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The Characters of a Maximal Parabolic Subgroup of $GL_n(F_q)$

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§0. Introduction.

Let G_n be the general linear group $GL_n(\boldsymbol{F}_q)$, that is, the group of nonsingular matrices of degree n with coefficients in the finite field $\boldsymbol{F}_q$ and P_n be the maximal parabolic subgroup of G_n consisting of matrices $(g_{ij}) \in G_n$ such that $g_{21}=g_{31}=\cdots=g_{n1}=0$.

In this paper we show an inductive method to calculate the irreducible characters of P_n and also determine the branching rules of irreducible characters for $G_n \to P_n$ and $P_n \to L_n$, where L_n is a Levi subgroup of P_n and hence L_n is isomorphic to $G_1 \times G_{n-1}$. J. A. Green [2] showed how to calculate the irreducible characters of G_n and A. V. Zelevinsky [5] determined the branching rules for $G_n \to H_n$ and $H_n \to G_{n-1}$, where $H_n = \{(g_{ij}) \in P_n \mid g_{11}=1\}$ is the group of affine transformations. Thus this paper can be viewed as an application of these two papers.

We use the following notation. We deal here only with complex characters and so for a finite group G, $\mathrm{Irr}\, G$ stands for the set of all irreducible complex characters of G and $\mathrm{ch}\, G$ is the ring of virtual complex characters of G. For a subgroup H of G and $\varphi \in \mathrm{ch}\, H$, $\mathrm{Ind}_H^G \varphi$ is the induced character of φ from H to G and for $\chi \in \mathrm{ch}\, G$, $\mathrm{Res}_H^G \chi$ is the restriction of χ to H. For $\chi_1, \chi_2 \in \mathrm{ch}\, G$,

$$(\chi_1, \chi_2)_G = |G|^{-1} \sum_{g \in G} \chi_1(g)\chi_2(g^{-1}) .$$

For $g \in G$, $Z_G(g)$ is the centralizer of g in G and for a fiinite set X, $|X|$ denotes its cardinality. Moreover we denote by $\boldsymbol{N}$ the set of natural numbers, so that $\boldsymbol{N}=\{0, 1, 2, \cdots\}$. Now let $V_n = \boldsymbol{F}_q{}^n$ be the n-dimensional vector space of column n-vectors with coefficients in $\boldsymbol{F}_q$ and $(e_1, e_2, \cdots, e_n)$ be the canonical basis of V_n. Then V_n is naturally a left G_n-module. For $f \in \mathrm{End}\, V_n$, $\mathrm{Im}\, f$ is the image of f and $\mathrm{Ker}\, f$ is the kernel of f. We also regard $\boldsymbol{F}_q$ to be the subset of $\mathrm{End}\, V_n$ identifying $a \in \boldsymbol{F}_q$ with $a \cdot 1_{V_n}$.

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This paper is organized as follows. In §1 we give a parametrization for the conjugacy classes of P_n and H_n, and in §2 after giving a parametrization of Irr P_n, we show an inductive formula for the value of irreducible characters of P_n and H_n. Finally in §3, applying a theorem in [5], we give branching rules for $G_n \to P_n$ and $P_n \to L_n$.

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§1. Conjugacy classes.

For $m \in \boldsymbol{N}$, a partition λ of m is a finite sequence of integers $\lambda = (l_1, l_2, \cdots, l_r)$ such that $l_1 \geqq l_2 \geqq \cdots \geqq l_r \geqq 0$ and $m = l_1 + l_2 + \cdots + l_r$. We identify $(l_1, l_2, \cdots, l_r)$ with $(l_1, \cdots, l_r, 0, \cdots, 0)$. The integers l_i are called the parts of λ and $r_j(\lambda)$ is the number of parts of λ which are equal to j. If $l_r > 0$ and $l_{r+1} = 0$ then r is called the rank of λ denoted by $r = r(\lambda)$. Also m is the degree of λ denoted by $m = |\lambda|$. Let $\boldsymbol{P}_m$ be the set of all partitions of degree m and put $\boldsymbol{P} = \bigcup_{m \in \boldsymbol{N}} \boldsymbol{P}_m$.

Let $\mathscr{F}$ be the set of all irreducible monic polynomials over $\boldsymbol{F}_q$ with nonzero constant term in the indeterminant t. For $n \in \boldsymbol{N}$, $S_n(\mathscr{F}, \boldsymbol{P})$ denotes the set of mappings ν from $\mathscr{F}$ to $\boldsymbol{P}$ such that

$$\sum_{f \in \mathscr{F}} |\nu(f)| \cdot \deg f = n .$$

We write $\operatorname{supp}(\nu) = \{f \in \mathscr{F} \mid |\nu(f)| > 0\}$.

It is well known that the conjugacy classes of G_n are parametrized by $S_n(\mathscr{F}, \boldsymbol{P})$. In fact, a representative c_ν of the conjugacy class corresponding to $\nu \in S_n(\mathscr{F}, \boldsymbol{P})$ is given as follows. For $f(t) = t^d + a_{d-1}t^{d-1} + \cdots + a_1 t + a_0 \in \mathscr{F}$, put

$$J(f) = J_1(f) = \begin{pmatrix} \cdot & 1 & & & \\ \cdot & \cdot & 1 & & \\ & \cdots & \cdots & \cdots & \\ \cdot & \cdot & \cdot & \cdot & 1 \\ -a_0 & -a_1 & \cdot & \cdot & -a_{d-1} \end{pmatrix}.$$

And for a positive integer m, put

$$J_m(f) = \begin{pmatrix} J(f) & 1_d & & \\ & J(f) & 1_d & \\ & \cdots & \cdots & \\ \cdot & \cdots & & J(f) \end{pmatrix} \in G_{md} ,$$

and moreover for $\lambda=(l_1, l_2, \cdots, l_r)\in \boldsymbol{P}$ with $r=r(\lambda)$, put

$$J_\lambda(f)=\mathrm{diag}(J_{l_1}(f), \cdots, J_{l_r}(f)) .$$

Then, if $\mathrm{supp}(\nu)=\{f_1, \cdots, f_N\}$, a representative c_ν of conjugacy class corresponding to ν is given by

$$c_\nu=\mathrm{diag}(J_{\nu(f_1)}(f_1), \cdots, J_{\nu(f_N)}(f_N)) .$$

If $f(t)=t-a$ for $a\in \boldsymbol{F}_q^\times=\boldsymbol{F}_q\backslash\{0\}$, we write $J_m(a)$, $J_\lambda(a)$ instead of $J_m(t-a)$, $J_\lambda(t-a)$. Now for a positive integer k, $0<k\leqq n$, $a\in \boldsymbol{F}_q^\times$ and $\mu\in S_{n-k}(\mathscr{F}, \boldsymbol{P})$, we define an element $c_n(a^{(k)}, \mu)$ of P_n by

$$c_n(a^{(k)}, \mu)=\mathrm{diag}(J_k(a), c_\mu) .$$

PROPOSITION 1.1. (i) *The set* $\{c_n(a^{(k)}, \mu) \mid 0<k\leqq n,\ a\in \boldsymbol{F}_q^\times,\ \mu\in S_{n-k}(\mathscr{F}, \boldsymbol{P})\}$ *is a complete system of representatives of conjugacy classes in* P_n. *In particular the conjugacy classes in* P_n *are parametrized by the set*

$$\bigcup_{l=0}^{n-1}(S_1(\mathscr{F}, \boldsymbol{P})\times S_l(\mathscr{F}, \boldsymbol{P})) .$$

(ii) *For* $\mu\in S_{n-k}(\mathscr{F}, \boldsymbol{P})$, *put* $r_i=r_i(\mu(t-a))=$ *the number of parts of* $\mu(t-a)\in \boldsymbol{P}$ *which are equal to* i, *and put*

$$e=e(c_n(a^{(k)}, \mu))=k-1+2\sum_{i=1}^{k-1} ir_i+(2k-1)\sum_{i=k}^{n-k} r_i .$$

Then we hvae

$$|Z_{P_n}(c_n(a^{(k)}, \mu))|=|Z_{G_{n-k}}(c_\mu)|(q-1)q^e .$$

PROOF. Let p be an element of P_n so that $pe_1=ae_1$ for some $a\in \boldsymbol{F}_q^\times$. Let k be the positive integer determined by $e_1\notin \mathrm{Im}(p-a)^k$ and $e_1\in \mathrm{Im}(p-a)^{k-1}$ and choose $v_i\in V_n$ $(i=1, \cdots, k)$ such that $v_1=e_1$, $(p-a)v_2=v_1, \cdots, (p-a)v_k=v_{k-1}$. Then if V' is the subspace spanned by $\{v_1, \cdots, v_k\}$, then $pV'=V'$ and so p induces an element $\bar{p}\in GL(V_n/V')$, which determines $\mu\in S_{n-k}(\mathscr{F}, \boldsymbol{P})$ such that $\bar{p}$ is conjugate to c_μ in $GL(V_n/V')$. Then it is easy to see that p is conjugate to $c_n(a^{(k)}, \mu)$ in P_n. The other assertion can be proved easily and so we omit the proof.

Next we consider the conjugacy classes of H_n. We start with the following lemma.

LEMMA 1.2. *Let* h_1 *and* h_2 *be elements of* H_n, *which are conjugate in* P_n. *Then they are also conjugate in* H_n.

PROOF. Let $h_1=\begin{pmatrix}1 & b\\ 0 & x\end{pmatrix}\in H_n$ $(x\in G_{n-1},\ {}^t b\in \boldsymbol{F}_q{}^{n-1})$, and $h_2=ph_1p^{-1}$ for $p\in P_n$. Since P_n is a semidirect product of the normal subgroup H_n and $M=\{(g_{ij})\in P_n \mid g_{ii}=1$ for $2\leqq i\leqq n$ and $g_{ij}=0$ if $i\neq j\}\simeq \boldsymbol{F}_q{}^{\times}$, $P_n=M\ltimes H_n$, it is sufficient to show that for $p=\begin{pmatrix}a & \\ & 1_{n-1}\end{pmatrix}$ with $a\in \boldsymbol{F}_q{}^{\times}$, we can find $h\in H_n$ such that $ph_1p^{-1}=hh_1h^{-1}$, which is satisfied by $h=\begin{pmatrix}1 & b\\ 0 & a^{-1}x\end{pmatrix}$.

Combining (1.1) and (1.2), we have

PROPOSITION 1.3. (i) *The set* $\{c_n(1^{(k)}, \mu) \mid 0<k\leqq n,\ a\in \boldsymbol{F}_q{}^{\times},\ \mu\in S_{n-k}(\mathscr{F}, \boldsymbol{P})\}$ *is a complete system of representatives of conjugacy classes in* H_n. *In particular, the conjugacy classes in* H_n *are parametrized by the set* $\bigcup_{l=0}^{n-1} S_l(\mathscr{F}, \boldsymbol{P})$.

(ii) $|Z_{H_n}(c_n(1^{(k)}, \mu))|=(q-1)^{-1}|Z_{P_n}(c_n(1^{(k)}, \mu))|$.

§2. Characters of P_n.

Firstly we recall the definition of generalized inductions and restrictions (cf. [5]). Let G be a finite group and A, S be subgroups of G such that A is normalized by S and $A\cap S=\{1\}$. Therefore S is a subgroup of $N=N_G(A)=$ the normalizer of A in G and in general N acts on $\mathrm{Irr}\,A$ by ${}^n\chi(a)=\chi(n^{-1}an)$ for $\chi\in \mathrm{Irr}\,A$, $a\in A$ and $n\in N$. Now let ψ be a linear character of A such that $S\subset \mathrm{Stab}_N(\psi)=\{n\in N \mid {}^n\psi=\psi\}$. Let (ρ, E) be a representation of S so that E is a finite dimensional vector space over $\boldsymbol{C}$ and ρ is a homomorphism from S to $GL(E)$. Then ρ can be extended to a representation ρ' of AS by $\rho'(as)=\psi(a)\rho(s)$ for $a\in A$ and $s\in S$. Then the generalized induction of (ρ, E) to G is defined by $\mathrm{Ind}_{AS}^G\rho'$ which will be denoted by $(i_{A,\psi})_S^G(\rho)$. Now let $(\tilde{\rho}, \tilde{E})$ be a representation of G. Then $\tilde{E}_\psi=\{v\in\tilde{E} \mid \tilde{\rho}(a)v=\psi(a)v$ for all $a\in A\}$ is an S-stable subspace of E and the representation $(\mathrm{Res}_S^G\tilde{\rho}, \tilde{E}_\psi)$ of S is called the generalized restriction of $(\tilde{\rho}, \tilde{E})$ to S and is denoted by $(r_{A,\psi})_S^G(\tilde{\rho})$. We also regard $(i_{A,\psi})_S^G$ (resp. $(r_{A,\psi})_S^G$) to be an additive homomorphism from $\mathrm{ch}\,S$ (resp. $\mathrm{ch}\,G$) to $\mathrm{ch}\,G$ (resp. $\mathrm{ch}\,S$). In particular, if A is trivial then $(i_{A,1})_S^G=\mathrm{Ind}_S^G$ and $(r_{A,1})_S^G=\mathrm{Res}_S^G$. Moreover we can prove the following property of the generalized inductions and restrictions without difficulty (cf. [5], 8.1 and 8.2).

PROPOSITION 2.1. *Let* G, A, S, ψ *be as above.*

(i) *For* $\chi\in\mathrm{ch}\,S$ *and* $\varphi\in\mathrm{ch}\,G$, *we have*

$$((i_{A,\psi})_S^G(\chi),\ \varphi)_G=(\chi,\ (r_{A,\psi})_S^G(\varphi))_S\,.$$

(ii) *Assume moreover that* A *is a normal subgroup of* G *and* $\mathrm{Stab}_G(\psi)=AS$. *Then* $(r_{A,\psi})_S^G \circ (i_{A,\psi})_S^G = \mathrm{id}_{\mathrm{ch}\, S}$.

(iii) *Let* G_1 *be a subgroup of* G *which contains* A, S *as its subgroups and* $S_1 \subset S$. *Then* $(r_{A,\psi})_{S_1}^{G_1} \circ \mathrm{Res}_{G_1}^G = \mathrm{Res}_{S_1}^S \circ (r_{A,\psi})_S^G$.

Furthermore, if $G=A \rtimes B$ is a semidirect product of an abelian normal subgroup A and a subgroup B, then we can apply the method of little groups to determine $\mathrm{Irr}\, G$ as follows.

THEOREM 2.2. *Let* $G=A \rtimes B$ *with* A *abelian.*

(i) *Let* $\psi \in \mathrm{Irr}\, A$ *and* $B_\psi = \mathrm{Stab}_B\, \psi$. *Then for all* $\sigma \in \mathrm{Irr}\, B_\psi$, $(i_{A,\psi})_{B_\psi}^G(\sigma) \in \mathrm{Irr}\, G$.

(ii) *For every* $\varphi \in \mathrm{Irr}\, G$, *there exist* $\psi \in \mathrm{Irr}\, A$ *and* $\sigma \in \mathrm{Irr}\, B_\psi$ *such that* $\varphi = (i_{A,\psi})_{B_\psi}^G(\sigma)$.

(iii) *For* $\varphi_j = (i_{A,\psi_j})_{B_{\psi_j}}^G(\sigma_j) \in \mathrm{Irr}\, G$ *where* $\psi_j \in \mathrm{Irr}\, A$ *and* $\sigma_j \in \mathrm{Irr}\, B_{\psi_j}$ $(j=1, 2)$, $\varphi_1=\varphi_2$ *if and only if* $\psi_1 = {}^g\psi_2$ *and* $\sigma_1 = {}^g\sigma_2$ *for some* $g \in G$.

For a proof we refer the reader to [1, § 11B].

Now we return to our original situation. Let $n \geqq 2$ and L_n and U_n be the subgroups of P_n defined by

$$L_n = \{(g_{ij}) \in P_n \mid g_{12}=g_{13}=\cdots=g_{1n}=0\}\ ,$$
$$U_n = \{(g_{ij}) \in P_n \mid g_{ii}=1 \text{ for } 1 \leqq i \leqq n \text{ and } g_{ij}=0 \text{ if } i \neq 1 \text{ and } i \neq j\}\ .$$

Then $L_n \simeq G_1 \times G_{n-1}$, $U_n \simeq F_q{}^{n-1}$ and $P_n = L_n \ltimes U_n$. Therefore we can apply (2.2) to determine $\mathrm{Irr}\, P_n$.

Let ψ be a nontrivial character of the additive group F_q, and define $\psi_n \in \mathrm{Irr}\, U_n$ by $\psi_n((u_{ij})) = \psi(u_{12})$ for $(u_{ij}) \in U_n$. Then it is easy to see that $\mathrm{Irr}\, U_n = \{1\} \cup \{{}^p\psi_n \mid p \in P_n\}$, where 1 is the trivial character of U_n. Put $S_n = \mathrm{Stab}_{L_n} \psi_n$. Then

$$S_n = \left\{ \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & a_{n3} & \cdots & a_{nn} \end{pmatrix} \in L_n \;\middle|\; a_{11}=a_{22} \right\} \simeq P_{n-1}\ .$$

Hereafter we identify P_{n-1} with S_n and we consider that P_{n-1} is a subgroup of P_n with this identification and so are P_k, H_k and G_k $(1 \leqq k \leqq n)$. By (2.2) there is a bijection from $\mathrm{Irr}\, L_n \cup \mathrm{Irr}\, P_{n-1}$ to $\mathrm{Irr}\, P_n$ and so by iteration from $\bigcup_{i=1}^n \mathrm{Irr}\, L_i$ to $\mathrm{Irr}\, P_n$. This bijection is described as follows. Firstly since $L_n \simeq P_n/U_n$ we can regard $\mathrm{Irr}\, L_n$ as a subset of $\mathrm{Irr}\, P_n$ and this inclusion is obtained by virtue of $(i_{U_n,1})^{P_n}_{L_n}$. Next if $\zeta\in\operatorname{Irr} P_{n-1}$ then $(i_{U_n,\psi_n})^{P_n}_{P_{n-1}}(\zeta)\in\operatorname{Irr} P_n$. Similar statement also holds for H_n and so we have the following theorem (cf. [5, 13.2]).

THEOREM 2.3. (i) *For* $1\leqq k\leqq n$, *let*

$$I^{P_n}_{L_k}=(i_{U_n,\psi_n})^{P_n}_{P_{n-1}}\circ\cdots\circ(i_{U_{k+1},\psi_{k+1}})^{P_{k+1}}_{P_k}\circ(i_{U_k,1})^{P_k}_{L_k}\ .$$

Then the mapping $I^n\colon \cup_{k=1}^n \operatorname{Irr} L_k\to\operatorname{Irr} P_n$ *defined by*

$$I^n(\zeta)=I^{P_n}_{L_k}(\zeta)\ ,\qquad for\quad \zeta\in\operatorname{Irr} L_k\ ,$$

is a bijection.

(ii) *For* $1\leqq k\leqq n$, *let*

$$I^{H_n}_{G_{k-1}}=(i_{U_n,\psi_n})^{H_n}_{H_{n-1}}\circ\cdots\circ(i_{U_{k+1},\psi_{k+1}})^{H_{k+1}}_{H_k}\circ(i_{U_k,1})^{H_k}_{G_{k-1}}\ .$$

Then the mapping $I'^n\colon \cup_{k=1}^n \operatorname{Irr} G_{k-1}\to\operatorname{Irr} H_n$ *defined by*

$$I'^n(\chi)=I^{H_n}_{G_{k-1}}(\chi)\ ,\qquad for\quad \chi\in\operatorname{Irr} G_{k-1}\ ,$$

is a bijection.

For brevity, we will also write $I^n_k(\zeta)=I^{P_n}_{L_k}(\zeta)$ if $\zeta\in\operatorname{Irr} L_k$. The character values of $I^n_n(\zeta)$ on P_n can be derived from the results of [2], while for the values of $I^n_k(\zeta)$ $(1\leqq k<n)$, we have the following inductive formula.

THEOREM 2.4. *Let* $1\leqq k,\ l<n$, $\zeta\in\operatorname{Irr} L_k$ *and* $\nu\in S_{n-l}(\mathscr{F}, \boldsymbol{P})$. *Put* $r_i=r_i(\nu(t-a))$ *for* $i=1,2,\cdots,n-l$. *Also for* $i=1,\cdots,n-l$ *define* $\nu_i\in S_{n-1-i}(\mathscr{F},\boldsymbol{P})$ *as follows: firstly* $\nu_{l-1}=\nu$, *and if* $i\neq l-1$, *then for* $f\in\mathscr{F}$, $f\neq t-a$, $\nu_i(f)=\nu(f)$ *and*

$$r_j(\nu_i(t-a))=\begin{cases} r_j\ , & if\quad j\neq i,\ l-1\ ,\\ r_j-1\ , & if\quad j=i\ ,\\ r_j+1\ , & if\quad j=l-1\ .\end{cases}$$

Then we have

$$\begin{aligned} &I^n_k(\zeta)(c_n(a^{(l)},\nu))\\ &\quad=\sum_{i=l}^{n-l}(q^{r_i}-1)q^{r_{i+1}+\cdots+r_{n-l}}I^{n-1}_k(\zeta)(c_{n-1}(a^{(i)},\nu_i))\\ &\qquad-q^{r_l+\cdots+r_{n-l}}I^{n-1}_k(\zeta)(c_{n-1}(a^{(l-1)},\nu))\ ,\end{aligned}$$

where if $l=1$, *we regard the last term on the right hand side to be equal to zero.*

PROOF. Let $x=c_n(a^{(l)}, \nu)$ and $\chi=I_k^{n-1}(\zeta)$. Then by the definition of I_k^n, we have

$$I_k^n(\zeta)(x)=|U_nS_n|^{-1}\sum_{g\in P_n}(\psi\cdot\chi)(g^{-1}xg) ,$$

where, as usual, $\psi\cdot\chi\in \mathrm{Irr}(U_nS_n)$ is extended to a complex valued function on P_n by $(\psi\cdot\chi)(y)=0$ for $y\in P_n\backslash U_nS_n$.

Since we have $(g^{-1}xg)e_1=ae_1$ for $g\in P_n$, $g^{-1}xg\in U_nS_n$ if and only if $(x-a)ge_2=be_1$ for some $b\in \boldsymbol{F}_q$. Let $v_0=e_1, v_1,\cdots, v_s$ be a basis of $\mathrm{Ker}(x-a)$, and put $l_0=l$ and $\nu(t-a)=(l_1,\cdots, l_s)=((n-l)^{r_{n-l}},\cdots, 1^{r_1})$. Then we may assume that $v_i\in \mathrm{Im}(x-a)^{l_i-1}$ and $v_i\notin \mathrm{Im}(x-a)^{l_i}$ for $i=0, 1,\cdots, s$.

Case 1. $(x-a)ge_2=0$. Then $ge_2\in \mathrm{Ker}(x-a)$ and so $ge_2=\sum_{i=0}^{s}a_iv_i$ for some $a_i\in \boldsymbol{F}_q$ $(i=0, 1, \cdots, s)$, which implies $g^{-1}xge_2=ae_2$ or equivalently $xge_2=age_2$. Let p_n be the natural homomorphism of vector spaces, $p_n\colon V_n\to V_n/\boldsymbol{F}_qe_1\simeq V_{n-1}$ and π_n be the canonical homomorphism of groups, $\pi_n\colon U_nS_n\to U_nS_n/U_n\simeq S_n\simeq P_{n-1}$. Then it can be shown without difficulty that the number of elements $g\in P_n$ such that $p_n(ge_2)\in \mathrm{Im}(\pi_n(x)-a)^{i-1}\backslash \mathrm{Im}(\pi_n(x)-a)^{i}$ is given by $(q^{r_i}-1)q^{r_{i+1}+\cdots+r_{n-l}}|S_nU_n|$, for $i=1, 2,\cdots, n-l$. Moreover in this case $\pi_n(g^{-1}xg)$ is conjugate to $c_{n-1}(a^{(i)}, \nu_i)$ in P_{n-1} and $(\psi\cdot\chi)(g^{-1}xg)=\chi(\pi_n(g^{-1}xg))=\chi(c_{n-1}(a^{(i)}, \nu_i))$. Therefore we have

$$|S_nU_n|^{-1}\sum(\psi\cdot\chi)(g^{-1}xg)=(q^{r_i}-1)q^{r_{i+1}+\cdots+r_{n-l}}\chi(c_{n-1}(a^{(i)}, \nu_i)) ,$$

for $i=1, 2,\cdots, n-l$, where the summation is all over $g\in P_n$ such that $g^{-1}xg\in S_nU_n$ and that $\pi_n(g^{-1}xg)$ is conjugate to $c_{n-1}(a^{(i)}, \nu_i)$.

Case 2. $(x-a)ge_2=be_1$ for some $b\in \boldsymbol{F}_q^{\times}$. Thus we have $e_1\in \mathrm{Im}(x-a)$ and hence $l>1$. Therefore $(x-a)e_2=e_1$ and so $ge_2-be_2\in \mathrm{Ker}(x-a)$, which implies that $ge_2-be_2=\sum_{i=0}^{s}a_iv_i$ for some $a_i\in \boldsymbol{F}_q$, $i=0, 1,\cdots, s$. Putting $g^{-1}e_1=ce_1$ for $c\in \boldsymbol{F}_q^{\times}$, we have

$$(g^{-1}xg)e_2=bce_1+ae_2$$

and so $(\psi\cdot\chi)(g^{-1}xg)=\psi(bc)\chi(\pi_n(g^{-1}xg))$, where $\pi_n(g^{-1}xg)$ is conjugate to some $c_{n-1}(a^{(i)}, \nu_i)$ for some $i=1,\cdots, l-1$. Therefore we have

$$|S_nU_n|^{-1}\sum(\psi\cdot\chi)(g^{-1}xg)$$
$$=\begin{cases}-(q^{r_i}-1)q^{r_{i+1}+\cdots+r_{n-l}}\chi(c_{n-1}(a^{(i)}, \nu_i)) , & \text{for } i=1,\cdots, l-2 ,\\ -q^{r_{l-1}+r_l+\cdots+r_{n-l}}\chi(c_{n-1}(a^{(l-1)}, \nu)) , & \text{for } i=l-1 ,\end{cases}$$

where the summation is all over $g\in P_n$ such that $g^{-1}xg\in S_nU_n$ and that $\pi_n(g^{-1}xg)$ is conjugate to $c_{n-1}(a^{(i)}, \nu_i)$. Thus adding these values in cases 1 and 2, we obtain the required results.

Let m be the multiplicity of the eigenvalue a of $x=c_n(a^{(l)}, \nu)$. Using the same notation as in the proof of (2.4), we have

$$m=\sum_{i=1}^{n-l} ir_i+l=\sum_{i=0}^{s} l_i .$$

The following corollaries are easy consequences of (2.4).

COROLLARY 2.5. *If* $I_k^n(\chi)(c_n(a^{(l)}, \nu))\neq 0$, *then* $m>n-k$.

COROLLARY 2.6. *Assume that* $m=\sum_{i=0}^{s} l_i>n-k$.

(i) *If* $l_0=l_1=\cdots=l_s=1$ *and so* $m=s+1$ and $\nu(t-a)=(1^s)$, *then*

(2.6.1) $$I_k^n(\chi)(c_n(a^{(l)}, \nu))=(q^s-1)\cdots(q^{s-n+k+1}-1)I_k^k(\chi)(c_k(a^{(l)}, \nu_k)) ,$$

where

$$\nu_k(f)=\begin{cases}\nu(f) , & if \quad f\neq t-a , \\ (1^{s-n+k}) , & if \quad f=t-a .\end{cases}$$

In particular, we have

(2.6.2) $$\deg I_k^n(\chi)=(q^{n-1}-1)\cdots(q^k-1)\deg \chi .$$

(ii) *If* $s=0$ *and hence* $m=l>n-k$, *then*

(2.6.3) $$I_k^n(\chi)(c_n(a^{(l)}, \nu))=(-1)^{n-k}I_k^k(\chi)(c_k(a^{(l-n+k)}, \nu)) .$$

In particular if $\chi=\chi_1\times\chi_{k-1}\in \mathrm{Irr}\, G_1\times \mathrm{Irr}\, G_{k-1}$ *and* $l-n+k=1$, *then*

(2.6.4) $$I_k^n(\chi)(c_n(a^{(l)}, \nu))=(-1)^{n-k}\chi_1(a)\chi_{k-1}(c_{k-1}(\nu)) .$$

REMARK 2.7. The formula (2.6.4) means that certain part of the character table of G_k multiplied by $(q-1)$th roots of unity appears as a part of the character table of P_n. In particular if $q=2$, then the matrix $(\chi_{k-1}(c_{k-1}(\nu)))$, where $\chi_{k-1}\in \mathrm{Irr}\, G_{k-1}$ and $\nu\in S_{k-1}(\mathscr{F}, P)$ and $\nu(t-1)=(0)$, appears as a submatrix of the character table of P_n.

Now we consider a relation between $\mathrm{Irr}\, P_n$ and $\mathrm{Irr}\, H_n$. By (2.3), every irreducible character of P_n (resp. H_n) can be expressed as $I_{L_{k+1}}^{P_n}(\zeta)$ (resp. $I_{G_k}^{H_n}(\chi)$) for some $\zeta\in \mathrm{Irr}\, L_{k+1}$ (resp. $\chi\in \mathrm{Irr}\, G_k$), $0\leqq k<n$. Since $L_{k+1}\simeq G_1\times G_k$, we have $\zeta=\alpha\cdot\chi$ for some $\alpha\in \mathrm{Irr}\, G_1$ and $\chi\in \mathrm{Irr}\, G_k$.

PROPOSITION 2.8. *With the notation as above, we have the following branching rule for* $P_n\to H_n$.

(i) $\mathrm{Res}_{H_n}^{P_n}\circ I_{L_{k+1}}^{P_n}(\alpha\cdot\chi)=I_{G_k}^{H_n}(\chi)$.

(ii) $\mathrm{Ind}_{H_n}^{P_n}\circ I_{G_k}^{H_n}(\chi)=\sum_{\alpha\in \mathrm{Irr}\, G_1} I_{L_{k+1}}^{P_n}(\alpha\cdot\chi)$.

PROOF. (i) Let $\varphi \in \operatorname{Irr} H_n$ be a component of $\operatorname{Res}^{P_n}_{H_n} \circ I^{P_n}_{L^n_{k+1}}(\alpha \cdot \chi)$. Then by a theorem of Clifford (cf. [1; 11.1]) and by (1.2), we have $\operatorname{Res}^{P_n}_{L^n_{k+1}} \circ I^{P_n}_{L^n_{k+1}}(\alpha \cdot \chi) = a\varphi$, for some positive integer a. On the other hand, we have

$$
\begin{aligned}
&(\operatorname{Res}^{P_n}_{H_n} \circ I^{P_n}_{L^n_{k+1}}(\alpha \cdot \chi),\ I^{H_n}_{G_k}(\chi))_{H_n} \\
&\quad = (\operatorname{Res}^{P_n}_{H_n} \circ I^{P_n}_{L^n_{k+1}}(\alpha \cdot \chi),\ (i_{U_n,\psi_n})^{H_n}_{H_{n-1}} \circ I^{H_{n-1}}_{G_k}(\chi))_{H_n} \\
&\quad = ((r_{U_n,\psi_n})^{H_n}_{H_{n-1}} \circ \operatorname{Res}^{P_n}_{H_n} \circ I^{P_n}_{L^n_{k+1}}(\alpha \cdot \chi),\ I^{H_{n-1}}_{G_k}(\chi))_{H_{n-1}} \\
&\quad = (\operatorname{Res}^{P_{n-1}}_{H_{n-1}} \circ (r_{U_n,\psi_n})^{P_n}_{P_{n-1}} \circ I^{P_n}_{L^n_{k+1}}(\alpha \cdot \chi),\ I^{H_{n-1}}_{G_k}(\chi))_{H_{n-1}} \\
&\quad = (\operatorname{Res}^{P_{n-1}}_{H_{n-1}} \circ I^{P_{n-1}}_{L^{n-1}_{k+1}}(\alpha \cdot \chi),\ I^{H_{n-1}}_{G_k}(\chi))_{H_{n-1}} \\
&\quad = \cdots\cdots \\
&\quad = (\operatorname{Res}^{P_{k+1}}_{H_{k+1}} \circ I^{P_{k+1}}_{L^{k+1}_{k+1}}(\alpha \cdot \chi),\ I^{H_{k+1}}_{G_k}(\chi))_{H_{k+1}} \\
&\quad = (\operatorname{Res}^{P_{k+1}}_{H_{k+1}} \circ I^{P_{k+1}}_{L^{k+1}_{k+1}}(\alpha \cdot \chi),\ (i_{U_{k+1},1})^{H_{k+1}}_{G_k}(\chi))_{H_{k+1}} \\
&\quad = ((r_{U_{k+1},1})^{H_{k+1}}_{G_k} \circ \operatorname{Res}^{P_{k+1}}_{H_{k+1}} \circ I^{P_{k+1}}_{L^{k+1}_{k+1}}(\alpha \cdot \chi),\ \chi)_{G_k} \\
&\quad = (\operatorname{Res}^{L_{k+1}}_{G_k} \circ (r_{U_{k+1},1})^{P_{k+1}}_{L^{k+1}_{k+1}} \circ (i_{U_{k+1},1})^{P_{k+1}}_{L^{k+1}_{k+1}}(\alpha \cdot \chi),\ \chi)_{G_k} \\
&\quad = (\operatorname{Res}^{L_{k+1}}_{G_k}(\alpha \cdot \chi),\ \chi)_{G_k} = (\chi,\ \chi)_{G_k} = 1\ .
\end{aligned}
$$

Thus $a=1$ and also the assertion follows.

(ii) Since $P_n/H_n \simeq G_1$, the second assertion follows from (i) and the Frobenius reciprocity.

§ 3. Branching rules.

In this section we consider the decomposition of $\operatorname{Res}^{G_n}_{P_n}(\zeta)$ for every $\zeta \in \operatorname{Irr} G_n$ and $\operatorname{Res}^{P_n}_{L_n}(\chi)$ for every $\chi \in \operatorname{Irr} P_n$. By the Frobenius reciprocity, this is equivalent to consider the decomposition of $\operatorname{Ind}^{G_n}_{P_n}(\chi)$ for every $\chi \in \operatorname{Irr} P_n$ and the decomposition of $\operatorname{Ind}^{P_n}_{L_n}(\rho)$ for every $\rho \in \operatorname{Irr} L_n$. For this purpose, we firstly recall a parametrization of $\operatorname{Irr} G_n$ due to J. A. Green [2]. Let $\bar{\boldsymbol{F}}_q$ be an algebraic closure of $\boldsymbol{F}_q$ and σ be the automorphism of $\bar{\boldsymbol{F}}_q$ defined by $\sigma(a)=a^q$ for all $a \in \bar{\boldsymbol{F}}_q$. Let $\mathscr{K}$ be the set of orbits of $\bar{\boldsymbol{F}}_q^\times$ under σ and so $\mathscr{K}$ naturally corresponds with $\mathscr{F}$ bijectively. Then $\operatorname{Irr} G_n$ can be parametrized by $S_n(\mathscr{K}, \boldsymbol{P}) = \{\lambda : \mathscr{K} \to \boldsymbol{P} \mid \sum_{O \in \mathscr{K}} |\lambda(O)|\,|O| = n\}$. For the details we refer the reader to [2]. For example, $\lambda \in S_n(\mathscr{K}, \boldsymbol{P})$ corresponds to a cuspidal representation of G_n if and only if $\lambda(O)=(1)$ for an orbit $O \in \mathscr{K}$ such that $|O|=n$ and $\lambda(O')=(0)$ for $O' \neq O$. We denote the irreducible character corresponding to λ by $[\lambda]$ and fix an injective homomorphism $\iota : \bar{\boldsymbol{F}}_q^\times \to \boldsymbol{C}^\times$, keeping it fixed throughout this paper. For $O \in \mathscr{K}$ and $\lambda \in S_n(\mathscr{K}, \boldsymbol{P})$, let $k(O)$ and $e(\lambda)$ be integers determined by

$$\iota(\prod_{\alpha \in O} \alpha) = \exp(2\pi\sqrt{-1}k(O)/(q-1)) , \quad \text{and}$$

$$e(\lambda) = \sum_{O \in \mathscr{X}} |\lambda(O)| \cdot k(O) .$$

Notice that the integers $k(O)$ and $e(\lambda)$ are determined only by mod $q-1$. Then $[\lambda]$ takes the following value on the central element $\tilde{\varepsilon} \cdot 1_n$ of G_n, where $\iota(\tilde{\varepsilon}) = \varepsilon = \exp(2\pi\sqrt{-1}/(q-1))$:

$$(3.1) \qquad [\lambda](\tilde{\varepsilon} \cdot 1_n) = \varepsilon^{e(\lambda)} \deg[\lambda] .$$

Next we recall a result of A. V. Zelevinsky [5], which plays a crucial role.

DEFINITION 3.2. (i) For $\lambda = (l_1, l_2, \cdots, l_r) \in \boldsymbol{P}$ with $r = r(\lambda)$, $\lambda^{\leftarrow}$ is the partition $(l_1 - 1, l_2 - 1, \cdots, l_r - 1)$.

(ii) For $\lambda = (l_1, \cdots)$, $\mu = (m_1, \cdots) \in \boldsymbol{P}$, $\mu \subset \lambda$ if and only if $m_i \leqq l_i$ for all $i = 1, 2, 3, \cdots$.

(iii) For λ, $\mu \in S(\mathscr{X}, \boldsymbol{P}) = \cup_{n \in N} S_n(\mathscr{X}, \boldsymbol{P})$, the relation $\mu \dashv \lambda$ holds if and only if $\lambda(O)^{\leftarrow} \subset \mu(O) \subset \lambda(O)$ for every $O \in \mathscr{X}$.

THEOREM 3.3 [5; 13.5]. (i) *Let* $\lambda \in S_n(\mathscr{X}, \boldsymbol{P})$. *Then*

$$\mathrm{Res}^{G_n}_{H_n}[\lambda] = \sum_{k=0}^{n-1} \sum_{\mu} I^{H_n}_{G_k}([\mu]) ,$$

where μ *runs over all* $S_k(\mathscr{X}, \boldsymbol{P})$ *such that* $\mu \dashv \lambda$.

(ii) *If* $\mu \in S_k(\mathscr{X}, \boldsymbol{P})$, $0 \leqq k < n$, *then*

$$\mathrm{Res}^{H_n}_{G_{n-1}} I^{H_n}_{G_k}[\mu] = \sum_{\lambda} [\lambda] ,$$

where λ *runs over all* $S_n(\mathscr{X}, \boldsymbol{P})$ *such that* $\mu \dashv \lambda$.

In particular, the restriction of any irreducible characters of G_n *(resp.* H_n*) to* H_n *(resp.* G_{n-1}*) is multiplicity free.*

Using (3.3) we can prove the following theorem.

THEOREM 3.4. (i) *Let* $\lambda \in S_n(\mathscr{X}, \boldsymbol{P})$. *Then*

$$\mathrm{Res}^{G_n}_{P_n}[\lambda] = \sum_{k=0}^{n-1} \sum_{\theta, \mu} I^{P_n}_{L_{k+1}}([\theta] \cdot [\mu]) ,$$

where the summation is over all $\theta \in S_1(\mathscr{X}, \boldsymbol{P})$ *and* $\mu \in S_k(\mathscr{X}, \boldsymbol{P})$, $0 \leqq k < n$, *such that* $\mu \dashv \lambda$ *and* $e(\lambda) \equiv e(\theta) + e(\mu)$ (mod $q-1$).

(ii) *Let* $\theta \in S_1(\mathscr{X}, \boldsymbol{P})$ *and* $\mu \in S_k(\mathscr{X}, \boldsymbol{P})$, $0 \leqq k < n$. *Then*

$$\mathrm{Res}^{P_n}_{L_n} I^{P_n}_{L_{k+1}}([\theta]\cdot[\mu]) = \sum_{\eta,\rho} [\eta]\cdot[\rho] ,$$

where the summation is over all $\eta \in S_1(\mathscr{K}, \boldsymbol{P})$ *and* $\rho \in S_{n-1}(\mathscr{K}, \boldsymbol{P})$ *such that* $\mu \dashv \rho$ *and* $e(\theta)+e(\mu) \equiv e(\eta)+e(\rho) \pmod{q-1}$.

PROOF. (i) Let $\lambda \in S_n(\mathscr{K}, \boldsymbol{P})$ and $\mu \in S_k(\mathscr{K}, \boldsymbol{P})$ for certain k such that $0 \leqq k < n$. Then we have

$$\begin{aligned}
&(\mathrm{Res}^{G_n}_{P_n}[\lambda], \sum_{\theta \in S_1(\mathscr{K}, P)} I^{P_n}_{L_{k+1}}([\theta]\cdot[\mu]))_{P_n} \\
&= (\mathrm{Res}^{G_n}_{P_n}[\lambda], \mathrm{Ind}^{P_n}_{H_n} I^{H_n}_{G_k}[\mu])_{P_n} && \text{(by (2.8.ii))} \\
&= (\mathrm{Res}^{G_n}_{H_n}[\lambda], I^{H_n}_{G_k}[\mu])_{H_n} && \text{(by Frobenius reciprocity)} \\
&= \begin{cases} 1 , & \text{if } \mu \dashv \lambda , \\ 0 , & \text{otherwise} . \end{cases} && \text{(by (3.3.i))}
\end{aligned}$$

Therefore if $\mu \dashv \lambda$, then there exists one and only one $\theta \in S_1(\mathscr{K}, \boldsymbol{P})$ such that

$$(\mathrm{Res}^{G_n}_{P_n}[\lambda], I^{P_n}_{L_{k+1}}([\theta]\cdot[\mu]))_{P_n} = 1 . \tag{3.3.1}$$

Now by (3.1), we have

$$\mathrm{Res}^{P_n}_{Z_n} \mathrm{Res}^{G_n}_{P_n}[\lambda] = \mathrm{Res}^{G_n}_{Z_n}[\lambda] = \deg[\lambda]\cdot\alpha ,$$

where $\alpha \in \mathrm{Irr}\, Z_n$ is defined by $\alpha(\tilde{\varepsilon}\cdot 1_n) = \varepsilon^{e(\lambda)}$. Similarly by (2.6.1), we have

$$\mathrm{Res}^{P_n}_{Z_n} I^{P_n}_{L_{k+1}}([\theta]\cdot[\mu]) = \deg(I^{P_n}_{L_{k+1}}([\theta]\cdot[\mu]))\cdot\beta ,$$

where $\beta \in \mathrm{Irr}\, Z_n$ is defined by $\beta(\tilde{\varepsilon}\cdot 1_n) = \varepsilon^{e(\theta)+e(\mu)}$. Thus the condition of θ for (3.3.1) to hold is given by

$$e(\lambda) \equiv e(\theta) + e(\mu) \pmod{q-1} .$$

Hence we have proved (i).

The assertion (ii) can be proved similarly using (2.8.i) and

$$\mathrm{Ind}^{L_n}_{G_{n-1}}[\rho] = \sum_{\eta \in S_1(\mathscr{K}, P)} [\eta]\cdot[\rho] ,$$

for $\rho \in S_{n-1}(\mathscr{K}, \boldsymbol{P})$. We omit the details.

Combining (3.3.i) and (3.3.ii), we have

COROLLARY 3.5. *Let* $\lambda \in S_n(\mathscr{K}, \boldsymbol{P})$. *Then*

$$\mathrm{Res}^{G_n}_{L_n}[\lambda] = \sum_{\eta,\rho} f(\lambda, \rho)[\eta]\cdot[\rho] ,$$

where η *(resp.* ρ*) runs over all* $S_1(\mathscr{K}, \boldsymbol{P})$ *(resp.* $S_{n-1}(\mathscr{K}, \boldsymbol{P})$*) such that* $e(\lambda)\equiv e(\eta)+e(\rho)$ (mod $q-1$) *and* $f(\lambda, \rho)$ *is the number of* $\mu\in S_k(\mathscr{K}, \boldsymbol{P})$, $0\leqq k<n$, *such that* $\mu\dashv\lambda$ *and* $\mu\dashv\rho$.

REMARK 3.6. The result (3.5) in different form was obtained by S. Nozawa [3], using fully the result of [2]. But one notices that by virtue of a result of E. Thoma [4], one can prove directly (3.5) by an argument used in the proof of (3.4) without going back to [2].

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