TOKYO J. MATH.
VOL. 14, NO. 1, 1991

Interpolation between Some Banach Spaces in Generalized Harmonic Analysis: The Real Method

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(Communicated by Y. Ito)

Introduction.

In [3], A. Beurling introduced the space $A^p(\boldsymbol{R}^1)$, $1<p<\infty$, as

$$A^p(\boldsymbol{R}^1)=\left\{f : \|f\|_{A^p(\boldsymbol{R}^1)}=\inf_{\omega\in\Omega}\left(\int_{-\infty}^{\infty}|f(x)|^p\omega(x)^{-(p-1)}dx\right)^{1/p}<\infty\right\},$$

where Ω is the class of functions ω on $\boldsymbol{R}^1$ such that ω is positive, even, nonincreasing with respect to $|x|$, and

$$\omega(0)+\int_{-\infty}^{\infty}\omega(x)dx=1 .$$

By regarding $A^p(\boldsymbol{R}^1)$ as an $L^1(\boldsymbol{R}^1)$ analog, Y. Chen and K. Lau [5] developed the H^1-theory analog. In particular, the maximal function characterization, the atomic decomposition, and the duality corresponding to Fefferman-Stein's H^1-BMO duality were shown. The $\boldsymbol{R}^n$ case was investigated by J. Garcia-Cuerva [6].

Recently, by regarding $A^p(\boldsymbol{R}^n)$ as an $L^p(\boldsymbol{R}^n)$ analog, K. Matsuoka [7] characterized the complex interpolation space $(A^{p_0}(\boldsymbol{R}^n), A^{p_1}(\boldsymbol{R}^n))_{[\theta]}$. His result is

$$(A^{p_0}(\boldsymbol{R}^n), A^{p_1}(\boldsymbol{R}^n))_{[\theta]}=(A^{p_0}(\boldsymbol{R}^n), A^{p_1}(\boldsymbol{R}^n))^{[\theta]}=A^p(\boldsymbol{R}^n) \qquad \text{(equal norms)},$$

where $1<p_0, p_1<\infty$, $0<\theta<1$, $1/p=(1-\theta)/p_0+\theta/p_1$. On the other hand, in the harmonic analysis, many real interpolation spaces have been studied by various authors: e.g.,

$$(L^{p_0}(\boldsymbol{R}^n), L^{p_1}(\boldsymbol{R}^n))_{\theta,p}=L^p(\boldsymbol{R}^n) \qquad \text{(equivalent quasi-norms)},$$

where $0<p_0, p_1<\infty$, $0<\theta<1$, $1/p=(1-\theta)/p_0+\theta/p_1$ (cf. J. Bergh and J. Löfström [2]).

In this paper, we will calculate the real interpolation space $(A^{p_0}(\boldsymbol{R}^n), A^{p_1}(\boldsymbol{R}^n))_{\theta,p}$, where $1<p_0, p_1<\infty$, $0<\theta<1$, $1/p=(1-\theta)/p_0+\theta/p_1$, and also show the related interpolation results.

Received May 30, 1990

§ 1. Preliminaries.

First, we will recall the definition of the real interpolation space (see C. Bennett and R. Sharpley [1], and J. Bergh and J. Löfström [2] for details).

Now, let A_0 and A_1 be two quasi-normed Abelian groups. Then we shall say that A_0 and A_1 are compatible if there is a Hausdorff topological vector space V such that $A_0 \hookrightarrow V$ and $A_1 \hookrightarrow V$. Here, the symbol "$\hookrightarrow$" means that the left hand side is continuously embedded in the right hand side.

DEFINITION 1.1. Let (A_0, A_1) be a couple of compatible quasi-normed Abelian groups. For any $a \in A_0 + A_1$ and $t > 0$, we put

$$K(t, a) = K(t, a; A_0, A_1) = \inf_{a = a_0 + a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}), \tag{1.1}$$

which is called the Peetre K-functional. Then, the real interpolation space $(A_0, A_1)_{\theta,q}$ is defined by

$$(A_0, A_1)_{\theta,q} = \{a \in A_0 + A_1 : \|a\|_{(A_0,A_1)_{\theta,q}} < \infty\}, \tag{1.2}$$

where

$$\|a\|_{(A_0,A_1)_{\theta,q}} = \begin{cases} \left(\int_0^\infty [t^{-\theta}K(t, a)]^q \frac{dt}{t}\right)^{1/q} & (0 < \theta < 1,\ 1 \le q < \infty) \\ \sup_{t>0} t^{-\theta}K(t, a) & (0 \le \theta \le 1,\ q = \infty). \end{cases} \tag{1.3}$$

Concerning the real interpolation space, there are the following three well-known theorems.

THEOREM 1.2 (The reiteration theorem). *Let $0 < \theta_0, \theta_1, \eta < 1$, $1 \le q_0, q_1, q \le \infty$ and (A_0, A_1) be a couple of compatible Banach spaces. Then*

$$((A_0, A_1)_{\theta_0,q_0}, (A_0, A_1)_{\theta_1,q_1})_{\eta,q} = (A_0, A_1)_{\theta,q} \quad (\textit{equivalent norms}), \tag{1.4}$$

where $\theta = (1-\eta)\theta_0 + \eta\theta_1$.

THEOREM 1.3 (The duality theorem). *Let $0 < \theta < 1$, $1 \le q < \infty$ and (A_0, A_1) be a couple of compatible Banach spaces such that $A_0 \cap A_1$ is dense in both A_0 and A_1. Then*

$$(A_0, A_1)^*_{\theta,q} = (A_0^*, A_1^*)_{\theta,q'} \quad (\textit{equivalent norms}), \tag{1.5}$$

where $1/q + 1/q' = 1$.

In the following theorem, for a quasi-normed Abelian group $(A, \|\cdot\|)$, the notation $(A)^\rho$ $(\rho > 0)$ means the space A provided with the quasi-norm $\|\cdot\|^\rho$.

THEOREM 1.4 (The power theorem). *Let $\rho_0, \rho_1 > 0$ and (A_0, A_1) be a couple of compatible quasi-normed Abelian groups. Put*

$$\theta=\frac{\eta\rho_1}{\rho}, \quad \rho=(1-\eta)\rho_0+\eta\rho_1, \quad q=\rho r.$$

Then

(1.6) $$((A_0)^{\rho_0}, (A_1)^{\rho_1})_{\eta,r}=((A_0, A_1)_{\theta,q})^{\rho} \quad \textit{(equivalent quasi-norms)},$$

where $0<\eta<1$ *and* $0<r\leq\infty$.

Next, we state the definitions of the so-called Beurling algebra A^p and the space B^p.

DEFINITION 1.5. For $1<p<\infty$, we shall define

(1.7) $$A^p=A^p(\boldsymbol{R}^n) = \left\{f : \|f\|_{A^p}=\inf_{\omega\in\Omega}\left(\int_{\boldsymbol{R}^n}|f(x)|^p\omega(x)^{-(p-1)}dx\right)^{1/p}<\infty\right\},$$

where Ω is the class of functions ω on $\boldsymbol{R}^n$ such that ω's are positive, radial, nonincreasing with respect to $|x|$, and

$$\omega(0)+\int_{\boldsymbol{R}^n}\omega(x)dx=1,$$

and

(1.8) $$B^p=B^p(\boldsymbol{R}^n) = \left\{f\in L^p_{loc}(\boldsymbol{R}^n) : \|f\|_{B^p}=\sup_{R\geq 1}\left(\frac{1}{|B(0, R)|}\int_{B(0,R)}|f(x)|^p dx\right)^{1/p}<\infty\right\},$$

where $B(0, R)$ is the open ball in $\boldsymbol{R}^n$, having center 0 and radius $R>0$.

It follows easily that A^p and B^p are Banach spaces, and that $C_c^\infty(\boldsymbol{R}^n)$, i.e. the class of C^∞ functions having compact support on $\boldsymbol{R}^n$, is dense in A^p (see e.g., Y. Chen and K. Lau [4] and J. Garcia-Cuerva [6]). Note also that

(1.9) $$L^1\cap L^{p_1}(\boldsymbol{R}^n)\supset A^{p_1}\supset A^{p_2} \quad \text{and} \quad B^{p_1}\supset B^{p_2}\supset L^\infty(\boldsymbol{R}^n)$$

if $1<p_1<p_2<\infty$.

The following result is a basic duality theorem.

PROPOSITION 1.6 (A. Beurling [3]). *For* $1<p, p'<\infty$, $1/p+1/p'=1$, $(A^p)^*$ *is isomorphic to* $B^{p'}$.

§ 2. Interpolation theorems.

In this section, we shall characterize the real interpolation spaces $(A^{p_0}, A^{p_1})_{\theta,p}$ and

$(B^{p_0}, B^{p_1})_{\theta, p}$, whose results in the complex method were shown by K. Matsuoka [7].

THEOREM 2.1. *Suppose* $1<p_1<\infty$ *and* $0<\theta<1$. *Then*

$$(L^1, A^{p_1})_{\theta,p}=A^p \qquad (\textit{equivalent norms}), \tag{2.1}$$

where $1/p=1-\theta+\theta/p_1$.

PROOF. The proof of this theorem is similar to the proof of Theorem 5.5.1 of J. Bergh and J. Löfström [2].

Using the power theorem 1.4,

$$(L^1, A^{p_1})^p_{\theta,p}=(L^1, (A^{p_1})^{p_1})_{\eta,1} \qquad \left(\eta=\frac{\theta p}{p_1}\right).$$

Hence, we shall prove that

$$(L^1, (A^{p_1})^{p_1})_{\eta,1}=(A^p)^p . \tag{2.2}$$

Now, we have

$$\begin{aligned} K(t, f)&=K(t, f; L^1, (A^{p_1})^{p_1}) \\ &=\inf_{\omega\in\Omega}\ \inf_{f=f_0+f_1}\int_{\mathbf{R}^n}(|f_0(x)|+t|f_1(x)|^{p_1}\omega(x)^{-(p_1-1)})dx \\ &=\inf_{\omega\in\Omega}\int_{\mathbf{R}^n}\inf_{f(x)=f_0(x)+f_1(x)}(|f_0(x)|+t|f_1(x)|^{p_1}\omega(x)^{-(p_1-1)})dx \\ &=\inf_{\omega\in\Omega}\int_{\mathbf{R}^n}|f(x)|F(t|f(x)|^{p_1-1}\omega(x)^{-(p_1-1)})dx , \end{aligned}$$

where

$$F(s)=\inf_{y_0+y_1=1}(|y_0|+s|y_1|^{p_1})\sim\min(1, s) . \tag{2.3}$$

Therefore, it follows that

$$\begin{aligned} \|f\|_{(L^1,(A^{p_1})^{p_1})_{\eta,1}}&=\int_0^\infty t^{-\eta}K(t, f)\frac{dt}{t} \\ &=\int_0^\infty s^{-\eta}F(s)\frac{ds}{s}\cdot\inf_{\omega\in\Omega}\int_{\mathbf{R}^n}|f(x)|^{1-\eta+\eta p_1}\omega(x)^{-(-\eta+\eta p_1)}dx . \end{aligned}$$

Since $1-\eta+\eta p_1=p$, and writing

$$c_0=\int_0^\infty s^{-\eta}F(s)\frac{ds}{s}, \tag{2.4}$$

we conclude that

(2.5) $$\|f\|_{(L^1,(A^{p_1})^{p_1})_{\eta,1}}=c_0\|f\|^p_{A^p},$$

which gives (2.2). This completes the proof. ■

THEOREM 2.2. *Suppose* $1<p_0, p_1<\infty$ *and* $0<\theta<1$. *Then*

(2.6) $$(A^{p_0}, A^{p_1})_{\theta,p}=A^p \qquad (\textit{equivalent norms}),$$

where $1/p=(1-\theta)/p_0+\theta/p_1$.

PROOF. Given $1<p_0, p_1<\infty$ and $0<\theta<1$, choose $\max(p_0, p_1)<r<\infty$ and $1/p_i=1-\eta_i+\eta_i/r$ $(i=0, 1)$, $\eta=(1-\theta)\eta_0+\theta\eta_1$. Then, from the reiteration theorem 1.2 and Theorem 2.1, we infer that

$$(A^{p_0}, A^{p_1})_{\theta,p}=((L^1, A^r)_{\eta_0,p_0}, (L^1, A^r)_{\eta_1,p_1})_{\theta,p}=(L^1, A^r)_{\eta,p}=A^p. \quad ■$$

THEOREM 2.3. *Suppose* $1<p_0<\infty$ *and* $0<\theta<1$. *Then*

(2.7) $$(B^{p_0}, L^\infty)_{\theta,p}=B^p \qquad (\textit{equivalent norms}),$$

where $1/p=(1-\theta)/p_0$.

PROOF. $L^1\cap A^{p_0'}$ is dense in both L^1 and $A^{p_0'}$. Thus, using the duality theorem 1.3, we obtain, by Proposition 1.6 and Theorem 2.1,

$$(B^{p_0}, L^\infty)_{\theta,p}=(A^{p_0'}, L^1)^*_{\theta,p'}=(L^1, A^{p_0'})^*_{1-\theta,p'}=(A^{p'})^*=B^p. \quad ■$$

THEOREM 2.4. *Suppose* $1<p_0, p_1<\infty$ *and* $0<\theta<1$. *Then*

(2.8) $$(B^{p_0}, B^{p_1})_{\theta,p}=B^p \qquad (\textit{equivalent norms}),$$

where $1/p=(1-\theta)/p_0+\theta/p_1$.

PROOF. $A^{p_0'}\cap A^{p_1'}$ is dense in both $A^{p_0'}$ and $A^{p_1'}$. Thus, just as in the proof of Theorem 2.3, the desired conclusion follows from Proposition 1.6, the duality theorem 1.3 and Theorem 2.2. ■

ACKNOWLEDGEMENT. I would like to express my deep gratitude to Professor Sumiyuki Koizumi of Keio University for his helpful comments and encouragement.

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