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Remarks on Blowing-Up of Solutions for Some Nonlinear Schrödinger Equations

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Abstract. We study the blowing-up conditions of solutions for nonlinear Schrödinger equations with interaction which does not satisfy known Glassey's condition [4]. We also give some remarks on the blowing-up conditions on an exterior domain with a star-shaped complement under the Dirichlet boundary condition and on a complement of a ball under the Neumann boundary condition. Finally, we show global existence of solutions for the equation: $i\frac{\partial u}{\partial t}=\Delta u+\left(\frac{1}{|x|^2}*|u|^2\right)u$.

§1. Introduction.

In this paper we are concerned with the problem of blowing-up of solutions to the following nonlinear Schrödinger equations:

$$i\frac{\partial u}{\partial t}=\Delta u+f(|u|^2)u\ , \qquad t\geqq 0,\ x\in\Omega\ , \tag{1.1}$$

$$i\frac{\partial u}{\partial t}=\Delta u+(V*|u|^2)u\ , \qquad t\geqq 0,\ x\in\Omega\ . \tag{1.1'}$$

Here $i=\sqrt{-1}$, Ω is a smooth domain in $\boldsymbol{R}^n$ $(n\geqq 3)$ and f, V are real valued functions. For the case $\Omega=\boldsymbol{R}^n$ we study the blowing-up conditions for the Cauchy problem (1.1) (or (1.1')) with initial data (1.2):

$$u(0, x)=u_0(x)\ , \qquad x\in\Omega\ . \tag{1.2}$$

For the case $\Omega\neq\boldsymbol{R}^n$ we study the blowing-up conditions for the mixed problem (1.1) (or (1.1')) with the initial condition (1.2) and the Dirichlet boundary condition (1.3) or the Neumann boundary condition (1.4):

$$u(t, x)=0\ , \qquad t\geqq 0,\ x\in\partial\Omega\ , \tag{1.3}$$

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(1.4) $$\frac{\partial u}{\partial n}=0\ , \qquad t\geqq 0,\ x\in\partial\Omega\ .$$

For local interaction $f(|u|^2)u$ Glassey [4] studied the blowing-up conditions and showed that if the nonlinear term $f(|u|^2)u$ satisfies the following condition (G), solutions of (1.1) and (1.2) blow up in a finite time for suitable initial data.

(G) "$f(t)$ is a real-valued function from $\boldsymbol{R}_+$ $(=[0, +\infty))$ to $\boldsymbol{R}^1$ and there is a constant $c>1+2/n$ such that $cF(t)\leqq tf(t)$ for all $t\geqq 0$, where $F(t)=\int_0^t f(s)ds$."

In particular, single power interaction $f(|u|^2)u=|u|^{p-1}u$ with $p>1+4/n$ satisfies the condition (G). (For $p=1+4/n$, see, e.g., Cazenave and Weisslar [1], Nawa [13], Nawa and M. Tsutsumi [14], M. Tsutsumi [18], Weinstein [20].) However the interaction

(1.5) $$f(|u|^2)u=|u|^{p-1}u+|u|^{q-1}u \qquad \text{with} \quad 1<p<1+\frac{4}{n}<q$$

does not satisfy Glassey's condition (G). For nonlocal interaction $(V*|u|^2)u$ Matsumoto and Mochizuki [9] showed that if $V(x)$ satisfies the following conditions:

(V.1) $$V(-x)=V(x)\ ,$$

(V.2) $$|V(x)|\leqq\frac{M}{|x|^\lambda}\ , \qquad 0<\lambda<n\ , \quad M>0\ ,$$

(V.3) $$z\cdot\nabla V(x)\leqq -cV(x) \qquad \text{for some} \quad c\geqq 2\ ,$$

for all $x\in\boldsymbol{R}^n$, then the blowing-up occurs for suitable initial data. In the case $V(x)=1/|x|^\lambda$ the condition (V.3) means $2\leqq\lambda<n$. The condition (V.3) $(c>2)$ corresponds to Glassey's condition (G). However the interaction

(1.6) $$\left(\left(\frac{1}{|x|^\lambda}+\frac{1}{|x|^\mu}\right)*|u|^2\right)u \qquad \text{with} \quad 0<\lambda<2<\mu<n$$

does not satisfiy the condition (V.3).

One of the main purposes of this paper is to show that even for interactions (1.5) and (1.6) solutions for the Cauchy problem (1.1) (or (1.1$'$)) and (1.2) blow up in a finite time for suitable initial data (see section 3).

The second purpose of this paper is to give the blowing-up conditions for the Dirichlet problem (1.1)-(1.3) in an exterior domain Ω with a star-shaped complement, and for the Neumann problem (1.1), (1.2) and (1.4) in the complement of a ball. Kavian [6] studied the blowing-up of solutions for the problem (1.1)-(1.3) with $F(|u|^2)u=|u|^{p-1}u$, $p\geqq 5$, on $\Omega=(\text{ball})^c$ and considered only some examples for the Neumann boundary condition. We shall prove the pseudo-conformal conservation law under the Neumann boundary condition and give the blowing-up conditions for the problem (1.1), (1.2) and (1.4).

In this paper we are mainly concerned with the blowing-up conditions. But we just mention that, recently, several articles have appeared ([1], [10]-[14], [19], [21], [22]) in which the behaviour of the solutions of (1.1)-(1.2) (with $\Omega=\boldsymbol{R}^n$) near the blow-up time is studied for some interactions.

Finally, we show that the global existence of solutions for the Cauchy problem (1.1′)-(1.2) with nonlocal interaction $(V*|u|^2)u=(1/|x|^2*|u|^2)u$ can be proved for small initial data. This is an analogue of Weinstein's work [20].

We use the following notations throughout this paper.

$$\nabla=(\nabla_1,\cdots,\nabla_n)\ ,\quad \nabla_j=\frac{\partial}{\partial x_j}\ ,\quad \alpha=\frac{n+2}{n-2}\ \ (n\geqq 3)\ ,$$

$$\|u\|_p=\|u\|_{L^p(\Omega)}\ \ (p>0)\ ,$$

$$(V*|u|^2)(x)=\int_\Omega V(x-y)|u|^2(y)dy\ ,$$

$$\Sigma_o=\{u\in H_o^1(\Omega)\ ;\ |x|u(x)\in L^2(\Omega)\}\ ,$$

$$\Sigma=\{u\in H^1(\Omega)\ ;\ |x|u(x)\in L^2(\Omega)\}\ ,$$

$$H_r^1=\{u\in H^1(\boldsymbol{R}^n)\ ;\ u \text{ is radially symmetric}\}\ .$$

§2. Preliminaries.

In this section we collect several identities which are used throughout this paper. First, we consider the problem (1.1)-(1.3). We assume that $f\in C^1([0,+\infty))$ is a real-valued function and that Ω is a smooth domain with compact boundary. We also assume that there is a solution $u\in C^1([0,T);L^2(\Omega))\cap C([0,T);H_o^1(\Omega)\cap H^2(\Omega))$ of the problem (1.1)-(1.3) for some $T>0$. We put $F(t)=\int_0^t f(s)ds$. Then we have the following lemma.

LEMMA 2.1. *Let $n\geqq 3$. We assume that $f(t^2)t^2$, $F(t^2)\leqq Ct^{\alpha+1}$ for large $t>0$ where $C>0$ is a constant. Let $u_o\in H_o^1(\Omega)$ and let $u(t)$ be a solution* *in* $C^1([0, T); L^2(\Omega))\cap C([0, T); H_0^1(\Omega)\cap H^2(\Omega))$ *of* (1.1)-(1.3) *for some* $T>0$. *Then for any* $t<T$ *we have*

$$\|u(t)\|_2=\|u_0\|_2\ , \tag{2.1}$$

$$E(u(t))=\frac{1}{2}\|\nabla u(t)\|_2^2-\frac{1}{2}\int_\Omega F(|u(t)|^2)dx=E(u_0)\ . \tag{2.2}$$

Moreover, if $u_0\in\Sigma_0\cap H^2(\Omega)$, *the solution* u *satisfies* $|x|u(t, x)\in L^2(\Omega)$ *for* $0<t<T$ *and the following identity holds*:

$$\begin{aligned}\|\,|x|u(t)\|_2^2=\|\,|x|u_0\|_2^2-4\Big(\mathrm{Im}\int_\Omega(x\cdot\nabla u_0(x))dx\Big)t\\ +4\int_0^t(t-s)\Big\{2\|\nabla u(s)\|_2^2+n\int_\Omega F(|u(s, x)|^2)dx\\ -n\int_\Omega f(|u(s, x)|^2)|u(s, x)|^2dx\\ -\int_{\partial\Omega}(x\cdot n(x))|\nabla u(s, x)|^2d\Gamma\Big\}ds\ .\end{aligned} \tag{2.3}$$

This lemma is well-known (see, e.g., Glassey [4], Ginibre and Velo [2] for $\Omega=\boldsymbol{R}^n$, Kavian [6], Ōtani [15] for general domains, and see also the proof of Lemma 2.2.).

Next, we consider the problem (1.1), (1.2) and (1.4). We assume that there is a solution $u\in C^1([0, T); H^1(\Omega))\cap C([0, T); H^3(\Omega))$ of the problem (1.1), (1.2) and (1.4) for some $T>0$. Recall that $\Sigma=\{u\in H^1(\Omega)\ ;\ |x|u(x)\in L^2(\Omega)\}$.

LEMMA 2.2. *Let* f *and* F *be as in Lemma* 2.1 *and* $u_0\in H^1(\Omega)$. *Let* $u(t)$ *be a solution in* $C^1([0, T); H^1(\Omega))\cap C([0, T); H^3(\Omega))$ *of* (1.1), (1.2) *and* (1.4) *for some* $T>0$. *Then the identities* (2.1) *and* (2.2) *also hold for all* $0\leqq t<T$. *Moreover, if* $u_0\in\Sigma\cap H^3(\Omega)$, *then* $|x|u(t, x)\in L^2(\Omega)$ *for* $0<t<T$ *and the solution* u *satisfies*

$$\begin{aligned}\|\,|x|u(t)\|_2^2=\|\,|x|u_0\|_2^2-4\Big(\mathrm{Im}\int_\Omega(x\cdot\nabla u_0(x))u_0(x)dx\Big)t\\ +4\int_0^t(t-s)\Big[2\|\nabla u(s)\|_2^2+n\int_\Omega F(|u(s, x)|^2)dx\\ -n\int_\Omega f(|u(s, x)|^2)|u(s, x)|^2dx+\int_{\partial\Omega}(x\cdot n(x))\Big\{|\nabla u(s, x)|^2\\ -F(|u(s, x)|^2)-\mathrm{Im}\Big(\frac{\partial u}{\partial s}(s, x)\overline{u(s, x)}\Big)\Big\}d\Gamma\Big]ds\ .\end{aligned} \tag{2.4}$$

PROOF. We have only to show (2.4). We prove it similarly to

Kavian [6]. Let $\phi \in C_0^4(\boldsymbol{R}^n)$ be a function satisfying $\phi(x) \geqq 0$, $\phi \not\equiv 0$. Put

$$V(t)=\frac{1}{2}\int_{\Omega}\phi(x)|u(t, x)|^2 dx . \tag{2.5}$$

From (1.1) we have

$$\begin{aligned} V'(t) &= \mathrm{Re}\int_{\Omega}\phi(x)\overline{u(t, x)}\frac{\partial u(t, x)}{\partial t}dx \\ &= \mathrm{Im}\int_{\Omega}\phi(x)\overline{u(t, x)}\Delta u(t, x)dx \\ &= -\mathrm{Im}\int_{\Omega}\nabla\phi(x)\cdot\nabla u(t, x)\overline{u(t, x)}dx . \end{aligned} \tag{2.6}$$

By integrating by parts and using (1.4) we have, for $h \in \boldsymbol{R}^1$,

$$\begin{aligned} &V'(t+h)-V'(t) \\ &\quad = -\mathrm{Im}\int_{\Omega}\nabla\phi\cdot\nabla(u(t+h, x)-u(t, x))(\overline{u(t+h, x)}-\overline{u(t, x)})dx \\ &\qquad +2\,\mathrm{Im}\int_{\Omega}(u(t+h, x)-u(t, x))\nabla\phi\cdot\nabla\overline{u(t, x)}dx \\ &\qquad +\mathrm{Im}\int_{\Omega}(u(t+h, x)-u(t, x))\Delta\phi\overline{u(t, x)}dx \\ &\qquad -\mathrm{Im}\int_{\partial\Omega}(u(t+h, x)-u(t, x))\overline{u(t, x)}\frac{\partial\phi}{\partial n}d\Gamma . \end{aligned} \tag{2.7}$$

Under the assumption on the regularity of the solution u, we can see from (2.7) that V is a C^2-function and satisfies

$$\begin{aligned} V''(t) &= 2\,\mathrm{Im}\int_{\Omega}\frac{\partial u(t, x)}{\partial t}\nabla\phi\cdot\nabla\overline{u(t, x)}dx \\ &\quad +\mathrm{Im}\int_{\Omega}\frac{\partial u(t, x)}{\partial t}\Delta\phi\overline{u(t, x)}dx \\ &\quad -\mathrm{Im}\int_{\partial\Omega}\frac{\partial u(t, x)}{\partial t}\overline{u(t, x)}\frac{\partial\phi}{\partial n}d\Gamma . \end{aligned} \tag{2.8}$$

The first term of the right hand side of (2.8) is equal to

$$\begin{aligned} &2\,\mathrm{Im}\int_{\Omega}\nabla\phi\cdot\nabla\overline{u(t, x)}(-i\Delta u(t, x)-if(|u(t, x)|^2)u(t, x))dx \\ &\quad = -2\,\mathrm{Re}\int_{\Omega}\nabla\phi\cdot\nabla\overline{u(t, x)}\Delta u(t, x)dx-\int_{\Omega}\nabla\phi\cdot\nabla F(|u(t, x)|^2)dx \\ &\quad = 2\,\mathrm{Re}\int_{\Omega}\nabla(\nabla\phi\cdot\nabla\overline{u(t, x)})\cdot\nabla u(t, x)dx-\int_{\Omega}\nabla\phi\cdot\nabla F(|u(t, x)|^2)dx \end{aligned}$$

$$
\begin{aligned}
&=2\int_{\Omega}(H(\phi)\nabla u(t, x)|\nabla u(t, x))dx+\int_{\partial\Omega}\frac{\partial\phi}{\partial n}|\nabla u(t, x)|^2 d\Gamma \\
&\quad-\int_{\Omega}\Delta\phi|\nabla u(t, x)|^2-\int_{\partial\Omega}\frac{\partial\phi}{\partial n}F(|u(t, x)|^2)d\Gamma \\
&\quad+\int_{\Omega}\Delta\phi F(|u(t, x)|^2)dx ,
\end{aligned}
$$

where

$$
(H(\phi)\nabla u(t, x)|\nabla u(t, x))=\sum_{1\leq j,k\leq n}\nabla_j\nabla_k\phi\nabla_j u(t, x)\nabla_k\overline{u(t, x)} .
$$

On the other hand, the second term of the right hand side of (2.8) is equal to

$$
\begin{aligned}
&\operatorname{Im}\int_{\Omega}\Delta\phi\overline{u(t, x)}(-i\Delta u(t, x)-if(|u(t, x)|^2)u(t, x))dx \\
&\quad=-\operatorname{Re}\int_{\Omega}\Delta\phi\overline{u(t, x)}\Delta u(t, x)dx-\int_{\Omega}\Delta\phi f(|u(t, x)|^2)|u(t, x)|^2 dx \\
&\quad=\int_{\Omega}\Delta\phi|\nabla u(t, x)|^2 dx+\frac{1}{2}\int_{\partial\Omega}|u(t, x)|^2 n(x)\cdot\nabla(\Delta\phi)d\Gamma \\
&\quad\quad-\frac{1}{2}\int_{\Omega}|u(t, x)|^2\Delta^2\phi dx-\int_{\Omega}\Delta\phi f(|u(t, x)|^2)|u(t, x)|^2 dx .
\end{aligned}
$$

Hence we have the following identity:

$$
\begin{aligned}
(2.9)\qquad V''(t)&=2\int_{\Omega}(H(\phi)\nabla u(t, x)|\nabla u(t, x))dx \\
&\quad+\int_{\Omega}\Delta\phi\{F(|u(t, x)|^2)-f(|u(t, x)|^2)|u(t, x)|^2\}dx \\
&\quad-\frac{1}{2}\int_{\Omega}\Delta^2\phi|u(t, x)|^2 dx+\int_{\partial\Omega}\frac{\partial\phi}{\partial n}\{|\nabla u(t, x)|^2-F(|u(t, x)|^2)\}d\Gamma \\
&\quad+\int_{\partial\Omega}\frac{\partial\Delta\phi}{\partial n}|u(t, x)|^2 d\Gamma-\operatorname{Im}\int_{\partial\Omega}\frac{\partial\phi}{\partial n}\frac{\partial u(t, x)}{\partial t}\overline{u(t, x)}d\Gamma .
\end{aligned}
$$

We take the function $\zeta: \boldsymbol{R}_+\to\boldsymbol{R}_+$ such that

$$
\zeta(-t)=\zeta(t) ,\quad \zeta(t)=1 \text{ for } |t|\leqq 1 ,\quad \zeta(t)=0 \text{ for } |t|\geqq 2 ,
$$
$$
\zeta'(t)\leqq 0 \text{ for } t\geqq 0 \text{ and } \zeta\in C^\infty .
$$

For each integer $m\geqq 1$ we put $\phi_m(x)=\frac{1}{2}|x|^2\zeta(|x|/m)$ and define $V_m(t)$ as $V(t)$ with $\phi=\phi_m$ in (2.5). Then $V_m(t)$ satisfies the assumption in the argument above and we can apply (2.9) for this function $V_m(t)$. Since $V_m(t)$ is a C^2-function we have

$$V_m(t) = V_m(0) + V'_m(0)t + \int_0^t (t-s) V''_m(s) ds \ . \tag{2.10}$$

By the properties of the function ζ and ϕ_m, we obtain as $m \to +\infty$

$$V_m(t) \to \frac{1}{4}\int_\Omega |x|^2 |u(t, x)|^2 dx \ , \tag{2.11}$$

$$V'_m(0) \to -\mathrm{Im}\int_\Omega (x\cdot\nabla u_0(x))\overline{u_0(x)} dx \ , \tag{2.12}$$

$$V''_m(t) \to W(t) \ , \tag{2.13}$$

where

$$W(t) = 2\int_\Omega |\nabla u(t, x)|^2 dx + n\int_\Omega \{F(|u(t, x)|^2) - |u(t, x)|^2 f(|u(t, x)|^2)\} dx$$
$$+\int_{\partial\Omega} (x\cdot n(x))\left\{|\nabla u(t, x)|^2 - F(|u(t, x)|^2) - \mathrm{Im}\left(\frac{\partial u(t, x)}{\partial t}\overline{u(t, x)}\right)\right\} dx \ .$$

Therefore we can conclude that identity (2.4) holds. □

Next, we consider the problem (1.1′), (1.2) and (1.3). We may assume that the origin o belongs to Ω. We assume that $V \in C^1(\Omega \setminus \{o\})$ satisfies the following conditions:

$$V(-x) = V(x) \ , \tag{W.1}$$

$$|V(x)| \leqq \frac{C}{|x|^\lambda} + \frac{D}{|x|^\mu} \quad \text{with} \quad 0 < \lambda \leqq \mu < \min(4, n) \ , \tag{W.2}$$

$$|\nabla V(x)| \leqq \frac{C}{|x|^{\lambda+1}} + \frac{D}{|x|^{\mu+1}} \tag{W.3}$$

for all $x \in \Omega \setminus \{o\}$, where C and D are positive constants. We also assume that for $u_0 \in \Sigma \cap H^2(\Omega)$ there is a solution $u \in C^1([0, T); L^2(\Omega)) \cap C([0, T); \Sigma \cap H^2(\Omega))$ of the problem (1.1′), (1.2) and (1.3) for some $T > 0$.

LEMMA 2.3. *Assume* (W.1)-(W.3) *and suppose in addition that* $\mu + 1 < n$. *Let* $u(t)$ *be a solution in* $C^1([0, T); L^2(\Omega)) \cap C([0, T); \Sigma \cap H^2(\Omega))$ *of* (1.1′), (1.2) *and* (1.3) *for some* $T > 0$. *Then* $u(t)$ *satisfies the following identities*:

$$\|u(t)\|_2 = \|u_0\|_2 \ , \tag{2.14}$$

$$E(u(t)) = \frac{1}{2}\|\nabla u(t)\|_2^2 - \frac{1}{2}P(u(t)) = E(u_0) \ , \tag{2.15}$$

where $P(u(t))=\frac{1}{2}\iint_{\Omega\times\Omega}V(x-y)|u(t,x)|^2|u(t,y)|^2dxdy$. *Moreover, we have*

$$\| |x|u(t)\|_2^2=\| |x|u_0\|_2^2-4\left(\mathrm{Im}\int_\Omega(x\cdot\nabla u_0(x))\overline{u_0(x)}dx\right)t \tag{2.16}$$
$$+4\int_0^t(t-s)\left\{2\|\nabla u(s)\|_2^2-\frac{1}{2}\int_\Omega((x\cdot\nabla V)*|u(s)|^2)|u(s,x)|^2dx\right.$$
$$\left.-\int_{\partial\Omega}(x\cdot n(x))|\nabla u(s,x)|^2d\Gamma\right\}ds\,.$$

The proof of Lemma 2.3 is analogous to the proof of Lemmas 2.1 and 2.2 (see also, e.g., Kavian [6], Matsumoto and Mochizuki [9]).

§3. Blowing-up condition for $f(|u|^2)u=|u|^{p-1}u+|u|^{q-1}u$.

In this section we consider the Cauchy problem (1.1) and (1.2) with the interaction:

$$f(|u|^2)u=|u|^{p-1}u+|u|^{q-1}u\,,\qquad 1<p<1+\frac{4}{n}<q<\alpha\ \ (n\geqq 3)\,. \tag{3.1}$$

For such nonlinearity it is known that (1.1)-(1.2) has a unique local solution in the class $C^1([0,T);H^{k-2}(\boldsymbol{R}^n))\cap C([0,T);\Sigma\cap H^k(\boldsymbol{R}^n))$ for $k=1, 2$ (see Ginibre and Velo [1], Kato [5]). We state the main result of this section.

THEOREM 3.1. *We assume that* $u_0\in\Sigma\cap H^2(\boldsymbol{R}^n)$ *satisfies either of the following conditions* (3.2) *and* (3.3):

$$E(u_0)<-\frac{e_0}{(q-1)n}\,, \tag{3.2}$$

$$E(u_0)\geqq-\frac{e_0}{(q-1)n}\,,\quad \mathrm{Im}\int_{\boldsymbol{R}^n}(x\cdot\nabla u_0(x))\overline{u_0(x)}dx>0\quad and \tag{3.3}$$
$$2\left|\mathrm{Im}\left(\int_{\boldsymbol{R}^n}(x\cdot\nabla u_0(x))\overline{u_0(x)}dx\right)\right|^2\geqq(e_0+(q-1)nE(u_0))\|\,|x|u_0\|_2^2$$

where e_0 *is a positive constant depending only on* n, p, q, *and* $\|u_0\|_2$. *Then the solution* u *of* (1.1)-(1.2) *blows up in a finite time, more precisely, there is a finite time* $T_1>0$ *such that*

$$\lim_{t\to T_1}\|\nabla u(t)\|_2=\infty\,. \tag{3.4}$$

REMARK 3.1. e_0 is explicitly given by

$$e_o = C_1 \|u_o\|_2^{\kappa}\,, \qquad \kappa = \frac{4\theta}{\{2-(\alpha+1)(1-\theta)\}}\,, \tag{3.5}$$

where θ is determined by $p+1=2\theta+(\alpha+1)(1-\theta)$, C_1 is a positive constant depending only on n, p and q and $C_1 \to +\infty$ as $q \to 1+4/n$ (see (3.11) and (3.12)). We can show that condition (3.2) is satisfied for suitable initial data at least for $p<p_o$, where p_o is some constant in $(1, 1+4/n)$. In fact, for any $v_o \in \Sigma \cap H^2(\boldsymbol{R}^n)$ we can show that there is a large constant $\lambda_o>0$ such that the initial data $u_o(x)=\lambda v_o(x)$ satisfies the condition (3.2) for all $\lambda \geqq \lambda_o$.

REMARK 3.2. The class of solution in Theorem 3.1 is $C^1([0, T); L^2(\boldsymbol{R}^n)) \cap C([0, T); H^2(\boldsymbol{R}^n))$. Note that for the initial data $u_o \in \Sigma \cap H^2(\boldsymbol{R}^n)$ we have $|x|u(t, x) \in L^2(\boldsymbol{R}^n)$ for $t>0$, and hence $u \in C([0, T); \Sigma \cap H^2(\boldsymbol{R}^n))$. For a weak solution $u \in C([0, T); \Sigma)$, satisfying the integral equation associated with (1.1)-(1.2), we can prove the same result by using the well-known pseudoconformal conservation law (see, e.g., Ginibre and Velo [2]).

REMARK 3.3. Glassey's condition is not satisfied for such $f(|u|^2)u$ as given in (3.1). It seems that the blowing-up of solution more likely occurs in the case of Theorem 3.1 than the single power nonlinearity case $f(|u|^2)u=|u|^{p-1}u$ with $p>1+4/n$, which satisfies Glassey's condition (G). The condition (3.2) or (3.3) for initial data is more restricted than that of the case $f(|u|^2)u=|u|^{p-1}u$ (see, e.g., Glassey [4]), but we do not know whether these conditions are optimal or not.

For $f(|u|^2)u=|u|^{p-1}u+|u|^{q-1}u$, $1<p<q$, we can summarize as follows:
(i) if $q<1+4/n$, then (1.1)-(1.2) has global solutions for all initial data;
(ii) if $p \geqq 1+4/n$, then Glassey's condition holds and the finite time blowing-up occurs;
(iii) if $p<1+4/n<q<\alpha$, then Glassey's condition does not hold, but Theorem 3.1 is applicable to this case;
(iv) if $p<1+4/n=q$, we do not know whether the finite time blowing-up occurs or not.

For $f(|u|^2)u=|u|^{p-1}u-|u|^{q-1}u$, $1<p \neq q$, we can summarize as follows:
(v) if $p<q<\alpha$ or $p<1+4/n$, then (1.1)-(1.2) has global solutions for all initial data;
(vi) if $p \geqq 1+4/n>q$ or $p>q \geqq 1+4/n$, then Glassey's condition holds and the finite time blowing-up occurs.

REMARK 3.4. Theorem 3.1 can be generalized as follows. We assume that (f.1) $$f(t)=f_1(t)+f_2(t)\ , \qquad f_j(t)\in C^1([0,\ +\infty))$$

and putting $F_j(t)=\int_0^t f_j(s)ds$ $(j=1,\ 2)$,

(f.2) $$f_1(t)t\leqq Ct^{(p+1)/2}\ , \qquad \frac{q+1}{2}F_2(t)\leqq f_2(t)t\ ,$$

with

$$1<p<1+\frac{4}{n}<q<\alpha\ ,$$

for sufficiently large $t>0$, where C is a constant. Note that (f.2) implies that for sufficiently large $t>0$ there is a constant $C>0$ such that

$$F_1(t)\leqq Ct^{(p+1)/2}\ , \qquad f_2(t)t\geqq Ct^{(q+1)/2}\ .$$

Under the assumptions (f.1) and (f.2), the solutions of (1.1)-(1.2) blow up in a finite time for suitable initial data.

In this section, unless otherwise stated, the region of integration in x is always understood to be $\boldsymbol{R}^n$.

PROOF (OF THEOREM 3.1). For the nonlinearity of $f(t)$ in Theorem 3.1, Lemma 2.1 implies that

(3.6) $$\|\,|x|u(t)\|_2^2=\|\,|x|u_0\|_2^2-4\left(\mathrm{Im}\int(x\cdot\nabla u_0(x))\overline{u_0(x)}dx\right)t$$
$$+4\int_0^t(t-s)W(s)ds\ ,$$

where

(3.7) $$W(s)=2\|\nabla u(s)\|_2^2+n\left\{\frac{2}{p+1}-1\right\}\|u(s)\|_{p+1}^{p+1}$$
$$+n\left\{\frac{2}{q+1}-1\right\}\|u(s)\|_{q+1}^{q+1}\ .$$

We note that Lemma 2.1 also implies that $u\in C([0,\ T);\ \Sigma\cap H^2(\boldsymbol{R}^n))$. Using the conservation of energy we can eliminate the L^{q+1}-norm from (3.7). Hence we obtain

(3.8) $$W(s)=\left\{2-\frac{(q-1)n}{2}\right\}\|\nabla u(s)\|_2^2$$
$$+\left\{\frac{n(q-p)}{(p+1)}\right\}\|u(s)\|_{p+1}^{p+1}+(q-1)nE(u_0)\ .$$

Note that $\delta\equiv\{2-(q-1)n/2\}$ is negative by the assumption $1+4/n<q$. By Gagliardo and Nirenberg's inequality, we have

$$\|u(s)\|_{p+1}^{p+1}\leqq M\|u(s)\|_2^{2\theta}\|\nabla u(s)\|_2^{(\alpha+1)(1-\theta)}\ , \tag{3.9}$$

where M is a constant depending only on n and p, and $\theta\in(0,1)$ is determined by $p+1=2\theta+(\alpha+1)(1-\theta)$. The assumption $p<1+4/n$ implies that

$$\eta\equiv(\alpha+1)(1-\theta)<2\ . \tag{3.10}$$

Here we consider the function $h(s)$:

$$h(s)=\delta s^2+\gamma M\|u_0\|^{2\theta}s^\eta\ , \tag{3.11}$$

where $\gamma\equiv(q-p)n/(p+1)>0$ and M is a constant in (3.9). By (3.10) $h(s)$ achieves its maximum

$$e_0=C_1\|u_0\|_2^{4\theta/(2-\eta)} \tag{3.12}$$

on $s\in(0,+\infty)$, where C_1 is a positive constant depending only on n, p and q. From (3.8) and the conservation of L^2-norm of u, we obtain

$$W(s)\leqq e_0+(q-1)nE(u_0)\ , \tag{3.13}$$

for all $s<T$. (3.6) and (3.13) yield the following inequality:

$$\begin{aligned}\|\,|x|u(t)\|_2^2\leqq\|\,|x|u_0\|_2^2-4\Big(\mathrm{Im}\int(x\cdot\nabla u_0(x))\overline{u_0(x)}dx\Big)t\\+2(e_0+(q-1)nE(u_0))t^2\ .\end{aligned} \tag{3.14}$$

Therefore, under the assumptions in Theorem 3.1 there is a finite time $T^*>0$ such that

$$\lim_{t\to T^*}\|\,|x|u(t)\|_2^2=0\ . \tag{3.15}$$

By the conservation of L^2-norm and the following inequality:

$$\|u(t)\|_2^2\leqq M'\|\,|x|u(t)\|_2\|\nabla u(t)\|_2\ ,\qquad u\in\Sigma\ , \tag{3.16}$$

we can conclude that there is a finite time $0<T^{**}$ $(\leqq T^*)$ such that

$$\lim_{t\to T^{**}}\|\nabla u(t)\|_2^2=+\infty\ . \tag{3.17}$$ □

Next, we consider the nonlocal version of Theorem 3.1. We consider the problem (1.1′)-(1.2) with the following nonlocal interaction:

$$(3.18)\qquad (V*|u|^2)u=(V_1*|u|^2)u+(V_2*|u|^2)u\ ,$$
$$V_1(x)=\frac{1}{|x|^\lambda}\quad\text{and}\quad V_2(x)=\frac{1}{|x|^\mu}$$

with $0<\lambda<2<\mu<\min(4, n)$ and $\mu+1<n$. We note that for any $u_0\in H^2(\boldsymbol{R}^n)$, there is a solution $u\in C^1([0, T); L^2(\boldsymbol{R}^n))\cap C([0, T); H^2(\boldsymbol{R}^n))$ for the problem (1.1′)-(1.2) and if $u_0\in\Sigma$, then $u(t)\in\Sigma$ for all $t>0$ (see Ginibre and Velo [3]).

THEOREM 3.2. *We assume that $u_0\in\Sigma\cap H^2(\boldsymbol{R}^n)$ and satisfies either of the following conditions* (3.19) *and* (3.20): *for some $e_1>0$*

$$(3.19)\qquad E(u_0)<-\frac{e_1}{2\mu}\ ,$$

$$(3.20)\qquad E(u_0)\geqq-\frac{e_1}{2\mu}\ ,\quad \mathrm{Im}\int(x\cdot\nabla u_0(x))u_0(x)dx>0\quad \textit{and}$$
$$2\left|\mathrm{Im}\int(x\cdot\nabla u_0(x))\overline{u_0(x)}dx\right|\geqq(e_1+2\mu E(u_0))\|\,|x|u_0\|_2^2\ ,$$

where e_1 is a positive constant depending only on n, λ, μ, and $\|u_0\|_2$. Then the solution u of (1.1′)-(1.2) *blows up in a finite time.*

This theorem can be proved in the same way as the proof of Theorem 3.1 by using Lemma 2.3, therefore we omit the details. We only note that we use the following inequalities to prove Theorem 3.2. Put

$$(3.21)\qquad P(u)=\frac{1}{2}\int_\Omega(V*|u|^2)|u|^2dx\ ,$$

for any $u\in H_0^1(\Omega)$, where Ω is a smooth domain in $\boldsymbol{R}^n$ and $V(x)=1/|x|^\lambda$ with $0<\lambda<\min(4, n)$. Then the Hardy-Littlewood-Sobolev inequality implies

$$(3.22)\qquad P(u)\leqq C\|u\|_2^{4-\lambda}\|\nabla u\|_2^\lambda\ ,$$

for some positive constant C. (3.22) is used instead of (3.9). We also remark that the fact $u(t)\in\Sigma$ for all $t>0$ and the assumption $\mu+1<n$ yield the following identity:

$$(3.23)\qquad \int x\cdot\nabla(V_j*|u(s)|^2)|u(s)|^2dx=\frac{1}{2}\int((x\cdot\nabla V_j)*|u(s)|^2)|u(s)|^2dx\ ,$$

for $j=1, 2$.

The positive constant e_1 is obtained as the maximum of the following function

$$h(s)=(2-\mu)s^2+\frac{1}{2}(\mu-\lambda)C\|u_o\|_2^{4-\lambda}s^\lambda , \tag{3.24}$$

where C is a positive constant appearing in the estimate

$$\int(V_1*|u(s)|^2)|u(s)|^2dx\leqq C\|u_o\|_2^{4-\lambda}\|\nabla u(s)\|_2^\lambda . \tag{3.25}$$

§ 4. Blowing-up conditions for general domains.

In this section we study the blowing-up conditions for solutions of (1.1)-(1.3) on an exterior domain Ω with a star-shaped complement Ω^c. We also study the blowing-up conditions for the solutions of the Neumann problem (1.1), (1.2) and (1.4) on a complement of a ball. For the problem (1.1)-(1.3) with Ω^c=ball Kavian [6] proved that if $f(|u|^2)u=|u|^{p-1}u$ with $5\leqq p$ then solutions of (1.1)-(1.3) blow up in a finite time for suitable initial data.

We assume that

(D) "Ω is an exterior domain in $\boldsymbol{R}^n$ $(n\geqq 3)$ with a bounded complement Ω^c which is star-shaped with respect to some point $x_0\in\Omega^c$ and the boundary $\partial\Omega$ is smooth."

Define D, $d>0$ and p_o as follows:

$$\begin{aligned} D&=\inf\{r>0;\ B(x_o, r)\supset\Omega^c\} ,\\ d&=\sup\{r>0;\ B(x_o, r)\subset\Omega^c\} ,\\ p_o&=\frac{\{n+4(n-1)(D/d)^n+4\}}{n} . \end{aligned} \tag{4.1}$$

We assume that there is a solution $u\in C^1([0, T); L^2(\Omega))\cap C([0, T); H_o^1(\Omega)\cap H^2(\Omega)\cap L^{p+1}(\Omega))$ of (1.1)-(1.3) with local interaction $f(|u|)^2u=|u|^{p-1}u$.

THEOREM 4.1. *Let* (D) *be satisfied. We assume that* $p\geqq p_o$, $u_o\in\Sigma_o\cap H^2(\Omega)$ *and* u_o *satisfies either of the following conditions* (4.2) *and* (4.3):

$$E(u_o)=\frac{1}{2}\|\nabla u_o\|_2^2-\frac{1}{p+1}\|u_o\|_{p+1}^{p+1}<0 , \tag{4.2}$$

$$E(u_o)\geqq 0 ,\quad \mathrm{Im}\int_\Omega(\nabla\Phi(x)\cdot\nabla u_o(x))\overline{u_o(x)}dx>0 \quad \textit{and} \tag{4.3}$$
$$\left|\mathrm{Im}\int_\Omega(\nabla\Phi(x)\cdot\nabla u_o(x))\overline{u_o(x)}dx\right|^2\geqq 4\left\{(n-1)\left(\frac{D}{d}\right)^n+1\right\}E(u_o)\int_\Omega\Phi|u_o|^2dx ,$$

where

$$\Phi(x)=\frac{1}{2}|x-x_o|^2+\Big(\frac{1}{n-2}\Big)D^n\frac{1}{|x-x_o|^{n-2}}\ . \tag{4.4}$$

Then the solution $u(t, x)$ of (1.1)-(1.3) blows up in a finite time.

Theorem 4.1 can be stated in a somewhat generalized form. Let Ω_1 and Ω_2 be domains which are star-shaped with respect to some point $x_o\in\Omega_2$, with $\Omega_1\supset\Omega_2$ and let Ω_2 be bounded (Ω_1 is not necessarily bounded). We also assume that there is an $r_o>0$ such that $\Omega_1\supset B(x_o, r_o)\supset\Omega_2$. Define D and $d>0$ as follows:

$$\begin{aligned} D&=\inf\{r>0;\ \Omega_1\supset B(x_o, r)\supset\Omega_2\}\ ,\\ d&=\sup\{r>0;\ B(x_o, r)\subset\Omega_2\}\ . \end{aligned} \tag{4.5}$$

Define p_o as in Theorem 4.1 by using (4.5).

THEOREM 4.1′. *Let Ω be a smooth domain such that $\Omega=\Omega_1\setminus\overline{\Omega_2}$, with Ω_1 and Ω_2 as above. Then the same statement as in Theorem 4.1 holds.*

REMARK 4.1. Our proof of Theorem 4.1 is a modification of the argument of Kavian [6]. We note that when $\Omega=(\text{ball})^c$ we have $p_o=5$. The function Φ is introduced by Kavian [6], and this function is a key of our proof of Theorem 4.1. Φ also plays an important role when we later consider the Neumann problem on the complement of a ball.

PROOF (OF THEOREM 4.1 AND THEOREM 4.1′). We shall give the proof of Theorem 4.1 with some comments on the proof of Theorem 4.1′. Without losing generality we may assume that $x_o=o$. Put

$$V(t)=\frac{1}{2}\int_\Omega\Phi(x)|u(t, x)|^2dx\ . \tag{4.6}$$

As can be seen by an approximation argument as in [6], we have only to obtain the upper bound for the following function $W(t)$:

$$\begin{aligned} W(t)=&2\int_\Omega\sum_{1\le j,k\le n}\nabla_j\nabla_k\Phi(x)\nabla_j u(t, x)\nabla_k\overline{u(t, x)}dx\\ &+n\Big\{\frac{2}{p+1}-1\Big\}\int_\Omega|u(t, x)|^{p+1}dx\\ &-\int_{\partial\Omega}|\nabla u(t, x)\cdot n(x)|^2\nabla\Phi(x)\cdot n(x)dx\ . \end{aligned} \tag{4.7}$$

Here $n(x)$ is an outward unit normal at $x\in\partial\Omega$. (Here we have used $\nabla\Phi=n$ in Ω.) By the assumption on the domain Ω and the definition of D, we have $x\cdot n(x)\leqq 0$ and $D/|x|\geqq 1$ for all $x\in\partial\Omega$. Since $\nabla_j\Phi(x)=\{1-(D/|x|)^n\}x_j$, we have

$$\nabla\Phi(x)\cdot n(x)=\left\{1-\left(\frac{D}{|x|}\right)^n\right\}(x\cdot n(x))\geqq 0 \tag{4.8}$$

for all $x\in\partial\Omega$. (In the proof of Theorem 4.1′ we note that for $x\in\partial\Omega_2$ we can argue as above and for $x\in\partial\Omega_1$ we also have (4.8) because of $x\cdot n(x)\geqq 0$ and $D/|x|\leqq 1$.)

Since

$$\nabla_j\nabla_k\Phi(x)=\delta_{jk}\left\{1-\left(\frac{D}{|x|}\right)^n\right\}+nD^n\frac{x_jx_k}{|x|^{n+2}}$$

and $|x|\geqq d$ for all $x\in\Omega$, we obtain

$$\sum_{1\leqq j,k\leqq n}\nabla_j\nabla_k\Phi(x)\nabla_j u(t,x)\nabla_k\overline{u(t,x)}\leqq\left\{(n-1)\left(\frac{D}{d}\right)^n+1\right\}|\nabla u(t,x)|^2, \tag{4.9}$$

for all $x\in\Omega$. Consequently we have

$$\begin{aligned}W(t)\leqq 2&\left\{(n-1)\left(\frac{D}{d}\right)^n+1\right\}\|\nabla u(t)\|_2^2\\&+n\left\{\frac{2}{p+1}-1\right\}\|u(t)\|_{p+1}^{p+1}.\end{aligned} \tag{4.10}$$

Using the conservation of energy we obtain the following estimate:

$$\begin{aligned}W(t)\leqq 4&\left\{(n-1)\left(\frac{D}{d}\right)^n+1\right\}E(u_0)\\&+\left[n\left\{\frac{2}{p+1}-1\right\}+\frac{4}{p+1}\left\{(n-1)\left(\frac{D}{d}\right)^n+1\right\}\right]\|u(t)\|_{p+1}^{p+1}.\end{aligned} \tag{4.11}$$

Under the assumption $p\leqq p_0$ the coefficient of the second term of the right hand side of (4.11) is non-positive. Therefore we have the following upper bound:

$$W(t)\leqq 4\left\{(n-1)\left(\frac{D}{d}\right)^n+1\right\}E(u_0). \tag{4.12}$$

From (4.12) we can assert that

$$0<V(t)\leqq V(0)+V'(0)t+2\left\{(n-1)\left(\frac{D}{d}\right)^n+1\right\}E(u_0)t^2. \tag{4.13}$$

Therefore under the assumption (4.2) or (4.3) there is a finite time $T^*>0$ such that

$$\lim_{t\to T^*} V(t)=0 . \tag{4.14}$$

We now recall the following inequalities:

$$\|u\|_2^2 \leqq C\| \, |x|u \, \|_2 \|\nabla u\|_2 \quad \text{for} \quad u\in\Sigma_0 , \tag{4.15}$$

$$\| \, |x|u(t) \, \|_2 \leqq \frac{1}{\sqrt{2}} V(t)^{1/2} , \tag{4.16}$$

where C is a constant depending only on n. (4.15) and (4.16) yield

$$\|u_0\|_2^2 \leqq C(V(t))^{1/2}\|\nabla u\|_2 . \tag{4.17}$$

Therefore by (4.14) and (4.17) we conclude that there is a finite time $(0<)T^{**}$ $(\leqq T^*)$ such that $\|\nabla u(t)\|_2 \to +\infty$ as $t\to T^{**}$. □

Next, we consider the Neumann problem (1.1′), (1.2) and (1.4) with local interaction $f(|u|^2)u=|u|^{p-1}u$ with $p\geqq 5$ on $\Omega=B(o, r)^c$. We assume that for an initial data $u_0\in H^3(\Omega)$ there is a solution $u\in C^1([0, t); H^1(\Omega))\cap C([0, T); H^3(\Omega))$. Then we have the following theorem.

THEOREM 4.2. *We assume that* $u_0\in H^3(\Omega)\cap\Sigma$ *and that* u_0 *satisfies either of the following conditions* (4.18) *and* (4.19).

$$E(u_0)<0 , \tag{4.18}$$

$$E(u_0)\geqq 0 , \quad \mathrm{Im}\int_\Omega (\nabla\Phi(x)\cdot\nabla u_0(x))\overline{u_0(x)}dx>0 \quad and$$
$$\left|\mathrm{Im}\int_\Omega (\nabla\Phi(x)\cdot\nabla u_0(x))\overline{u_0(x)}dx\right|^2 \geqq 4nE(u_0)\|\Phi u_0\|_2^2 , \tag{4.19}$$

where

$$\Phi(x)=\frac{1}{2}|x-x_0|^2+\frac{1}{(n-2)}r^n|x|^{-(n-2)} . \tag{4.20}$$

Then the solution u *of* (1.1′), (1.2) *and* (1.4) *blows up in a finite time.*

PROOF. Applying the argument used in the proof of Lemma 2.2 we have

$$V(t)=\frac{1}{2}\|\Phi u_0\|_2^2-\left(\mathrm{Im}\int_\Omega (\nabla\Phi(x)\cdot\nabla u_0(x))\overline{u_0(x)}dx\right)t$$
$$+\int_0^t (t-s)W(s)ds , \tag{4.21}$$

where $V(t)=\frac{1}{2}\int_{\Omega}\Phi(x)|u(t, x)|^2dx$ and

$$\begin{aligned} W(s)=&2\|\nabla u(s)\|_2^2+\left\{\frac{2}{(p+1)}-1\right\}n\|u(s)\|_{p+1}^{p+1} \\ &+\int_{\partial\Omega}\frac{\partial\Phi(x)}{\partial n}\left\{|\nabla u(s, x)|^2-\frac{2}{(p+1)}|u(s, x)|^{p+1}\right\}d\Gamma \\ &-\mathrm{Im}\int_{\partial\Omega}\frac{\partial\Phi(x)}{\partial n}\frac{\partial u(s, x)}{\partial s}\overline{u(s, x)}d\Gamma . \end{aligned}$$

Here we note that $\nabla_j\Phi(x)=(x-x_0)_j(1-(r/|x-x_0|)n)$ $(1\leqq j\leqq n)$ implies

$$\frac{\partial\Phi(x)}{\partial n}=0 , \tag{4.22}$$

for all $x\in\partial\Omega$. Conservation of energy, (4.22) and the assumption $p\geqq 5$ yield

$$\begin{aligned} W(s)&\leqq\left\{\frac{6}{(p+1)}-1\right\}\|u(s)\|_{p+1}^{p+1}+4nE(u_0) \\ &\leqq 4nE(u_0) . \end{aligned} \tag{4.23}$$

Hence we obtain

$$\begin{aligned} V(t)\leqq&\frac{1}{2}\|\Phi u_0\|_2^2 \\ &-\left(\mathrm{Im}\int_{\Omega}(\nabla\Phi(x)\cdot\nabla u_0(x))\overline{u_0(x)}dx\right)t+2nE(u_0)t^2 . \end{aligned} \tag{4.24}$$

By assumption (4.18) or (4.19) we can conclude that there is a finite time $T^*>0$ such that $V(t)\to 0$ as $t\to T^*$, and hence $\|u(t)\|_{H^1(\Omega)}\to+\infty$ as $t\to T^{**}$ $(\leqq T^*)$. □

We close this section by giving several remarks on some related results on blowing-up conditions for local or nonlocal interactions.

REMARK 4.2. Kavian [6] studied the blowing-up condition for solutions of the problem (1.1)-(1.3) in a star-shaped domain only for single power interaction. It can be immediately extended to general local interactions which satisfy Glassey's condition. For local interaction such as $f(|u|^2)u=|u|^{p-1}u+|u|^{q-1}u$ with $1<p<1+4/n<q$, which does not satisfy Glassey's condition, we can obtain the blowing-up result similar to Theorem 3.1. Moreover, for a domain as in Theorem 4.1 (or Theorem 4.1′) and for the interaction $f(|u|^2)u=|u|^{p-1}u+|u|^{q-1}u$ with $1<p<p_0<q$, we can obtain the result analogous to Theorem 4.1.

The following remark is a nonlocal version of Kavian [6] (see also Matsumoto and Mochizuki [9]).

REMARK 4.3. Let Ω be a domain in $\boldsymbol{R}^n$ $(n\geqq 3)$ which is star-shaped with respect to some point $x_0\in\Omega$. We consider the problem (1.1′), (1.2) and (1.3) with $V(x)=1/|x|^\lambda$, $2<\lambda<\min(4, n)$, $\lambda+1<n$. Assume that an initial data u_0 satisfies the following condition (4.25) or (4.26):

$$E(u_0)=\frac{1}{2}\|\nabla u_0\|_2^2-\frac{1}{4}\int_\Omega (V*|u_0|^2)|u_0|^2dx<0\ , \tag{4.25}$$

$$E(u_0)\geqq 0\ ,\quad \mathrm{Im}\int_\Omega (x-x_0)\cdot\nabla u_0(x)\overline{\nabla u_0(x)}dx>0 \quad \text{and} \tag{4.26}$$

$$\left|\mathrm{Im}\int_\Omega (x-x_0)\cdot\nabla u_0(x)\overline{u_0(x)}dx\right|^2\geqq E(u_0)\int_\Omega |x-x_0|^2|u_0(x)|^2dx\ .$$

Then the solution $u\in C^1([0, T); L^2(\Omega))\cap C([0, T); \Sigma_0\cap H^2(\Omega))$ blows-up in a finite time.

The following remark is a nonlocal version of Theorem 3.1 and Remark 4.2.

REMARK 4.4. We consider the problem (1.1′), (1.2) and (1.3) for nonlocal interaction $(V*|u|^2)u=(V_1*|u|^2)u+(V_2*|u|^2)u$, where $V_1(x)=1/|x|^\lambda$ and $V_2(x)=1/|x|^\mu$ with $0<\lambda<2<\mu<\min(4, n)$ and $\lambda+1<n$. Let Ω be a domain in $\boldsymbol{R}^n$ $(n\geqq 3)$ which is star-shaped with respect to some point $x_0\in\Omega$. Then there is a positive constant e_3 such that if the following condition

$$\begin{aligned} E(u_0)=\frac{1}{2}\|\nabla u_0\|_2^2&-\frac{1}{4}\int_\Omega (V_1*|u|^2)|u|^2dx \\ &-\frac{1}{4}\int_\Omega (V_2*|u|^2)|u|^2dx<-e_3 \end{aligned} \tag{4.27}$$

is satisfied, the solution $u\in C^1([0, T); L^2(\Omega))\cap C([0, T); \Sigma_0\cap H^2(\Omega))$ blows-up in a finite time.

§ 5. Remark on global solution for nonlocal interaction.

In this section we briefly state the global existence of solutions for the following Cauchy problem:

$$i\frac{\partial u}{\partial t}=\Delta u+(V*|u|^2)u\ ,\qquad t\geqq 0\ ,\quad x\in\boldsymbol{R}^n\ , \tag{5.1}$$

$$u(o, x)=u_0(x)\ , \qquad x\in \boldsymbol{R}^n\ , \tag{5.2}$$

where $V(x)=1/|x|^2$. Matsumoto and Mochizuki [9] proved that solutions of (5.1)-(5.2) blow-up in a finite time for suitable initial data. For small initial data, we can prove the global existence of the weak solution for (5.1), (5.2).

First, we consider the following minimizing problem:

$$\Theta=\inf\left\{Q(u)\equiv\frac{\|u\|_2^2\|\nabla u\|_2^2}{P(u)}\ ;\ u\in H^1(\boldsymbol{R}^n),\ u\neq 0\right\} \tag{5.3}$$

where $P(u)=\int_{\boldsymbol{R}^n}(V*|u|^2)|u|^2dx$. We note that Θ is also represented as

$$\Theta=\inf\left\{\frac{1}{2}\|u\|_2^2;\ E(u)=0,\ u\in H^1(\boldsymbol{R}^n)\right\}\ , \tag{5.4}$$

where $E(u)=\frac{1}{2}\|\nabla u\|_2^2-\frac{1}{4}P(u)$. For the minimizing problem (5.3) we can prove the existence of a minimizer $U(x)$ by using Schwarz's symmetrization and Strauss' compactness lemma (see, e.g., Lieb [7], Strauss [17]). The fact that $Q(u)$ is invariant under the transformation

$$u(x)\longrightarrow v(x)=au(bx)\ , \qquad a,\ b>0\ , \tag{5.5}$$

also plays an important role in the proof.

We have the following theorem.

THEOREM 5.1. (1) *There is a minimizer R for Θ which satisfies the nonlinear elliptic equation*:

$$(1-\Delta)R=(V*|R|^2)R\ , \qquad R>0\ , \quad \text{in}\quad \boldsymbol{R}^n\ . \tag{5.6}$$

Moreover, Θ is given by $\frac{1}{2}\|R\|_2^2$.

(2) *If $u_0\in H^1(\boldsymbol{R}^n)$ $(n\geqq 3)$ and $\|u_0\|_2<\|R\|_2$, then there is a unique solution $u\in C([0, +\infty);\ H^1(\boldsymbol{R}^n))$ of* (5.1)-(5.2).

REMARK 5.1. The existence of a positive solution of (5.6) was proved by Lions [8], but his proof depends on the critical point theory and the relation to the minimizing problem (5.3) is not stated in the paper [8].

Local existence of H^1-solution can be proved by the contraction mapping principle (see Ginibre and Velo [3]). Hence we have only to obtain the uniform H^1-estimate. By Theorem 5.1 (1) and the conservation of L^2-norm and energy we obtain the following estimate:

$$\frac{1}{2}\left\{1-\left(\frac{\|u_0\|_2}{\|R\|_2}\right)^2\right\}\|\nabla u(t)\|_2^2 \leqq E(u_0)\,. \tag{5.7}$$

Hence we have Theorem 5.1 (2).

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