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On the Bernstein-Nikolsky Inequality

Ha Huy BANG* and Mitsuo MORIMOTO

Institute of Mathematics (Hanoi) and Sophia University

1. Introduction.

It is well-known that while trigonometric polynomials are good means of approximation of periodic functions, entire functions of exponential type may serve as a mean of approximation of nonperiodic functions, given on n-dimensional space. Some properties of entire functions of exponential type, bounded on the real space $\boldsymbol{R}^n$ have been considered in [1]. These results are very important in the imbedding theory, the approximation theory and applications. The present paper is a continuation of this direction.

2. Results.

Let $1 \leq p \leq \infty$ and $\sigma = (\sigma_1, \cdots, \sigma_n)$, $\sigma_j > 0$, $j = 1, \cdots, n$. Denote by $M_{\sigma,p}$ the space of all entire functions of exponential type σ which as functions of a real $x \in \boldsymbol{R}^n$ belong to $L_p(\boldsymbol{R}^n)$. The well-known Bernstein-Nikolsky inequality reads as follows (see [1], p. 114): Let $f(x) \in M_{\sigma,p}$. Then

$$\sigma^{-\alpha} \| D^\alpha f \|_p \leq \| f \|_p , \qquad \alpha > 0 . \tag{1}$$

We have the following result:

THEOREM 1. *Given* $1 \leq p < \infty$ *and* $f(x) \in M_{\sigma,p}$. *Then*

$$\lim_{|\alpha| \to \infty} \sigma^{-\alpha} \| D^\alpha f \|_p = 0 . \tag{2}$$

To prove this theorem we need the following results:

LEMMA 1. *Let* $0 < r \leq p \leq q \leq \infty$. *Then* $L_r(\boldsymbol{R}^n) \cap L_q(\boldsymbol{R}^n) \subset L_p(\boldsymbol{R}^n)$ *and*

$$\| f \|_p \leq \| f \|_r^t \| f \|_q^{1-t}$$

for any $f(x) \in L_r(\boldsymbol{R}^n) \cap L_q(\boldsymbol{R}^n)$, *where* $t = (1/p - 1/q)/(1/r - 1/q)$.

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This is a corollary to Hölder's inequality. See for example [2], p. 227.

LEMMA 2. *Let* $1<p<q<\infty$. *Then* $M_{\sigma,p}$ *is dense in* $M_{\sigma,q}$.

PROOF. Given $\varepsilon>0$ and $f(x)\in M_{\sigma,q}$. There exists a function $g(x)\in L_p(\boldsymbol{R}^n)\cap L_q(\boldsymbol{R}^n)$ such that $\|f-g\|_q<\varepsilon$. Hence (see [3], p. 100),

$$\|S_\sigma(f-g)\|_q\le A_q\|f-g\|_q<\varepsilon A_q\,,$$

where $S_\sigma h=\mathscr{F}^{-1}\chi_\sigma\mathscr{F}h$, $\mathscr{F}$ is the Fourier transform, χ_σ is the characteristic function of $\Delta_\sigma=\{\xi\,;\,|\xi_j|\le\sigma_j, j=1,\cdots,n\}$ and the constant A_q depends only on q. Consequently, taking account of $S_\sigma g\in M_{\sigma,p}$ (because of $1<p<\infty$) and $S_\sigma f=f$, we conclude $M_{\sigma,p}$ is dense in $M_{\sigma,q}$. (q.e.d.)

PROOF OF THEOREM 1. We divide the proof into four cases.

Case 1 ($p=2$). This case is easy: Given $\varepsilon>0$. We choose $\lambda>1$ so that

$$\int_{\Delta_\sigma\setminus\lambda^{-1}\Delta_\sigma}|\tilde{f}(\xi)|^2d\xi<\varepsilon\,,$$

where $\tilde{f}=\mathscr{F}f$. Hence, it follows from Parseval's theorem for $D^\alpha f$ that

$$\sigma^{-2\alpha}\|D^\alpha f\|_2^2<\sigma^{-2\alpha}\int_{\lambda^{-1}\Delta_\sigma}\xi^{2\alpha}|\tilde{f}(\xi)|^2d\xi+\varepsilon$$
$$\le\lambda^{-2|\alpha|}\|f\|_2^2+\varepsilon\,.$$

Therefore,

$$\limsup_{|\alpha|\to\infty}\sigma^{-2\alpha}\|D^\alpha f\|_2^2\le\varepsilon$$

and since $\varepsilon>0$ is arbitrarily chosen, we get (2).

Case 2 ($1<p<2$). We fix $1<r<p$. We notice that $M_{\sigma,r}\subset M_{\sigma,p}\subset M_{\sigma,2}$ (it follows from the Nikolsky inequality ([1], p. 125)). At first we show (2) for all $f(x)\in M_{\sigma,r}$. Applying Lemma 2 (with $q=2$), we have

$$\|D^\alpha f\|_p\le\|D^\alpha f\|_r^t\|D^\alpha f\|_2^{1-t}\,,\qquad \alpha\ge0$$

for each $f(x)\in M_{\sigma,r}$. Therefore, by (1) it follows that

$$\sigma^{-\alpha}\|D^\alpha f\|_p\le(\sigma^{-\alpha}\|D^\alpha f\|_r)^t(\sigma^{-\alpha}\|D^\alpha f\|_2)^{1-t}$$
$$\le\|f\|_r^t(\sigma^{-\alpha}\|D^\alpha f\|_2)^{1-t}\,,$$

which together with proved Case 1 implies (2).

Now let $f(x)\in M_{\sigma,p}$. For given $\varepsilon>0$, by Lemma 2, there is a function $g(x)\in M_{\sigma,r}$ such that $\|f-g\|_p<\varepsilon$. On the other hand, we have

$$\sigma^{-\alpha}\|D^\alpha f\|_p-\sigma^{-\alpha}\|D^\alpha g\|_p\le\sigma^{-\alpha}\|D^\alpha(f-g)\|_p\le\|f-g\|_p\,.$$

Therefore, taking account of

$$\lim_{|\alpha|\to\infty} \sigma^{-\alpha}\|D^\alpha g\|_p=0 ,$$

which was shown above, we get

$$\limsup_{|\alpha|\to\infty} \sigma^{-\alpha}\|D^\alpha f\|_p\le\varepsilon .$$

Case 2 is proved.

Case 3 ($2<p<\infty$). Invoking the density of $M_{\sigma,2}$ in $M_{\sigma,p}$, proved Case 1 and the last part of the proof of Case 2, we deduce (2) for all $f(x)\in M_{\sigma,p}$.

Case 4 ($p=1$). To prove this case we cannot invoke above proved cases. Given $f(x)\in M_{\sigma,1}$. Then

$$f(x)=\int_{\Delta_\sigma} e^{ix\xi}\tilde{f}(\xi)d\xi ,$$

where $\tilde{f}(\xi)\in C(\boldsymbol{R}^n)$ and $\tilde{f}(\xi)$ is vanishing in $\boldsymbol{R}^n\setminus\Delta_\sigma$. Further, let $\lambda>1$. Then taking account of

$$\int_{\Delta_{\lambda^{-1}\sigma}} e^{ix\xi}\tilde{f}(\lambda\xi)d\xi=\int_{\Delta_\sigma} e^{ix\xi}\tilde{f}(\lambda\xi)d\xi ,$$

we get

$$f(x)=\int_{\Delta_{\lambda^{-1}\sigma}} e^{ix\xi}\tilde{f}(\lambda\xi)d\xi+\int_{\Delta_\sigma} e^{ix\xi}(\tilde{f}(\xi)-\tilde{f}(\lambda\xi))d\xi . \tag{3}$$

Put

$$g(x)=\int_{\Delta_{\lambda^{-1}\sigma}} e^{ix\xi}\tilde{f}(\lambda\xi)d\xi ,$$

$$h(x)=\int_{\Delta_\sigma} e^{ix\xi}(\tilde{f}(\xi)-\tilde{f}(\lambda\xi))d\xi .$$

Then the type of exponential function $g(x)$ is $\lambda^{-1}\sigma$. Therefore

$$\limsup_{|\alpha|\to\infty} \sigma^{-\alpha}\|D^\alpha g\|_1\le\|g\|_1 \limsup_{|\alpha|\to\infty} \lambda^{-|\alpha|}=0 . \tag{4}$$

Invoking $\mathscr{F}^{-1}(\tilde{f}(\lambda\xi))=\lambda^{-n}f(\lambda^{-1}x)$, the type of exponential function $f(x)-\lambda^{-n}f(\lambda^{-1}x)$ is σ because the type of exponential function $f(\lambda^{-1}x)$ is $\lambda^{-1}\sigma<\sigma$, and by the Bernstein-Nikolsky inequality, we have

$$\begin{aligned}\sigma^{-\alpha}\|D^\alpha h\|_1 &= \sigma^{-\alpha}\|D^\alpha(f(x)-\lambda^{-n}f(\lambda^{-1}x))\|_1 \\ &\le \|f(x)-\lambda^{-n}f(\lambda^{-1}x)\|_1 \\ &\le \lambda^{-n}(\lambda^n-1)\|f\|_1+\lambda^{-n}\|f(x)-f(\lambda^{-1}x)\|_1 .\end{aligned} \tag{5}$$

For fixed $\varepsilon>0$, there is a number $\lambda_1>1$ such that

$$\lambda^{-n}(\lambda^n-1)\|f\|_1<\varepsilon, \qquad 1<\lambda\le\lambda_1 . \tag{6}$$

Further, we can choose $\delta>0$ such that for some $\lambda_2>1$

$$\|f(x)-f(\lambda^{-1}x)\|_{L_1(\boldsymbol{R}^n\setminus\Delta_\delta)}<\varepsilon, \qquad 1<\lambda\le\lambda_2 . \tag{7}$$

Then, it follows from the uniform continuity of the function $f(x)$ on Δ_δ we get a number $\lambda_3>1$ such that

$$\|f(x)-f(\lambda^{-1}x)\|_{L_1(\Delta_\delta)}<\varepsilon, \qquad 1<\lambda\le\lambda_3 . \tag{8}$$

Combining (5)–(8), we have

$$\sigma^{-\alpha}\|D^\alpha h\|_1<3\varepsilon, \qquad 1<\lambda\le\lambda_4=\inf\{\lambda_1, \lambda_2, \lambda_3\} . \tag{9}$$

Finally, put $\lambda=\lambda_4$. Then combining (3), (4) and (9) we get

$$\sigma^{-\alpha}\|D^\alpha f\|_1\le\sigma^{-\alpha}\|D^\alpha g\|_1+\sigma^{-\alpha}\|D^\alpha h\|_1<\lambda_4^{-|\alpha|}\|g\|_1+3\varepsilon .$$

Therefore,

$$\limsup_{|\alpha|\to\infty}\sigma^{-\alpha}\|D^\alpha f\|_1\le 3\varepsilon ,$$

and since $\varepsilon>0$ is arbitrarily chosen, we get

$$\lim_{|\alpha|\to\infty}\sigma^{-\alpha}\|D^\alpha f\|_1=0 .$$

The proof of Theorem 1 is complete.

REMARK 1. It easily follows from the Bernstein-Nikolsky inequality and Theorem 1 that $\sigma^{-\alpha}\|D^\alpha f\|_p$ converges decreasingly to 0.

REMARK 2. Theorem 1 does not hold if $p=\infty$. Actually, let

$$f(x)=\prod_{j=1}^{n}\sin\sigma_j x_j .$$

Then $f(x)\in M_{\sigma,\infty}$ and $\|D^\alpha f\|_\infty=\sigma^\alpha$, $\alpha\ge 0$.

Let us now consider the Bernstein-Nikolsky inequality for the directional derivatives of entire function of exponential type.

Suppose that $a=(a_1, \cdots, a_n)$ is an arbitrary real unit vector. Then

$$D_a f(x)=f'_a(x)=\sum_{j=1}^{n} a_j \frac{\partial f}{\partial x_j}(x)$$

is the derivative of f at the point x in the direction a, and

$$f_a^{(l)}(x)=D_a f_a^{(l-1)}(x)=\sum_{|\alpha|=l} a^\alpha f^{(\alpha)}(x), \qquad l=1, 2, \cdots$$

is the derivative of order l of f at x in the direction a.

Further, let $1 \leq p \leq \infty$ and $K \subset \boldsymbol{R}^n$ be a compact set. Denote by $M(K, p)$ the space of all functions $f(x) \in L_p(\boldsymbol{R}^n)$ such that $\operatorname{supp} \mathscr{F} f \subset K$.

We put

$$h_K(a)=\sup_{\xi \in K} |a\xi| .$$

Then we have the following result:

THEOREM 2. *Let $f(x) \in M(K, p)$. Then*

$$\|D_a^m f\|_p \leq [h_K(a)]^m \|f\|_p , \qquad m \geq 0 .$$

PROOF. We introduce the transformation

$$x=(x_1, \cdots, x_n) \to (\xi_1, \cdots, \xi_n)=\xi ,$$

where $\xi_1, \cdots, \xi_n$ are the coordinates of x in the new rectangular system of coordinates, which is chosen such a way that the increase of ξ_1 for fixed $\xi_2, \cdots, \xi_n$ will lead to a motion of the point x in the direction a. The coordinate transformation

$$x_k=\sum_{s=1}^{n} \alpha_{k,s} \xi_s , \qquad k=1, \cdots, n$$

is defined by a real orthogonal matrix $A=(\alpha_{k,s})$. Here, evidently we have

$$a_j=\alpha_{j,1} , \qquad j=1, \cdots, n \quad \text{and} \quad |\det A|=1 .$$

Put $g(\xi)=f(x)$. Then

$$\frac{\partial^m}{\partial \xi_1^m} g(\xi)=f_a^{(m)}(x) , \qquad m=1, 2, \cdots .$$

Now we show that $|y_1| \leq h_K(a)$ for each point $y \in \operatorname{supp} \tilde{g}(y)$. Actually, it is clear that for any function $f \in L_p(\boldsymbol{R}^n)$ $(1 \leq p \leq \infty)$, there exists a sequence of infinitely differentiable finite functions f_m such that $f_m \to f$ in the topology of $\mathscr{S}'$ (see, for example [1], p. 44). For this sequence we put

$$g_m(\xi)=f_m(x) , \qquad m=1, 2, \cdots .$$

Then $g_m \to g$ in the topology of $\mathscr{S}'$.

$$\tilde{g}(y) \leftarrow \tilde{g}_n(y) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{-i\xi y} g_n(\xi) d\xi$$
$$= \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{-iyA^{-1}x} f_n(x) dx$$
$$= \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{-ix^tA^{-1}y} f_n(x) dx$$
$$= \tilde{f}_n({}^tA^{-1}y) \to \tilde{f}({}^tA^{-1}y) .$$

Therefore,

$$\operatorname{supp} \tilde{g} = \{{}^tAx \ ; \ x \in \operatorname{supp} \tilde{f}\} \subset \{{}^tAx \ ; \ x \in K\} .$$

Denote by $({}^tAx)_1$ the first parameter of ${}^tAx$. Then

$$({}^tAx)_1 = \sum_{k=1}^{n} \alpha_{k,1} x_k = \sum_{k=1}^{n} a_k x_k .$$

Therefore

$$|y_1| \leq h_K(a) , \qquad y \in \operatorname{supp} \tilde{g}(y) .$$

Hence, using the Bernstein-Nikolsky inequality for the function $g(\xi)$, we get for $m = 1, 2, \cdots$

$$\|D_a^m f(x)\|_p = \left\| \frac{\partial_m}{\partial \xi_1^m} g(\xi) \right\|_p \leq [h_K(a)]^m \|g\|_p = [h_K(a)]^m \|f\|_p .$$

The proof of Theorem 2 is completed.

Using Theorem 1 and the proof of Theorem 2 we have the following result:

THEOREM 3. *Let* $f(x) \in M(K, p)$, $1 \leq p < \infty$. *Then*

$$\lim_{m \to \infty} [h_K(a)]^{-m} \|D_a^m f\|_p = 0 .$$

REMARK 3. Let $n = 1$. Then it was shown in [4] that: If $1 \leq p \leq \infty$ and $f(x) \in C^\infty(\mathbf{R}^1)$ such that $D^k f(x) \in L_p(\mathbf{R}^1)$, $k = 0, 1, \cdots$. Then there always exists the limit

$$d_f = \lim_{k \to \infty} \|D^k f\|_p^{1/k} ,$$

and moreover

$$d_f = \sigma_f = \sup\{|\xi| \ ; \ \xi \in \operatorname{supp} \tilde{f}(\xi)\} .$$

Therefore, if $\sigma_f < \infty$, using Theorem 1 and Remark 1, we get the following representation:

$$\|D^k f\|_p = \gamma_k \sigma_f^k \|f\|_p\,, \qquad k \geq 0\,,$$
$$0 < \gamma_{k+1} \leq \gamma_k \leq 1\,,$$
$$\lim_{k\to\infty} \gamma_k^{1/k} = 1\,,$$

and

$$\lim_{k\to\infty} \gamma_k = 0\,, \qquad \text{if} \quad 1 \leq p < \infty\,.$$

This representation says us about the speed of the convergence to 0 of the sequence $\sigma_f^{-k}\|D^k f\|_p$, $k=0, 1, \cdots$.

For the directional derivatives we also have the following representation: Let $f(x) \in M(K, p)$. Then

$$\|D_a^m f\|_p = \gamma_m [h_K(a)]^m \|f\|_p\,, \qquad m=0, 1, \cdots\,,$$
$$0 < \gamma_{m+1} \leq \gamma_m \leq 1\,,$$
$$\lim_{m\to\infty} \gamma_m^{1/m} = 1\,,$$

and

$$\lim_{m\to\infty} \gamma_m = 0\,, \qquad \text{if} \quad 1 \leq p < \infty\,.$$

We can prove the following theorem:

THEOREM 4. *Let $1 \leq p \leq \infty$ and I be some unbounded set of multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\alpha_j \geq 0, j=1, \cdots, n$, $0 \in I$. Let $f(x)$ be a nonconstant measurable function such that its generalized derivatives $D^\alpha f(x)$ belong to $L_p(\boldsymbol{R}^n)$ for all $\alpha \in I$. Then*

$$\lim_{|\alpha|\to\infty} \inf(|\xi^{-\alpha}|\|D^\alpha f\|_p)^{1/|\alpha|} \geq 1$$

for any point $\xi \in \operatorname{supp} \tilde{f}(\xi)$.

REMARK 4. Generalizing a result obtained in [5], we can prove Theorem 4. (The proof is long and will be published elsewhere.) We notice that this result is dual with the Bernstein-Nikolsky inequality. In this inequality the bound 1 cannot be improved.

From (1) and Theorem 4 we get

COROLLARY 1. *Let $1 \leq p \leq \infty$. Let $f(x) \in M_{\sigma,p}$ be not a constant and* supp $\tilde{f}(\xi)$ *contains at least one vertex of the parallelepiped Δ_σ. Then*

$$\lim_{|\alpha|\to\infty} (\sigma^{-\alpha}\|D^\alpha f\|_p)^{1/|\alpha|}=1 .$$

Here we cannot drop the assumption that $\mathrm{supp}\tilde{f}(\xi)$ contains at least one vertex of Δ_σ.

COROLLARY 2. *Let $1\le p\le\infty$ and let $f(x)$ be the function defined in Corollary 1. Then we have*

$$\|D^\alpha f\|_0=\gamma_\alpha\sigma^\alpha\|f\|_p , \qquad \alpha\ge 0 ,$$

$$0<\gamma_\beta\le\gamma_\alpha , \qquad \alpha\le\beta ,$$

$$\lim_{|\alpha|\to\infty} \gamma_\alpha^{1/|\alpha|}=1$$

and

$$\lim_{|\alpha|\to\infty} \gamma_\alpha=0 , \qquad \text{if} \quad 1\le p<\infty .$$

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Present Address:

HA HUY BANG
INSTITUTE OF MATHEMATICS
P. O. BOX 631, BO HO, 10.000, HANOI, VIETNAM

MITSUO MORIMOTO
DEPARTMENT OF MATHEMATICS, SOPHIA UNIVERSITY
KIOICHO, CHIYODA-KU, TOKYO 102, JAPAN