$. From $\mathring{\mathscr{E}}$, we obtain $\mathscr{E} := i_*(\mathring{\mathscr{E}})$, where $i: X-Z \to X$ is the inclusion. Since all components of H are meromorphic functions on X, $\mathscr{E}$ is coherent. Moreover, $\mathscr{E}$ is locally free outside Z whose codimension is two. Thus $\mathscr{E}$ is reflexive on X.

DEFINITION. We call the above $\mathscr{E}$ *a reflexive sheaf of Grassmann type* or *a reflexive sheaf of type* (G) associated with $(\mathscr{L}, s)$, where $s=(s_0, \cdots, s_r) \in \Gamma(X, \mathscr{L})^{\oplus r+1}$. If $\mathscr{E}$ is locally free, we replace "reflexive sheaf" by "vector bundle".

T. Hosoh told us about another definition of reflexive sheaves of type (G). Let X, $\mathscr{L}$, s be as above. But we do *not* need to assume that $(s_0)_0$ is reduced and irreducible. Then we obtain the exact sequence of sheaves:

$$\text{(A)} \qquad 0 \longrightarrow \mathscr{L}^{\vee} \xrightarrow{\otimes s} \mathcal{O}^{\oplus r+1} \longrightarrow \mathscr{F} \longrightarrow 0 \quad (\mathscr{L}^{\vee} := \mathscr{H}om(\mathscr{L}, \mathcal{O}))\,.$$

$\mathscr{F}^{\vee\vee}$ is a reflexive sheaf of type (G) and all reflexive sheaf of type (G) is obtained in this way, because $i_*(\mathscr{E}|_{X-Z}) = \mathscr{E}$ if $\mathscr{E}$ is reflexive and $\operatorname{codim} Z \geqq 2$.

Using the above definition, we obtain some properties of reflexive sheaves of type (G).

Taking the dual of (A)$\otimes\mathscr{L}$, define an ideal $\mathscr{I}_Y$ as follows:

$$\text{(B)} \qquad 0 \longrightarrow \mathscr{F}^{\vee} \otimes \mathscr{L}^{\vee} \longrightarrow (\mathscr{L}^{\vee})^{\oplus(r+1)} \longrightarrow \mathscr{I}_Y \longrightarrow 0\,,$$

$$\text{(C)} \qquad 0 \longrightarrow \mathscr{F}^{\vee} \otimes \mathscr{L}^{\vee} \longrightarrow (\mathscr{L}^{\vee})^{\oplus(r+1)} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_Y \longrightarrow 0\,,$$

where Y is the subvariety determined by $\mathscr{I}_Y$. Our assumption on s is that codim $Y \geqq 2$.

T. Fukui pointed out the following fact, using ring theory.

PROPOSITION 1.1. *The reflexive sheaf* $\mathscr{E}$ $(=\mathscr{F}^{\vee\vee})$ *is locally free if and only if one of the following holds*:

(1) $Y=\emptyset$,

(2) codim $Y=2$ *and* $\mathcal{O}_{Y,y}$ *is a Cohen-Macaulay* $\mathcal{O}_y$*-module for any* $y \in Y$.

PROOF. In the case (1), $\mathscr{F}$ is locally free. Thus we only prove the proposition when $Y \neq \emptyset$. Recall the theorem of Auslander-Buchsbaum (i.e., $\mathrm{p.d.}_{\mathcal{O}} \mathcal{O}_Y = \dim X - \mathrm{depth}\, \mathcal{O}_Y$), and the definition of Cohen-Macaulay module (i.e., $\mathrm{depth}\, \mathcal{O}_Y = \dim \mathcal{O}_Y$). Then by the exact sequence (C) and our assumption that codim $Y \geqq 2$, we have the conclusion. (q.e.d.)

REMARK 1.2. In the same notation as above, let X be a surface and the rank r of $\mathscr{E}$ be two. Assume that $\mathscr{L}$ is ample and $\deg Y < \mathscr{L}^2/2$. Then $\mathscr{F}^{\vee}$ is (so $\mathscr{E}$ is) $\mathscr{L}$-stable.

We next define the bundles of configuration type. Let X be a complex manifold and take the following data (D1)$\sim$(D3):

(D1) A reduced and *reducible* divisor Y on X of the form $Y = \cup_{i=1}^{m} Y_i$, $m \geqq 2$, where Y_i are the irreducible components of Y and satisfy the following conditions: each Y_i is smooth, Y_i and Y_j intersect transversally if $i \neq j$, and $Y_i \cap Y_j \cap Y_k$ is of codimension at least 3 for three different indices.

Define the following subvarieties from the above data: $Z_{ij} := Y_i \cap Y_j$, $Z := \cup_{i \neq j} Z_{ij}$, $\mathring{X} := X \backslash Z$, $\mathring{Y}_i := Y_i \backslash Z$, $\mathring{Y} := \cup_i \mathring{Y}_i = Y \backslash Z$ and $N_0 := \mathring{X} \backslash \mathring{Y}$.

(D2) An open neighborhood N_1 of $\mathring{Y}$ in $\mathring{X}$ of the form $N_1 = \cup_i N_1^i$, where N_1^i is an open neighborhood of $\mathring{Y}_i$ in $\mathring{X}$ and does not meet the others.

(D3) A matrix $H \in M_r(\Gamma(N_1, \mathcal{O}))$ of the form: $H = \begin{bmatrix} I_{r-1} & \begin{matrix} f_1 \\ \vdots \end{matrix} \\ 0 & f_r \end{bmatrix}$, where I_m denotes the unit matrix of $m \times m$ and f_i are holomorphic functions on N_1 satisfying the following: $(f_r)_0 = \mathring{Y}$ and there exists a unique meromorphic function f_{ij} on Y_i such that $f_{ij}|\mathring{Y}_i = f_j|\mathring{Y}_i$ for $j \neq r$.

Now we have data to construct a vector bundle $\mathring{\mathscr{E}}$ on $\mathring{X}$ and the reflexive sheaf $\mathscr{E} := i_*(\mathring{\mathscr{E}})$, where $i: \mathring{X} \to X$ in the same manner as type (G).

DEFINITION. We call the above $\mathscr{E}$ *a reflexive sheaf of configuration* *type* or *a reflexive sheaf of type* (C). If $\mathscr{E}$ is locally free, we replace "reflexive sheaf" by "vector bundle".

REMARK 1.3. Two important bundles are of type (C). One is the Horrocks-Mumford bundle on $\boldsymbol{P}^4$ and the other is the null correlation bundle on $\boldsymbol{P}^3$ (for the definitions see [5] and [1]). We see only the latter here. Let (z_0, z_1, z_2, z_3) be homogeneous coordinates in $\boldsymbol{P}^3$, $Y_i := (z_i)_0$ for $i=1$ and 3, $Y := Y_1 \cup Y_3$. Take $N_1 = N_1^1 \cup N_1^2$ as in (D2) and a matrix H as follows: $H|N_1^i := \begin{bmatrix} 1 & z_{i+1}/z_{i+2} \\ 0 & z_i/z_{i+2} \end{bmatrix}$, $i=1$ and 3. Then the corresponding sheaf $\mathscr{E}$ is locally free and moreover it is the null correlation bundle twisted by $\mathscr{O}(1)$. This fact is verified by H. Kaji ([16] Theorem 1.1.1).

From now on, we consider the sheaves of *rank 2* of type (G) and (C) only on a *surface* S. Since every reflexive sheaf on a surface is locally free, all the reflexive sheaves of type (G) and (C) are vector bundles.

REMARK 1.4. When $\mathscr{E}$ is a vector bundle of rank 2 of type (C) or (G) on a surface S, we have the following sufficient condition for the simpleness of $\mathscr{E}$: $c_1(\mathscr{E})^2 - 2c_2(\mathscr{E}) < 0$ ([15], lemma 4.5.1, lemma 4.6.1, lemma 4.6.3).

Our main results are the following:

THEOREM 1.5. *Let* $\mathscr{L}$ *be a line bundle and* d *the degree of* S *(with respect to the hyperplane bundle* $\mathscr{O}_S(1)$*). Assume that* $m \gg 0$ *and* c_2 *is an integer with* $f(m) \leqq c_2 \leqq \mathscr{L}(m)^2$, *where* $f(m)$ *is a polynomial of* $\sqrt{m}$ *of the form* $f(m) = \sqrt{2}\, dm^{3/2} + O(m)$. *Then there exists a vector bundle* $\mathscr{E}$ *of type* (G) *such that* $c_1(\mathscr{E}) = \mathscr{L}(m)$ *and* $c_2(\mathscr{E}) = c_2$.

This theorem (cf. Theorem 2.11) implies the theorem of Schwarzenberger ([15]):

For each line bundle $\mathscr{L}$ *and* $c_2 \in \boldsymbol{Z}$, *there exists a vector bundle* $\mathscr{E}$ *such that* $c_1(\mathscr{E}) = \mathscr{L}$ *and* $c_2(\mathscr{E}) = c_2$.

THEOREM 1.6. *For each integer* $m \geqq 2$ *and for each integer* c_2 *such that* $m(m-1)/2 \leqq c_2 \leqq m(m-1)$, *there exists a vector bundle of type* (C) *on* $\boldsymbol{P}^2$ *such that* $c_1(\mathscr{E}) = m$ *and* $c_2(\mathscr{E}) = c_2$.

As a corollary of this theorem, we obtain that each stable 2-bundle on $\boldsymbol{P}^2$ is a deformation of a vector bundle of type (C) twisted by $\mathscr{O}(n)$.

§ 2. Vector bundles of Grassmann type.

Let S be a projective surface, $\mathscr{L}$ be a line bundle on S and s_i be a global section of $\mathscr{L}$ $(i=0, 1, 2)$.

Throughout this section, we fix a very ample line bundle $\mathcal{O}_S(1)$ and denote by d the degree of S (with respect to $\mathcal{O}_S(1)$). We also assume that s_0, s_1, s_2 are linearly independent over $\boldsymbol{C}$.

From now on, we treat the following problem:

Determine Chern classes which vector bundles of type (G) have.

For the problem, we will give a partial but meaningful answer in (2.9).

For simplicity, we use the following notation:

$[n, l] := \{k \in \boldsymbol{Z} \mid n \leqq k \leqq l\}$, $h^0(\mathcal{L}) := \dim \Gamma(S, \mathcal{L})$, $Y_s := \bigcap_{i=0}^{2} (s_i)_0$.

PROPOSITION 2.1. *Let $\mathcal{E}$ be a vector bundle of type* (G) (*of rank* 2 *on* S) *associated with* $(\mathcal{L}, s)$. *Then* $c_1(\mathcal{E}) = \mathcal{L}$ *and* $c_2(\mathcal{E}) = \mathcal{L}^2 - \deg Y_s$.

PROOF. $c_1(\mathcal{E}) = \overset{2}{\wedge} \mathcal{E} = \mathcal{L}$. $c_2(\mathcal{E})$ is the zero of a global section of $\mathcal{E}$, hence we immediately obtain the result. (q.e.d.)

Thanks to (2.1), the problem is reduced to the problem below:

Determine the two subsets $D(\mathcal{L})$ and $C_2(\mathcal{L})$ of $\boldsymbol{Z}$, where

$$D(\mathcal{L}) := \{t \in \boldsymbol{Z} \mid \deg Y_s = t \text{ for some } s \in \Gamma(S, \mathcal{L})^{\oplus 3} \text{ such that } \operatorname{codim} (s_2)_0 \cap (s_0)_0 = 2 \text{ and } (s_0)_0 \text{ is reduced and irreducible}\}$$

and

$$C_2(\mathcal{L}) := \{c_2 \in \boldsymbol{Z} \mid c_1(\mathcal{E}) = \mathcal{L} \text{ and } c_2(\mathcal{E}) = c_2 \text{ for some vector bundle } \mathcal{E} \text{ of type (G)}\}.$$

REMARK 2.2. Both $D(\mathcal{L})$ and $C_2(\mathcal{L})$ are included in $[0, \mathcal{L}^2]$. The set $C_2(\mathcal{L})$ (resp. $D(\mathcal{L})$) contains $\mathcal{L}^2$ (resp. 0) and does not contain 1 (resp. $\mathcal{L}^2 - 1$) when S is a complete intersection in a projective space and $\mathcal{L} = \mathcal{O}(n)$ for a large n. The former is obtained by taking general $s \in \Gamma(S, \mathcal{L})^{\oplus 3}$. The latter is followed from the theorem of Cayley-Bacharach ([4], chap. 5, § 2). These facts suggest the difficulty of determinating the sets $C_2(\mathcal{L})$ and $D(\mathcal{L})$.

The following lemma, which holds also in characteristic >0, is useful for the problem.

LEMMA 2.3 (General Position Lemma). *Let X be a curve in $\boldsymbol{P}^n$ which is not contained in any $\boldsymbol{P}^{n-1}$, and let r be an integer with $2 \leqq r \leqq n-1$. Assume that for almost all choices of r points $p_1, \cdots, p_r$ on X, the linear* *space spanned by the p_i contains at least one further point of X. Then X is a strange curve.*

For the proof, see Laksov ([8], lemma 4).

Since there are no strange curves but $\boldsymbol{P}^1$ in char. $=0$, we easily obtain the following corollary.

COROLLARY 2.4. *Let X be a surface in $\boldsymbol{P}^n$ $(n\geqq 3)$ which is not contained in any $\boldsymbol{P}^{n-1}$. Then for almost all choices of $n-2$ points $p_1, \cdots, p_{n-2}$ on X, the linear space L^{n-3} spanned by the p_i does not contain any further point of X.*

COROLLARY 2.5. *Let S be a non-singular projective surface and $\mathscr{L}$ a very ample line bundle on S. Then $D(\mathscr{L})\supset[0, h^0(\mathscr{L})-3]$.*

PROOF. Embedding S by the complete linear system $|\mathscr{L}|$ in $\boldsymbol{P}^N$ where $N=h^0(\mathscr{L})-1$, we apply (2.4). (q.e.d.)

Thus, we have the following:

COROLLARY 2.6. *Under the same assumption as in* (2.5),

$$C_2(\mathscr{L})\supset[\mathscr{L}^2-h^0(\mathscr{L})+3,\ \mathscr{L}^2]\,.$$

We will sharpen the estimate of $C_2(\mathscr{L})$ in (2.6). Let $\mathscr{L}$ be a very ample line bundle on S and n a non-negative integer. We use certain special reducible elements in $\Gamma(S, \mathscr{L}(n))$ to get information about $D(\mathscr{L}(n))$. Set

$$N(n, l; \mathscr{L}):=n(\mathscr{L}\cdot\mathcal{O}_s(1))+dl(n+l) \qquad \text{for integers } n\geqq l\geqq 0\,.$$

PROPOSITION 2.7. *Let n, l be integers with $n\geqq l\geqq 0$. Then*

$$D(\mathscr{L}(n))\supset[N(n, l; \mathscr{L}),\ N(n, l; \mathscr{L})+h^0(\mathscr{L})-3].$$

PROOF. Let $k\in[N(n, l; \mathscr{L}), N(n, l; \mathscr{L})+h^0(\mathscr{L})-3]$ and $j=k-N(n, l; \mathscr{L})$. Thanks to (2.5), we have $s'=(s'_0, s'_1, s'_2)\in\Gamma(S, \mathscr{L})^{\oplus 3}$ such that $\deg Y_{s'}=j$. Choose $h_i\in\Gamma(S, \mathcal{O}_S(1))$ $(i=1, 2, \cdots, 2n)$ which satisfy the following conditions:

$$(2.8)\quad\begin{cases}(1) & (h_p)_0\cap(h_q)_0\cap(h_r)_0=\varnothing\,, \quad \text{if} \quad p<q<r\,.\\ (2) & (h_p)_0\cap(h_q)_0\cap(s'_r)_0=\varnothing\,, \quad \text{if} \quad p<q\,.\\ (3) & (h_p)_0\cap(s'_q)_0\cap(s'_r)_0=\varnothing\,, \quad \text{if} \quad q<r\,.\end{cases}$$

Set $t=(t_0, t_1, t_2)\in\Gamma(S, \mathcal{O}_S(n))^{\oplus 3}$ as below:

$$\begin{cases} t_0 := h_1 \otimes \cdots \otimes h_n \\ t_1 := h_{n+1} \otimes \cdots \otimes h_{2n} \\ t_2 := h_1 \otimes \cdots \otimes h_l \otimes h_{n+l+1} \otimes \cdots \otimes h_{2n} . \end{cases}$$

We define $s_i := s_i' \otimes t_i$ $(i=0, 1, 2)$.

By the conditions (2.8),

$$Y_s = Y_{s'} + \sum_{i=1}^{l} (s_1')_0 \cdot (h_i)_0 + \sum_{i=l+1}^{n} (s_0') \cdot (h_{n+i})_0 + \sum_{i,j=0}^{n} (h_i)_0 \cdot (h_{n+j})_0 - \sum_{i=l+1}^{n} \sum_{j=1}^{l} (h_i)_0 \cdot (h_{n+j})_0 .$$

Since $\deg(h_i)_0 \cdot (h_j)_0 = d$ and $\deg(s_i')_0 \cdot (h_j)_0 = \mathscr{L} \cdot \mathscr{O}_S(1)$, we obtain $\deg Y_s = k$.

It is easy to see that s_0, s_1, s_2 are linearly independent. A general linear combination of s_0, s_1, s_2 is both reduced and irreducible by the theorem of Bertini, thus we obtain the result. (q.e.d.)

Rewriting (2.7), we obtain the following theorem for existence of vector bundles of type (G).

THEOREM 2.9. *Let $\mathscr{L}$ be a very ample line bundle. Then*

$$C_2(\mathscr{L}) \subset \bigcup_{n \in A} \bigcup_{l=0}^{n} [\mathscr{L}^2 - N(n, l; \mathscr{L}) - h^0(\mathscr{L}(-n)) + 3, \mathscr{L}^2 - N(n, l; \mathscr{L})] ,$$

where $A := \{n \in \boldsymbol{Z} \mid n \geqq 0$ and $\mathscr{L}(-n)$ is very ample$\}$.

The set of the right side in (2.9) is rather complicated and in general it is not equal to $[a, \mathscr{L}^2]$ for any integer a. We estimate

$$a_m(\mathscr{L}) := \max\{M \in \boldsymbol{Z} \mid [0, M] \subset D(\mathscr{L}(m))\} \quad \text{and}$$
$$b_m(\mathscr{L}) := \min\{M \in \boldsymbol{Z} \mid [M, \mathscr{L}(m)^2] \subset C_2(\mathscr{L}(m))\}$$

as applications of the previous theorem.

PROPOSITION 2.10. *Let $\mathscr{L}$ be a very ample line bundle on S and $m \gg 0$. Then $a_m(\mathscr{L}) > dm^2 - 2dm^{3/2} + g(\sqrt{m})$, where $g(x)$ is a polynomial with degree at most 2. In fact, $a_m(\mathscr{L}) > dm^2 - \sqrt{2}\, dm^{3/2} + O(m)$.*

PROOF. Let m_0 be a positive integer such that for each integer $p \geqq m_0$, $\mathscr{L}(p)$ is very ample and $H^i(S, \mathscr{L}(p)) = 0$ $(i>0)$. We take an integer m which satisfies $2\sqrt{m} \geqq m_0$.

Let n be an integer such that $0 \leqq n \leqq m - 2\sqrt{m}$. We simply denote $N(n, 0; \mathscr{L}(m))$ and $a_m(\mathscr{L})$ by N_n and a_m respectively. Then for each integer $k \in [N_n, N_n + h^0(\mathscr{L}(m-n)) - 3]$ there exists $s \in \Gamma(S, \mathscr{L}(m-n))^{\oplus 3}$ such that $\deg Y_s = k$, because of (2.7). Since N_n is monotonically increasing, we get the following:

If $N_i+h^0(\mathscr{L}(m-i))-3\geqq N_{i+1}$ for $0\leqq i\leqq n$, then $a_m\geqq N_{n+1}+h^0(\mathscr{L}(m-n-1))-3$.

By the choice of m,

$$h^0(\mathscr{L}(m))=\chi(\mathscr{L}(m))=-m^2+\frac{d}{2}f(m)\ ,$$

where $\chi(\mathscr{L}(m))$ is the Hilbert polynomial of $\mathscr{L}$ and $f(m)$ is a polynomial of m with degree at most one. Hence,

$$h^0(\mathscr{L}(m-n))-3>\frac{d}{3}(m-n)^2 \qquad \text{for} \quad m\gg 0\ .$$

Since

$$N_{n+1}-N_n=d(m-2n-1)+\mathscr{L}\cdot\mathscr{O}_S(1)<\frac{4}{3}dm \qquad \text{for} \quad m\gg 0$$

and $m-n\geqq 2\sqrt{m}$,

$$h^0(\mathscr{L}(m-n))-3+N_n-N_{n+1}>\frac{d}{3}(m-n)^2-\frac{4}{3}dm\geqq 0 \qquad \text{for} \quad m\gg 0\ .$$

Hence,

$$\begin{aligned} a_m &> N_n \\ &> dm(m-2\sqrt{m})+(m-2\sqrt{m})\mathscr{L}\cdot\mathscr{O}_S(1) \\ &= dm^2-2dm^{3/2}+g(\sqrt{m})\ . \end{aligned}$$

For the second part, we have

$$h^0(\mathscr{L}(m-n))-3+N_n-N_{n-1}=\frac{1}{2}d(m-n)^2-dm+o(m)\ .$$

Thus, if $m-n>\sqrt{2m}$ then $a_m=dm^2-\sqrt{2}\,dm^{3/2}+O(m)$. (q.e.d.)

By the definition of $b_m(\mathscr{L})$ $(=\mathscr{L}(m)^2-a_m(\mathscr{L}))$, we thus have the following theorem:

THEOREM 2.11. *For $m\gg 0$, $b_m(\mathscr{L})=\sqrt{2}\,dm^{3/2}+O(m)$. What is almost the same thing, for any c_2 such that $\mathscr{L}(m)^2\geqq c_2>2dm^{3/2}+f(\sqrt{m})$, where $f(x)$ is a polynomial of degree at most 2, there exists a vector bundle $\mathscr{E}$ of type* (G) *such that $c_1(\mathscr{E})=\mathscr{L}(m)$ and $c_2(\mathscr{E})=c_2$.*

COROLLARY 2.12.

$$\lim_{m\to\infty}\frac{b_m(\mathscr{L})}{\mathscr{L}(m)^2}=0 .$$

REMARK 2.13. If $l:=\mathscr{L}^2\geqq\mathscr{L}\cdot\mathscr{O}_S(1)>0$ and $H^i(S,\mathscr{L})=0$ for $i>0$, then $h^0(\mathscr{L}(m))>dm^2/2$ and $N_{n+1}-N_n<dm$, hence we can explicitly estimate $b_m(\mathscr{L})$. For example if $m>l/d+1$ then $b_m(\mathscr{L})<\mathscr{L}(m)^2/2$. If $m>3l/(2d)+1$, then $b_m(\mathscr{L})<\mathscr{L}(m)^2/4$. For a general $\mathscr{L}$, we can obtain explicit but very complicated estimate of $b_m(\mathscr{L})$.

We derive the theorem of Schwarzenberger from the previous proposition.

THEOREM 2.14 (Schwarzenberger ([17])). *Let S be a non-singular projective surface. For each line bundle $\mathscr{L}$ on S and each integer c_2, there exists an integer l and a vector bundle $\mathscr{E}$ of type* (G) *such that* $c_1(\mathscr{E}(l))=\mathscr{E}$, $c_2(\mathscr{E}(l))=c_2$.

PROOF. At first we think about c_1. Let m be an even integer such that $\mathscr{L}(m)$ is very ample. If $\mathscr{E}$ is a vector bundle of type (G) associated with $(\mathscr{L}(m), s)$ for general $s\in\Gamma(S,\mathscr{L}(m))^{\oplus 3}$, then $c_1(\mathscr{E})=\mathscr{L}(m)$ because of (2.1). Hence $c_1(\mathscr{E}(-m/2))=\mathscr{L}$.

Next we consider about c_2. As in the proof of (2.10), a_m denotes $a_m(\mathscr{L})$. Let $k\in[0, a_m]$ and $\mathscr{E}$ be a vector bundle of type (G) associated with $(\mathscr{L}(m), s)$, where $\deg Y_s=k$. Then $c_1(\mathscr{E})^2-4c_2(\mathscr{E})=\mathscr{L}(m)^2-4(\mathscr{L}(m)^2-k)$ from (2.1). On the other hand, $-3\mathscr{L}(m)^2\to-\infty$ as $m\to\infty$ and $\mathscr{L}(m)^2-4(\mathscr{L}(m)^2-a_m)\to\infty$ as $m\to\infty$ by (2.10). Since $c_1^2-4c_2$ is invariant under twisting and $c_1(\mathscr{E})^2\equiv c_1(\mathscr{E}(l))^2$ modulo 4, there exist an even integer m and a vector bundle $\mathscr{E}$ of type (G) associated with $(\mathscr{L}(m), s)$ such that $c_1(\mathscr{E})^2-4c_2(\mathscr{E})=\mathscr{L}^2-4c_2$. Setting $l:=-m/2$, $c_1(\mathscr{E}(l))=\mathscr{L}$ and $c_2(\mathscr{E}(l))=c_2$. Thus $\mathscr{E}$ and l are required. (q.e.d.)

§ 3. Vector bundles of configuration type.

A. General arguments. Let C be a reduced and reducible curve on S satisfying the properties of Y in the definition of type (C) in § 1. The symbols C_i, $\mathring{C}_i$, $\cdots$ denote Y_i, $\mathring{Y}_i$, $\cdots$ in the definition of type (C) and take data (D1)$\sim$(D3) to define a vector bundle $\mathscr{E}$ of type (C) of rank 2 on S. Note that $\mathscr{E}$ has frames $\boldsymbol{e}^0$, $\boldsymbol{e}^1$ on N_0, N_1 respectively, with $\boldsymbol{e}^0=\boldsymbol{e}^1H$ on $N_0\cap N_1$. The following properties are clear from the construction.

PROPOSITION 3.1.

$$e^0=(e_1, e_2)\in\Gamma(\mathring{S}, \mathscr{E}_{\mathring{S}})^{\oplus 2} \quad \textit{and} \quad \det(\mathscr{E}_{\mathring{S}}):=\wedge^2\mathscr{E}_{\mathring{S}}=\mathscr{O}_{\mathring{S}}(\mathring{C}) .$$

We here discuss the structure of bundles of type (C). Set $\mathscr{L}:=\mathscr{O}(C)$ and take a global section s of $\mathscr{L}$ such that $(s)_0=C$. The following lemma is fundamental for our argument.

LEMMA 3.2. *There exists an embedding of $\mathscr{E}$ into $\mathscr{L}^{\oplus 2}$.*

PROOF. Define an $\mathscr{O}_{\mathring{S}}$-morphism $\mathring{\theta}$: $\mathscr{E}_{\mathring{S}}\to\mathscr{L}^{\oplus 2}|\mathring{S}$ using the frames e^0, e^1 as follows:

$$\text{in } N_0, \quad \mathring{\theta}: e^0=(e_1, e_2)\longmapsto\left(\begin{bmatrix} s\\ 0\end{bmatrix}\begin{bmatrix} 0\\ s\end{bmatrix}\right),$$

$$\text{in } N_1, \quad \mathring{\theta}: e^1=(e_1', e_2')\longmapsto\left(\begin{bmatrix} s\\ 0\end{bmatrix}\begin{bmatrix} g_1\\ g_2\end{bmatrix}\right),$$

where $g_1:=-sf_1/f_2$ and $g_2:=s/f_2$. In fact, this is a well-defined embedding, for $\mathring{\theta}(e^0)=\mathring{\theta}(e^1)H$ in $N_0\cap N_1$. So $\mathring{\theta}(\mathscr{E}_{\mathring{S}})$ is a submodule of $\mathscr{L}^{\oplus 2}|\mathring{S}$ isomorphic to $\mathscr{E}_{\mathring{S}}$. Taking the direct image sheaf, we obtain $\mathscr{E}\simeq i_*\mathring{\theta}(\mathscr{E}_{\mathring{S}})\subset\mathscr{L}^{\oplus 2}$. (q.e.d.)

Let $\mathscr{E}'$ be the submodule $i_*\mathring{\theta}(\mathscr{E}_{\mathring{S}})$ of $\mathscr{L}^{\oplus 2}$. We study local structure of $\mathscr{E}$ by studying that of $\mathscr{E}'$ as a submodule of $\mathscr{L}^{\oplus 2}$. Concerning a local frame for $\mathscr{E}'$ we have the following fact.

LEMMA 3.3. *Let $\varphi=(\varphi_1, \varphi_2)\in\mathscr{E}_p^{\oplus 2}$ $(p\in S)$. Then $\theta(\varphi)=(\theta(\varphi_1), \theta(\varphi_2))$ is an $\mathscr{O}_p$-free basis for $\mathscr{E}_p'$ if and only if $(\theta(\varphi_1)\wedge\theta(\varphi_2))_0=C_p$.*

PROOF. φ is an $\mathscr{O}_p$-free basis for $\mathscr{E}_p$ if and only if $(\varphi_1\wedge\varphi_2|\mathring{S}_p)_0=\varnothing$. On the other hand, for any $\varphi=(\varphi_1, \varphi_2)\in\mathscr{E}_p^{\oplus 2}$, we know easily $(\theta(\varphi_1|\mathring{S}_p)\wedge\theta(\varphi_2|\mathring{S}_p))_0=\mathring{C}_p+(\varphi_1|\mathring{S}_p\wedge\varphi_2|\mathring{S}_p)_0$. (q.e.d.)

For each $i=1, \cdots, m$, let D_i be the divisor $(f_{i1})_\infty$ on C_i (cf. (D3)). Take $h_{i2}\in\Gamma(\mathscr{O}_{C_i}(D_i))$ whose zero divisor is D_i and put $h_{i1}:=(f_{i1}|C_i)h_{i2}$. Then $h_{ij}\in\Gamma(\mathscr{O}_{C_i}(D_i))$. Set $\Phi_i:={}^t(-h_{i1}, h_{i2})\in\Gamma(\mathscr{O}_{C_i}(D_i))^{\oplus 2}$ and $\mathscr{M}_i:=\mathscr{O}_{C_i}(D_i)^\vee\otimes\mathscr{L}|_{C_i}$ an invertible sheaf on C_i. Define an $\mathscr{O}_{C_i}$-homomorphism Φ_i: $\mathscr{M}_i\to\mathscr{L}|_{C_i}^{\oplus 2}$ by $\tau_i\mapsto\Phi_i(\tau_i)=\tau_i\otimes\Phi_i$.

The following proposition characterizes $\mathscr{E}'$.

PROPOSITION 3.4. *Let $\varphi\in\mathscr{L}_p^{\oplus 2}$ $(p\in S)$. Then $\varphi\in\mathscr{E}_p'$ if and only if $\varphi|C_i=\Phi_i(\tau_i)$ for some $\tau_i\in\mathscr{M}_{i,p}$, $i=1, \cdots, m$.*

PROOF. Take an open neighborhood U of p. Then one easily sees that $\varphi={}^t(\varphi_1, \varphi_2)\in \Gamma(U, \mathscr{L}^{\oplus 2})$ is in $\Gamma(U, \mathscr{E}')$ if and only if the condition

(a) $$\varphi|\mathring{C}_i=\psi(g_1, g_2)|\mathring{C}_i \qquad \text{for some } \psi_i\in\Gamma(U\cap\mathring{C}_i, \mathscr{O}_{\mathring{C}_i})$$

holds for all $i=1,\cdots,m$. Since $(g_1/g_2)|\mathring{C}_i=-f_1|\mathring{C}_i$ has a unique meromorphic extension f_{i1} on C_i, one can write (a) as the condition

(b) $$\varphi|C_i=(\varphi_2|C_i){}^t(-h_{i1}/h_{i2}, 1)\ .$$

This implies $\varphi_2|C_i\equiv 0\ (\mathrm{mod}(h_{i2})_0)$ and one sees

(c) $\varphi|C_i=\tau_i{}^t(-h_{i1}, h_{i2})$ with some element $\tau_i\in\Gamma(U\cap C_i, \mathscr{M}_i)$. (q.e.d.)

B. Vector bundles of null-correlation type and their Chern classes. In this paragraph, we apply the results of the former paragraph to some special bundle of type (C), which we will call "of type (NC)." We first define it.

Let S be a surface and fix a line bundle $\mathscr{L}$ on S. Assume that $\mathscr{L}$ has sections $s_1,\cdots,s_m$ $(m\geqq 2)$ which satisfy the following condition:

Setting $C_i:=(s_i)_0$ and $C:=\cup_i C_i$, C satisfies the condition of (D1). Set $\sigma_i:=s_1\cdots s_{i-1}s_{i+1}\cdots s_m$ $(i=1,\cdots,m)$. Take N_0, $N_1=\cup_i N_1^i$, $\mathring{S},\cdots$, etc. as in (D2). Take $t_i\in\Gamma(S, \mathscr{L}^{\otimes m-1})$ for $i=1,\cdots,m$ and set

$$H|N_1^i:=\begin{bmatrix}1 & t_i/\sigma_i|N_1^i\\ 0 & s_i/s_{i+1}|N_1^i\end{bmatrix}, \qquad \text{with} \quad s_{m+1}:=s_1\ .$$

Then H is a matrix of type (C). Consequently, we obtain the vector bundle $\mathscr{E}$ of type (C) determined by H on S with the frames e^0 and e^1 on N_0 and N_1 satisfying $e^0=e^1H$.

DEFINITION. A vector bundle of null correlation type, or briefly of type (NC), is a vector bundle of type (C) defined by data as above. We use the term "matrix of type (NC)" also.

REMARK 3.5. We saw before that the null correlation bundle twisted by $\mathscr{O}(1)$ is of type (C) (cf. §1). A matrix of type (NC) is a generalization of the one to construct that bundle (cf. (1.3)).

REMARK 3.6. We also have another definition of type (NC) which is similar to the definition of type (G) by Hosoh. As the same notation as above, we define a reflexive sheaf $\mathscr{E}$ of type (NC) to be the double dual of $\mathscr{F}$ which is defined from the following exact sequence:

$$0\longrightarrow(\mathscr{L}^{\otimes-(m-1)})^{\oplus m}\xrightarrow{H}(\mathscr{L}^{\otimes-(m-2)})^{\oplus m}\oplus\mathscr{O}^{\oplus 2}\longrightarrow\mathscr{F}\longrightarrow 0\ ,$$

where

$$H=\begin{bmatrix} \sigma_2 & & & \\ & \ddots & & 0 \\ & & \sigma_m & \\ 0 & & & \sigma_1 \\ \sigma_1 & \cdots & \sigma_{m-1} & \sigma_m \\ f_1 & \cdots & f_{m-1} & f_m \end{bmatrix},$$

and $(\mathscr{L}^{\otimes-(m-2)})^{\oplus m}$ is regarded as a subsheaf of $\mathscr{O}^{\oplus m}$ through the injection $j\colon (\mathscr{L}^{\otimes-(m-2)})^{\oplus m}\to\mathscr{O}^{\oplus m}$ by tensoring $(\sigma_{12}, \sigma_{23}, \cdots, \sigma_{m1})$ $(\sigma_{ij}:=\sigma_i/s_j=(\prod_{k=1}^{m} s_k)/(s_is_j))$.

In this definition, we do not need to assume the conditions in (D1). But if we do not assume them, the calculation of the second Chern class of the associated bundle is very complex and delicate.

This definition is easily generalized when X is higher dimensional and $\mathscr{E}$ has a higher rank.

Concerning existence of bundles of type (NC), we obtain the following results on $S=\boldsymbol{P}^2$.

THEOREM 3.7. *For each integer* $m\geqq 2$ *and for each integer* c_2 *such that* $m(m-1)/2\leqq c_2\leqq m(m-1)$, *there exists a vector bundle* $\mathscr{E}$ *of type* (NC) *on* $\boldsymbol{P}^2$ *such that* $c_1(\mathscr{E})=m$ *and* $c_2(\mathscr{E})=c_2$.

In the rest of this paper, we prove Theorem 3.7. For this we first calculate the Chern classes of a bundle of type (NC) by using the method of (3.2) and (3.3). Let $\mathscr{L}$, $s_1, \cdots, s_m$ be as above. Remark that $\mathscr{L}^{\otimes m}$ and $s_1\otimes\cdots\otimes s_m$ play the roles of $\mathscr{L}$ and s in (3.2) respectively. $\theta\colon \mathscr{E}\xrightarrow{\sim}\mathscr{E}'\subset(\mathscr{L}^{\otimes m})^{\oplus 2}$ is the "multiplication by s" and $D_i=(t_i/\sigma_i|C_i)_\infty$ for $i=1, \cdots, m$ (cf. (3.3)). As to the first Chern class, $c_1(\mathscr{E})=\det(\mathscr{E})=\mathscr{L}^{\otimes m}$. The second Chern class c_2 is the zero locus of its global section. Thus it suffices to investigate the zero locus of e_1, which vanishes only on $Z=\bigcup_{i\neq j} Z_{ij}=\bigcup_{i\neq j}((s_i)_0\cap(s_j)_0)$.

LEMMA 3.8. *Under the situation in the definition of type* (NC), *let* $p\in Z_{ij}$.

(I) *If neither* t_i *nor* t_j *vanishes at* p, *then the section* e_1 *has zero at* p *of order* 2, *and*

(II) *if both* t_i *and* t_j *have zero of order* 1 *at* p, *then the section* e_1 *has zero at* p *of order* 1 *for general* t_i *and* t_j.

PROOF. For our discussion is local, we may assume that $Z_{ij}=\{p\}$ and C_α is defined by the equation $x_\beta=0$, $\alpha, \beta=i, j$, $\alpha\neq\beta$, where (x_i, x_j)

is a local coordinate at p. Let $\varphi \in \mathscr{E}'_p$. Then (3.5) implies that there exist $\tau_i \in \mathscr{M}_{i,p}$ and $\tau_j \in \mathscr{M}_{j,p}$ such that $\varphi|C_\alpha = \tau_\alpha \Phi_\alpha$ for $\alpha = i, j$. Clearly,

$$\text{(E)} \qquad \Phi_i(\tau_i)|p = \Phi_j(\tau_j)|p \quad \text{in} \quad (\mathscr{L}|p)^{\oplus 2} .$$

Conversely, let $\tau_\alpha \in \mathscr{M}_{\alpha,p}$ $(\alpha = i, j)$ satisfy this equation, then $\Phi_\alpha(\tau_\alpha) \in (\mathscr{L}|C_\alpha)_p^{\oplus 2}$, so there exists $\varphi \in \mathscr{E}'_p$ with $\varphi|C_\alpha = \Phi_\alpha(\tau_\alpha) \in (\mathscr{L}|C_\alpha)_p^{\oplus 2}$. Define an $\mathscr{O}_{\{p\}}$-homomorphism $(\delta\Phi)_{ij}$ from $(\mathscr{M}_i \oplus \mathscr{M}_j)|p$ to $\mathscr{L}|p^{\oplus 2}$ by $(\delta\Phi)_{ij}(\tau_i, \tau_j)|p := -\Phi_i(\tau_i|p) + \Phi_j(\tau_j|p)$. Put the matrix $A_{ij} := [-\Phi_i, \Phi_j]$ and the $\mathscr{O}_p$-module $\mathscr{N}_{ij} := \ker(\delta\Phi)_{ij}$. Then (E) is equivalent to that

$$\text{(E')} \qquad (\tau_i, \tau_j)|p \in \mathscr{N}_{ij} .$$

Let μ_α be a local frame of $\mathscr{M}_\alpha$ at p $(\alpha = i, j)$. If $\operatorname{rank} A_{ij} = 2$, then $\mathscr{N}_{ij} = 0$. Hence τ_α can be written as $\tau_\alpha = x_\alpha \tau'_\alpha \mu_\alpha$ with $\tau'_\alpha \in \mathscr{O}_{C_\alpha,p}$ for x_α is a local coordinate of C_α at p. Thus $\varphi \in \mathscr{L}_p^{\oplus 2}$ is in $\mathscr{E}'_p$ if and only if $\varphi|C_\alpha = x_\alpha \tau'_\alpha \mu_\alpha \Phi_\alpha$. If $\operatorname{rank} A_{ij} = 1$, then $\mathscr{N}_{ij} \cong \boldsymbol{C}$.

If $1 \in \boldsymbol{C}$ corresponds to $(a_i \mu_i|p, a_j \mu_j|p)$, $a_\alpha \in \boldsymbol{C}^*$, then τ_α can be written as $\tau_\alpha = (\tau_{ij} a_\alpha + x_\alpha \tau'_\alpha)\mu_\alpha$ with $\tau_{ij} \in \boldsymbol{C}$, $\tau'_\alpha \in \mathscr{O}_{C_\alpha,p}$. Thus $\varphi \in \mathscr{L}_p^{\oplus 2}$ is in $\mathscr{E}'_p$ if and only if $\varphi|C_\alpha = (\tau_{ij} a_\alpha + x_\alpha \tau'_\alpha)\mu_\alpha$, $\alpha = i, j$.

(I). In this case, $D_\alpha = (\sigma_\alpha|C_\alpha)_0$, $\Phi_\alpha = {}^t(-t_\alpha, \sigma_\alpha)|C_\alpha \in \Gamma(\mathscr{O}_{C_\alpha}((\sigma_\alpha|C_\alpha)_0))^{\oplus 2}$ and $\mathscr{M}_\alpha = \mathscr{O}_{C_\alpha}(-D_\alpha) \otimes \mathscr{L}|C_\alpha^{\otimes m} = \mathscr{L}|C_\alpha$ $(\alpha = i, j)$ (around p). Furthermore, $\Phi_\alpha|p = {}^t(-t_\alpha, 0)|p$ $(\alpha = i, j)$, hence $(\delta\Phi)_{ij}$: $(\mathscr{L}|C_i \oplus \mathscr{L}|C_j)|p \to (\mathscr{L}^{\otimes m})^{\oplus 2}|p$ brings ${}^t(\tau_i, \tau_j)|p$ to ${}^t(-t_i\tau_i + t_j\tau_j, 0)|p$. Then, the rank of the map is 1 and $\mathscr{N}_{ij} \cong \boldsymbol{C}$. Let λ be a local frame of $\mathscr{L}$ at p such that $\mu_\alpha = \lambda|C_\alpha$. Then $t_\alpha = \tilde{t}_\alpha \lambda^{\otimes m-1}$, $s_\alpha = \tilde{s}_\alpha \lambda$ and $\tau_\alpha = \tilde{\tau}_\alpha (\lambda|C_\alpha)^{\otimes m}$ $(\alpha = i, j)$ for $\tilde{t}_\alpha \in \mathscr{O}_p$, $\tilde{s}_\alpha \in \mathscr{O}_p$ and $\tilde{\tau}_\alpha \in \mathscr{O}_{C_\alpha,p}$. $\tau_i \otimes \Phi_i|p = \tau_j \otimes \Phi_j|p$ if and only if $\tilde{t}_i \otimes \tilde{\tau}_i(p) = \tilde{t}_j \otimes \tilde{\tau}_j(p)$. For $(\tilde{t}_i, -\tilde{t}_j)(p) \neq (0, 0)$, we can put $\tilde{\tau}_{ij} := \tilde{\tau}_i/\tilde{t}_j(p) = \tilde{\tau}_j/\tilde{t}_i(p) \in \boldsymbol{C}$. Let α be the one of i and j and β the other. Now $\tilde{\tau}_\alpha(p) = \tilde{t}_\beta(p)\tilde{\tau}_{ij}$. So $\tilde{\tau}_\alpha(p)$ can be written as $\tilde{\tau}_\alpha = \tilde{t}_\beta(p)\tilde{\tau}_{ij} + x_\alpha \xi_\alpha$ with some $\xi_\alpha \in \mathscr{O}_{C_\alpha,p}$. We write $\tilde{t}_\alpha = \tilde{t}_\alpha(p) + x_\alpha t'_\alpha$ with some $t'_\alpha \in \mathscr{O}_{C_\alpha}$. Write $\sigma_\alpha = \tilde{\sigma}_\alpha \lambda^{\otimes m-1}$, $s = \tilde{s}\lambda^{\otimes m}$ and $\tilde{\sigma}_\alpha = x_\alpha \tilde{\sigma}'_\alpha$, where $\tilde{s} \in \mathscr{O}_p$, $\tilde{\sigma}'_\alpha \in \mathscr{O}_{C_\alpha,p}$. Using these elements, $\varphi \in \mathscr{E}'_p \subset \mathscr{L}_p^{\oplus 2}$ is written as

$$\varphi = \left\{ \tilde{\tau}_{ij} \begin{pmatrix} (\tilde{t}_i(p)\tilde{t}_j(p)) \\ 0 \end{pmatrix} + \sum_{\beta \neq \alpha} x_\alpha \begin{pmatrix} t'_\beta(\tilde{t}_\alpha(p) + x_\alpha \xi_\beta) + \tilde{t}_\beta(p)\xi_\beta \\ -\tilde{\sigma}'_\beta \tilde{t}_\alpha(p)\tilde{\tau}_{ij} + x_\beta \xi_\beta \end{pmatrix} + x_i x_j \begin{pmatrix} * \end{pmatrix} \right\} \lambda^{\otimes m} ,$$

where $(*)$ is not important for our arguments, so we neglect it.

Setting $\tilde{\tau}_{ij} = 1$ and $\xi_i = \xi_j = 0$,

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \lambda^{\otimes m} := \left\{ \begin{pmatrix} \tilde{t}_i(p)\tilde{t}_j(p) \\ 0 \end{pmatrix} + \sum_{\beta \neq \alpha} x_\alpha \begin{pmatrix} \tilde{t}'_\beta \tilde{t}_\alpha(p) \\ -\tilde{\sigma}'_\beta \tilde{t}_\alpha(p) \end{pmatrix} + x_i x_j \begin{pmatrix} * \end{pmatrix} \right\} \lambda^{\otimes m} .$$

Recall that the morphism θ: $\mathscr{E} \to (\mathscr{L}^{\otimes m})^{\oplus 2}$ brings $\boldsymbol{e}^0 = (e_1, e_2)$ to

$\left(\begin{bmatrix} s \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ s \end{bmatrix}\right)$. Now, $(\eta, \theta(e_2))$ forms a local frame at p. Indeed, since $\eta \wedge \theta(e_2) = x_i x_j \times$ unit, the condition of (3.3) is satisfied. Represent $\theta(e_1)$ by the local frame:

$$\theta(e_1) = \frac{\tilde{s}}{\eta_1}\eta - \frac{\eta_2}{\eta_1}\theta(e_2) \ ,$$

where

$$\frac{\tilde{s}}{\eta_1} = \text{constant} \cdot x_i x_j + (\text{higher order terms})$$

and

$$\frac{\eta_2}{\eta_1} = \text{constant} \cdot (x_i + x_j) + (\text{higher order terms}) \ .$$

The multiplicities of $\tilde{s}/\eta_1$ and η_2/η_1 are 2 and 1 respectively. Therefore e_1 has zero at p of order 2.

(II). In this case, $D_\alpha = \varnothing$ and $\Phi_\alpha = {}^t(-t_\alpha/\sigma_\alpha, 1)|C_\alpha$ around p, $\alpha = i, j$. Then

$$(\delta\Phi)_{ij} : (\tau_i, \tau_j)|p \longmapsto \left(\frac{t_i}{\sigma_i}\tau_i - \frac{t_j}{\sigma_j}\tau_j \, , \ -\tau_i + \tau_j\right)\Big| p$$

and no coefficients of τ_i and τ_j vanish at p. Then the rank of the homomorphism is 2 for general t_α. In this case $\mathscr{N}_{ij} \simeq 0$ and $(x_j\Phi_i, x_i\Phi_j)$ forms a local frame of $\mathscr{E}'$. Indeed $x_j\Phi_i \wedge x_i\Phi_j = x_i x_j \Phi_i \wedge \Phi_j$ satisfies the condition of (3.3). Represent $\theta(e_1)$ by the local frame:

$$\theta(e_1) = -\frac{s}{x_j\Delta}x_j\Phi_i + \frac{s}{x_i\Delta}x_i\Phi_j \ ,$$

where Δ is the determinant of the matrix $[\Phi_i, \Phi_j]$. Both multiplicities of $s/(x_j\Delta)$ and $s/(x_i\Delta)$ are 1. Therefore the order of zero of e_1 at p is 1. (q.e.d.)

Note that the number of points of Z is $(m(m-1)/2)\mathscr{L}^2 =: l$. Then Lemma 3.8 directly implies the following.

PROPOSITION 3.9. *If the case* (I) *occurs at d points of Z and the case* (II) *occurs at the other $l-d$ points, the then second Chern class $c_2(\mathscr{E})$ of $\mathscr{E}$ is $c_2(\mathscr{E}) = d + l$ (as an integer).*

PROOF OF THEOREM 3.7. When $S = \boldsymbol{P}^2$ and $\mathscr{L} = \mathscr{O}(1)$, for each $m \geqq 2$

there are sections $s_1, \cdots, s_m$ of $\mathscr{L}$ such that $C_i=(s_i)_0 \cong \boldsymbol{P}^1$ satisfy the condition in the definition of type (NC). In this case, each C_i meets the other C_j $(j \neq i)$ at $m-1$ points and Z consists of $M := m(m-1)/2$ points. For any choice of d $(0 \leqq d \leqq M)$ points of them, we can find homogeneous polynomials t_i of degree $m-1$ so that they make the situation as follows: the case (I) takes place at that d points and the case (II) takes place at the other $M-d$ points. Therefore, we obtain the theorem. (q.e.d.)

Note finally that for such m and c_2 as above, c_2-4m^2 takes any negative integers and recall that bundles are simple provided $c_1^2-2c_2<0$ (cf. (1.4)) and that simpleness and stability are equivalent in the case of rank 2 on $\boldsymbol{P}^n$. So the irreducibility of the moduli of stable vector bundles ([2], [7], [13]) implies

COROLLARY 3.10. *Each stable 2-bundle on $\boldsymbol{P}^2$ is a deformation of a vector bundle of type* (NC) *twisted by $\mathcal{O}(n)$.*

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