TOKYO J. MATH.
VOL. 14, NO. 1, 1991

Hölder Continuity of Sample Paths of Some Self-Similar Stable Processes

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Dedicated to Professor Tatsuo Kawata on his eightieth birthday

1. Introduction and results.

A stochastic process $\{X(t)\}$ is said to be H-self-similar (H-ss) for $H>0$ if for any $c>0$, all finite-dimensional distributions of $\{X(ct)\}$ are the same as those of $\{c^H X(t)\}$, and to have stationary increments (si) if any finite-dimensional distribution of $\{X(t+b)-X(t)\}$ does not depend on b. It is also said to be α-stable if any finite-dimensional distribution of $\{X(t)\}$ is α-stable.

In this paper, we examine the Hölder continuity of H-ss si α-stable processes.

There are two main classes of H-self-similar α-stable processes with stationary increments: the linear fractional stable processes and the harmonizable fractional stable processes. In [T], Takashima showed the Hölder continuity of the linear fractional stable processes when $1<\alpha<2$ and $1/\alpha<H<1$, and also pointed out that the exponent in the Hölder continuity cannot be bigger than $H-1/\alpha$. However, we can get a better Hölder continuity for the harmonizable fractional stable processes as follows. The harmonizable fractional stable process is a complex-valued process defined by

$$X(t)=\int_{-\infty}^{\infty}\frac{e^{it\lambda}-1}{i\lambda}|\lambda|^{1-H-1/\alpha}d\tilde{M}_\alpha(\lambda)\,,$$

where $0<H<1$ and $\tilde{M}_\alpha$ is a complex rotationally invariant α-stable motion, (see [CM]). This is an H-ss si rotationally invariant α-stable process.

THEOREM 1. *Let $0<H<1$ and $0<\alpha<2$. For the harmonizable fractional stable process, there exists a version X^* such that*

Received May 18, 1990

* Research completed while the second author was visiting the Center for Stochastic Processes, University of North Carolina at Chapel Hill, supported by the Air Force Office of Scientific Research Contract No. F49620 85C 0144.

$$\lim_{\delta \downarrow 0} \sup_{\substack{t,s \in [0,1] \\ |t-s| < \delta}} \frac{|X^*(t) - X^*(s)|}{|t-s|^H |\log|t-s||^{1/\alpha + 1/2 + \varepsilon}} = 0$$

for any $\varepsilon > 0$.

In [KM], we gave a partial result on the Hölder continuity of the harmonizable fractional stable process, where H is replaced by any $\gamma < H$.

As mentioned above, Takashima [T] showed that the sample paths of the linear fractional stable process have the Hölder continuity of $|t-s|^{H-1/\alpha}$ with the help of some slowly varying function, if $1 < \alpha < 2$ and $1/\alpha < H < 1$. In the following theorem, we can see that this is also true for general H-ss si α-stable processes with $1 < \alpha < 2$ and $1/\alpha < H < 1$.

THEOREM 2. *Let* $X = \{X(t)\}_{t \geq 0}$ *be* H*-ss si* α*-stable with* $1 < \alpha < 2$ *and* $1/\alpha < H < 1$. *Then there exists a version* X^* *of* X *on* $[0, 1]$ *such that*

$$\lim_{\delta \downarrow 0} \sup_{\substack{t,s \in [0,1] \\ |t-s| < \delta}} \frac{|X^*(t) - X^*(s)|}{|t-s|^{H-1/\alpha} |\log|t-s||^{1/\alpha + 1 + \varepsilon}} = 0$$

for any $\varepsilon > 0$.

α-stable processes have γ-th moments for any $\gamma < \alpha$. Then, by the property of H-ss si, we have

$$\begin{aligned} E[|X(t) - X(s)|^\gamma] &= E[|t-s|^{H\gamma} |X(1)|^\gamma] \\ &= C|t-s|^{H\gamma}. \end{aligned}$$

If $1 < \alpha < 2$ and $1/\alpha < H < 1$, then we can find $1 < \gamma < \alpha$ such that $H\gamma > 1$. This means that H-ss si α-stable processes with $1 < \alpha < 2$ and $1/\alpha < H < 1$ satisfy Kolmogorov's moment condition

$$E[|X(t) - X(s)|^\gamma] \leq K|t-s|^{H\gamma}, \tag{1.1}$$

where $\gamma > 1$, $K > 0$, $H\gamma > 1$. It follows from Proposition 1.3 of [B] that (1.1) implies the existence of a version X^* of X satisfying

$$\lim_{\delta \downarrow 0} \sup_{\substack{t,s \in [0,1] \\ |t-s| < \delta}} \frac{|X^*(t) - X^*(s)|}{|t-s|^{H-1/\gamma} |\log|t-s||^{1/\gamma + \varepsilon}} = 0, \qquad \text{for any} \quad \varepsilon > 0. \tag{1.2}$$

However, it is noted that (1.2) is not enough to get Theorem 2, because we cannot replace γ in (1.2) by α.

The proofs of Theorems 1 and 2 are given in the subsequent sections.

2. Proof of Theorem 1.

The basic idea to prove Theorem 1 is to use the LePage representation of complex-valued rotationally invariant stable processes. The LePage representation allows us to regard stable processes as conditionally Gaussian processes and we next use the known results for Gaussian processes.

We state results for the LePage representation and Gaussian processes as lemmas.

Let ψ be an arbitrary probability measure equivalent to Lebesgue measure on $\boldsymbol{R}$ and let φ be its Radon-Nikodym derivative, $\psi(d\lambda)=\varphi(\lambda)d\lambda$. Let $\{\xi_j\}_{j\geq 1}$ be a sequence of iid random variables with the distribution ψ, and let $\{g_j\}_{j\geq 1}$ be a sequence of iid rotationally invariant complex-valued random variables with $E[g_1]=0$ and $E[|\mathrm{Re}\, g_1|^\alpha]=1$. Let $\{\Gamma_j\}_{j\geq 1}$ be a sequence of Poisson arrival times with unit rate. Suppose that $\{\xi_i\}$, $\{g_j\}$, $\{\Gamma_j\}$ are mutually independent.

LEMMA 1. *Let $0<\alpha<2$ and suppose $X=\{X(t)\}_{t\geq 0}$ is represented as*

$$X(t)=\int_{-\infty}^{\infty} f(t,\lambda)d\tilde{M}_\alpha(\lambda)\,.$$

Then $\{X(t)\}_{t\geq 0}$ has the same finite-dimensional distributions as $\{Y(t)\}_{t\geq 0}$ defined by

$$Y(t)=C\sum_{j=1}^{\infty} g_j\Gamma_j^{-1/\alpha}\varphi(\xi_j)^{-1/\alpha}f(t,\xi_j)\,, \tag{2.1}$$

where the last series converges almost surely for each t.

This result was shown in [MP]. However, there is a small gap in their proof, which is filled in [KM].

The next lemma due to [K] was shown for real-valued processes, but it is easily seen to be valid also for the complex-valued case. More precisely, the lemma can be given from Theorem 1, Corollary 1 and the comment at the end of the proof of Theorem 1 of [K].

LEMMA 2. *Let $\{Y(t)\}_{t\in[0,1]}$ be a centered Gaussian process satisfying*

$$E[|Y(t)-Y(s)|^2]\leq\sigma^2(|t-s|)\,,$$

where $\sigma(x)$ is a non-decreasing function defined on $(0,\infty)$ and $\sigma(x)|\log x|^{1/2}$ is also non-decreasing near the origin. Then

$$\lim_{\delta\downarrow 0}\sup_{|t-s|<\delta}\frac{|Y(t)-Y(s)|}{\sigma(|t-s|)|\log|t-s||^{1/2}}\leq\sqrt{2}\qquad\text{a.s.}$$

PROOF OF THEOREM 1. Recall that

$$X(t)=\int_{-\infty}^{\infty} f(t,\lambda)d\tilde{M}_\alpha(\lambda)\,,$$

where

$$f(t, \lambda)=\frac{e^{it\lambda}-1}{i\lambda}|\lambda|^{1-H-1/\alpha}.$$

Take

$$\varphi(\lambda)=\frac{a_\eta}{|\lambda||\log|\lambda||^{1+\eta}},$$

where $\eta>0$ and a_η is the normalization for $\int\varphi(\lambda)d\lambda=1$, and fix $\{\xi_j\}$ and $\{\Gamma_j\}$ in (2.1) to regard Y as a conditionally Gaussian process.

We denote the expectations with respect to $\{g_j\}$ and $\{\xi_j\}$ by E_g and E_ξ, respectively. In what follows, C denotes a positive constant which may differ from one inequality to another. We then have

$$\begin{aligned}(2.2)\qquad E_g[|Y(t)-Y(s)|^2]&=CE_g[|g_1|^2]\sum_{j=1}^{\infty}\Gamma_j^{-2/\alpha}\varphi(\xi_j)^{-2/\alpha}|f(t,\xi_j)-f(s,\xi_j)|^2\\&=Ca^2(|t-s|),\end{aligned}$$

where

$$\begin{aligned}a^2(z)&\le\sum_{j=1}^{\infty}\Gamma_j^{-2/\alpha}\varphi(\xi_j)^{-2/\alpha}\sup_{|t-s|<z}|f(t,\xi_j)-f(s,\xi_j)|^2\\&\le C\sum_{j=1}^{\infty}\Gamma_j^{-2/\alpha}\varphi(\xi_j)^{-2/\alpha}\{|z\xi_j|^2\wedge 1\}|\xi_j|^{-2(H+1/\alpha)}.\end{aligned}$$

Then we can prove

$$(2.3)\qquad \sum_{n=1}^{\infty}\frac{a^2(2^{-n})}{b^2(2^{-n})}<\infty\qquad \text{a.s. } (\xi, \Gamma),$$

where

$$b(t)=t^H|\log|t||^{(1+\eta)/\alpha}.$$

We are going to show (2.3). We have

$$\begin{aligned}E_\xi[a^2(z)]&\le C\sum_{j=1}^{\infty}\Gamma_j^{-2/\alpha}\int_0^\infty\varphi(x)^{1-2/\alpha}\{|zx|^2\wedge 1\}|x|^{-2(H+1/\alpha)}dx\\&=C\sum_{j=1}^{\infty}\Gamma_j^{-2/\alpha}\left\{\int_0^{1/z}+\int_{1/z}^{\infty}\right\}\\&=:C\sum_{j=1}^{\infty}\Gamma_j^{-2/\alpha}\{I_1+I_2\},\end{aligned}$$

where

$$I_1 \leq Cz^2 \int_0^{1/z} (x|\log x|^{1+\eta})^{-(1-2/\alpha)} x^{-2(H+1/\alpha)+2} dx$$
$$\leq Cz^{2H} |\log z|^{-(1+\eta)(1-2/\alpha)}$$

and

$$I_2 \leq C \int_{1/z}^{\infty} (x|\log x|^{1+\eta})^{-(1-2/\alpha)} x^{-2(H+1/\alpha)} dx$$
$$\leq Cz^{2H} |\log z|^{-(1+\eta)(1-2/\alpha)} .$$

Therefore

$$E_\xi[a^2(z)] \leq \left(C \sum_{j=1}^{\infty} \Gamma_j^{-2/\alpha} \right) z^{2H} |\log z|^{-(1+\eta)(1-2/\alpha)} ,$$

and thus

$$E_\xi\left[\sum_{j=1}^{\infty} \frac{a^2(2^{-n})}{b^2(2^{-n})} \right] \leq \left(C \sum_{j=1}^{\infty} \Gamma_j^{-2/\alpha} \right) \sum_{n=1}^{\infty} n^{-1-\eta} < \infty ,$$

which implies (2.3). Hence

$$\lim_{z \downarrow 0} \frac{a(z)}{b(z)} = 0 \qquad \text{a.s. } (\xi, \Gamma) ,$$

and for small $z > 0$,

$$a(z) \leq Cz^H |\log|z||^{(1+\eta)/\alpha} \qquad \text{a.s. } (\xi, \Gamma) .$$

This combined with (2.2) gives us

$$E_g[|Y(t) - Y(s)|^2] \leq C|t-s|^{2H} |\log|t-s||^{2(1+\eta)/\alpha} .$$

If we regard this right-hand side as $\sigma^2(|t-s|)$ in Lemma 2, it satisfies the conditions in Lemma 2. Therefore by Lemma 2, almost surely with respect to (ξ, Γ),

$$\lim_{\delta \downarrow 0} \sup_{|t-s|<\delta} \frac{|Y(t) - Y(s)|}{|t-s|^H |\log|t-s||^{1/\alpha + 1/2 + \varepsilon}} = 0$$

for any $\varepsilon > 0$. The proof is thus completed. □

3. Proof of Theorem 2.

We need a real variable lemma.

LEMMA 3. *Let $\{f(t)\}_{t \in [0,1]}$ be a real-valued continuous function. Then we have*

$$\sup_{|t-s|<2^{-n}} |f(t)-f(s)| \le 3 \sum_{r=n}^{\infty} \max_{1 \le k \le 2^r} |f((k+1)2^{-r})-f(k2^{-r})| .$$

PROOF. Write the binary expansion of $t \in [0, 1]$ as

$$t = \sum_{j=0}^{\infty} a_j(t) 2^{-j} , \qquad a_j(t) = 0 \text{ or } 1 ;$$

and put

$$t_r = \sum_{j=0}^{r} a_j(t) 2^{-j} .$$

Then for t, s satisfying $|t-s| \le 2^{-n}$,

$$\begin{aligned} |f(t)-f(s)| &\le \sum_{r=n}^{\infty} |f(t_{r+1})-f(t_r)| + |f(t_n)-f(s_n)| + \sum_{r=n}^{\infty} |f(s_{r+1})-f(s_r)| \\ &\le 3 \sum_{r=n}^{\infty} \max_{1 \le k \le 2^r} |f((k+1)2^{-r})-f(k2^{-r})| . \end{aligned}$$

This concludes the lemma. □

PROOF OF THEOREM 2. As mentioned in Section 1, H-ss si α-stable processes with $1<\alpha<2$ and $1/\alpha<H<1$ satisfy moment condition (1.1). Hence there exists a version X^* with continuous sample paths. We write it X for simplicity of the notation. We restrict $X(t)$ on $\{t \mid t \in [0, 1]\}$. Put

$$\Delta_n(X) = \max_{1 \le k \le 2^n} |X((k+1)2^{-n})-X(k2^{-n})| .$$

By Lemma 3, we see

$$\sup_{|t-s|<2^{-n}} |X(t)-X(s)| \le 3 \sum_{r=n}^{\infty} \Delta_r(X) . \tag{3.1}$$

Let $\Phi(x)$ be a nonnegative, nondecreasing convex function defined on $[0, \infty)$ satisfying $\Phi(0)=0$ and

$$\Phi(x) \sim \frac{x^{\alpha}}{(\log x)^{1+\eta}} \qquad \text{as} \quad x \to \infty$$

for some η with $0<\eta<\varepsilon\alpha$. Denote the inverse function of $\Phi(x)$ by $\Phi^{-1}(x)$. $\Phi^{-1}(x)$ is a nonnegative, nondecreasing concave function on $[0, \infty)$ and satisfies

$$\Phi^{-1}(x) \sim \frac{1}{\alpha} x^{1/\alpha} (\log x)^{(1+\eta)/\alpha} \qquad \text{as} \quad x \to \infty .$$

Since X is α-stable with $\alpha<2$, we know

$$P\{|X(1)|>x\}\sim Cx^{-\alpha} \qquad \text{as} \quad x\to\infty\,, \tag{3.2}$$

and therefore

$$E[\Phi(|X(1)|)]<\infty\,. \tag{3.3}$$

For simplicity, we put $\beta:=H-1/\alpha$ (>0) below.

We now have

$$\begin{aligned}
E\left[\sum_{n=1}^{\infty}\frac{\sup_{|t-s|\le 2^{-n}}|X(t)-X(s)|}{2^{-n\beta}n^{1/\alpha+1+\varepsilon}}\right] &\le 3\sum_{n=1}^{\infty}\frac{1}{2^{-n\beta}n^{1/\alpha+1+\varepsilon}}\sum_{r=n}^{\infty}2^{-rH}E\left[\frac{\Delta_r(X)}{2^{-rH}}\right] \qquad \text{(by (3.1))}\\
&=3\sum_{n=1}^{\infty}\frac{1}{2^{-n\beta}n^{1/\alpha+1+\varepsilon}}\sum_{r=n}^{\infty}2^{-rH}E\left[\Phi^{-1}\circ\Phi\left(\frac{\Delta_r(X)}{2^{-rH}}\right)\right]\\
&\leqq 3\sum_{n=1}^{\infty}\frac{1}{2^{-n\beta}n^{1/\alpha+1+\varepsilon}}\sum_{r=n}^{\infty}2^{-rH}\Phi^{-1}\left(E\left[\Phi\left(\frac{\Delta_r(X)}{2^{-rH}}\right)\right]\right)
\end{aligned}$$

(by Jensen's inequality), where we have

$$\begin{aligned}
E\left[\Phi\left(\frac{\Delta_r(X)}{2^{-rH}}\right)\right] &\leqq \sum_{k=1}^{2^r}E\left[\Phi\left(\frac{|X((k+1)2^{-r})-X(k2^{-r})|}{2^{-rH}}\right)\right]\\
&=\sum_{k=1}^{2^r}E[\Phi(|X(1)|)] \qquad \text{(by H-ss si)}\\
&=C2^r \qquad \text{(by (3.3))}\,.
\end{aligned}$$

Hence we have

$$E\left[\sum_{n=1}^{\infty}\frac{\sup_{|t-s|\le 2^{-n}}|X(t)-X(s)|}{2^{-n\beta}n^{1/\alpha+1+\varepsilon}}\right]\le C\sum_{n=1}^{\infty}\frac{1}{2^{-n\beta}n^{1/\alpha+1+\varepsilon}}\sum_{r=n}^{\infty}2^{-rH}\Phi^{-1}(C2^r)$$

$$\le C\sum_{n=1}^{\infty}\frac{1}{2^{-n\beta}n^{1/\alpha+1+\varepsilon}}\sum_{r=n}^{\infty}2^{-rH}2^{r/2}(\log 2^r)^{(1+\eta)/\alpha} \qquad \text{(by (3.2))}$$

$$\le C\sum_{n=1}^{\infty}\frac{1}{2^{-n\beta}n^{1/\alpha+1+\varepsilon}}2^{-n(H-1/\alpha)}n^{(1+\eta)/\alpha}<\infty\,,$$

implying

$$\sum_{n=1}^{\infty}\frac{\sup_{|t-s|\le 2^{-n}}|X(t)-X(s)|}{2^{-n\beta}n^{1/\alpha+1+\varepsilon}}<\infty \qquad \text{a.s.}$$

Therefore, there exists an N such that for any $n \geq N$,

$$\sup_{|t-s| \leq 2^{-n}} |X(t)-X(s)| < 2^{-n\beta} n^{1/\alpha+1+\varepsilon} \quad \text{a.s.} \tag{3.4}$$

For any t, s satisfying $|t-s| < 2^{-N}$, take $n \geq N$ such that $2^{-n} \leq |t-s| < 2^{-n+1}$. Then we have by (3.4)

$$|X(t)-X(s)| < |t-s|^{\beta} |\log|t-s||^{1/\alpha+1+\varepsilon} \quad \text{a.s.}$$

and hence

$$\sup_{|t-s| \leq 2^{-N}} \frac{|X(t)-X(s)|}{|t-s|^{\beta} |\log|t-s||^{1/\alpha+1+\varepsilon}} < 1 \quad \text{a.s.}$$

Since $\varepsilon > 0$ can be arbitrarily taken, we conclude Theorem 2. □

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