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Invariant Measures for the Multitype Voter Model

Yuki SUZUKI

Keio University

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1. Introduction.

The voter model was introduced independently by Clifford and Sudbury [1] and by Holley and Liggett [2]. For this model, a complete description of all invariant measures and ergodic theorems were obtained in [2] and [3]. On the other hand, Spitzer [7] introduced a generalized voter model as a class of infinite systems with locally interacting components, and the same problems were discussed in Liggett and Spitzer [4].

In this paper, we study the multitype voter model which is described as follows. Let S be a countable set and $p(x, y)$ the transition probability for an irreducible Markov chain on S. We regard S as a collection of voters, each having one of countably many possible positions on an issue. Every voter x waits an exponential time with parameter one and then, he chooses a voter y with probability $p(x, y)$ and adopts the position of y. For a set of positions on an issue we take $\boldsymbol{Z}_+=\{0, 1, 2, \cdots\}$, since the structure of the set does not play an important role in our model.

In the case there are only two possible positions, it is a genuine voter model. Our model corresponds to the generalized voter model of Spitzer [7] whose parameter is particularly zero. This case is not treated in [4].

Our model is defined on a state space

$$X=(\boldsymbol{Z}_+)^S=\{\eta : S\to \boldsymbol{Z}_+\} .$$

Here we equip $\boldsymbol{Z}_+$ with the discrete topology and X with the product topology. X is then a complete separable metric space. For $\eta\in X$ and $x\in S$, $\eta(x)\in \boldsymbol{Z}_+$ represents the position of the voter x. So $\eta\in X$ is regarded as a configuration of positions of the voters. First we construct a Markov process on X describing the time evolution of our model. We do this by using a stochastic differential equation associated with Poisson random measures. Next we find all extremal invariant measures for this model and determine the domain of attraction of each of them. To do this, we make use of Shiga's method in [6], who discussed the continuous time multi-allelic stepping stone models which

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were closely related to our model. The invariant measures for our model can be described in the same way as those models.

In Section 2 we state our results. In Section 3 we prove our theorem about the construction of a Markov process. In Section 4 we summarize the results on the two-position voter model in [2] and [3] for the later use. In Section 5 we prove our theorems about the invariant measures.

2. Results.

Given an irreducible transition function $p(x, y)$ on a countable set S, we first construct a Markov process on X describing the time evolution of our model explained in the introduction. Roughly speaking, it is a Markov process with generator

$$Lf(\eta)=\sum_{x\in S}\sum_{y\in S} p(x, y)\{f(\eta^{x,y})-f(\eta)\}\,, \tag{2.1}$$

where, for $\eta\in X$ and $x, y\in S$, $\eta^{x,y}\in X$ is defined by

$$\eta^{x,y}(z)=\begin{cases}\eta(y) & \text{if} \quad z=x\,, \\ \eta(z) & \text{if} \quad z\neq x\,, \quad z\in S\,.\end{cases}$$

For the construction of such a Markov process, we use a stochastic differential equation. Let $(\Omega, \mathscr{F}, P)$ be a probability space, equipped with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ of increasing sub-σ-fields of $\mathscr{F}$, and denote by E the expectation with respect to P. For $x\in S$, let $N_x(dsdy)$ be an $\mathscr{F}_t$-adapted Poisson random measure on $(0, \infty)\times S$ with intensity measure $dsp(x, y)dy$ and let $\{N_x(dsdy), x\in S\}$ be independent.

THEOREM 2.1. *For any $\eta\in X$, there exists a unique solution of*

$$\eta_t(x)=\eta(x)+\int_{(0,t]\times S}\{\eta_{s-}(y)-\eta_{s-}(x)\}N_x(dsdy)\,, \qquad x\in S\,, \quad t\geq 0\,. \tag{2.2}$$

Let $\boldsymbol{D}=\boldsymbol{D}([0, \infty)\to X)$ be the set of all functions on $[0, \infty)$ with values in X which are right continuous and have left limits. For a generic element $\xi.$ of $\boldsymbol{D}$, the value $(\in X)$ of $\xi.$ at time t is denoted by ξ_t. For $t\in[0, \infty)$, let $\mathscr{G}_t$ be the smallest σ-field on $\boldsymbol{D}$ with respect to which all ξ_s, $0\leq s\leq t$, are measurable, and put $\mathscr{G}=\bigvee_t\mathscr{G}_t$. For each $\eta\in X$, the unique solution of (2.2) defines a random variable $\eta.$ with values in $\boldsymbol{D}$. The distribution of $\eta.$ is denoted by P^η. This is a probability measure on $(\boldsymbol{D}, \mathscr{G})$. Denote by E^η the expectation with respect to P^η. Let $C_b(X)$ be the Banach space of all bounded continuous functions on X with the supremum norm, and let

$$\mathscr{D}=\{f\in C_b(X) : f \text{ depends only on finitely many coordinates}\}\,.$$

For $f\in\mathscr{D}$, Lf in (2.1) is well-defined. By Itô's formula, we have, for $f\in\mathscr{D}$,

$$f(\eta_t)-f(\eta)=\sum_{x\in S}\int_{(0,t]\times S}\{f(\eta_{s-}^{x,y})-f(\eta_{s-})\}N_x(dsdy)\,.$$

Therefore we see that P^η solves the following martingale problem:

$$\begin{cases} P^\eta\{\eta_0=\eta\}=1\,; \\ f(\eta_t)-f(\eta)-\displaystyle\int_0^t Lf(\eta_s)ds \text{ is a } \{\mathscr{G}_t\}\text{-martingale under } P^\eta \text{ for all } f\in\mathscr{D}\,. \end{cases}$$

In this sense $\{\eta_t, t\geq 0, P^\eta\}$ is associated with L and may be regarded as a Markov process describing the time evolution of our model. For $f\in C_b(X)$, define

$$S(t)f(\eta)=E^\eta[f(\eta_t)]\,,\qquad t\geq 0\,.$$

Then $\{S(t), t\geq 0\}$ is a contraction semigroup on $C_b(X)$ which corresponds to the Markov process $\{\eta_t, t\geq 0, P^\eta\}$. Note that this is not strongly continuous.

Let $\mathscr{P}(X)$ be the set of all probability measures on X with the topology of weak convergence. For $\mu\in\mathscr{P}(X)$, define $\mu S(t)\in\mathscr{P}(X)$ by the relation

$$\langle\mu S(t), f\rangle=\langle\mu, S(t)f\rangle$$

for all $f\in C_b(X)$, where $\langle\mu, f\rangle=\int_X f(\eta)\mu(d\eta)$. The probability measure $\mu S(t)$ is interpreted as the distribution at time t of the process when the initial distribution is μ. Let $\mathscr{I}$ be the set of all invariant measures for $\{S(t), t\geq 0\}$, i.e.,

$$\mathscr{I}=\{\mu\in\mathscr{P}(X) : \mu S(t)=\mu \text{ for all } t\geq 0\}\,.$$

$\mathscr{I}$ is a non-empty, closed and convex subset of $\mathscr{P}(X)$. Denote by $\mathscr{I}_e$ the set of all extreme points for $\mathscr{I}$. Then $\mathscr{I}$ is the closed convex hull of $\mathscr{I}_e$. Let

$$\mathscr{H}=\left\{\alpha : S\to[0, 1] : \sum_{y\in S}p(x, y)\alpha(y)=\alpha(x) \text{ for all } x\in S\right\},$$

and define

$$p_t(x, y)=e^{-t}\sum_{n=0}^{\infty}\frac{t^n}{n!}p^{(n)}(x, y)\,,\qquad x, y\in S\,,$$

where $p^{(n)}(x, y)$ are the n-step transition probabilities associated with $p(x, y)$. Let $W=\boldsymbol{D}([0, \infty)\to S)$ and denote by P_x, $x\in S$, the probability measure on W such that $\{w(t), t\geq 0, P_x\}$ is a continuous time Markov chain on S with transition function $p_t(\cdot, \cdot)$ starting at x. Define

$$g(x, y)=P_x\otimes P_y\{(w, z)\in W\times W : w(t)=z(t) \text{ for some } t\geq 0\}\,,\qquad x, y\in S\,.$$

Then, for any $x, y\in S$, $\{g(w(t), z(t)), t\geq 0, P_x\otimes P_y\}$ is a nonnegative supermartingale, so there exists

$$G=\lim_{t\to\infty} g(w(t), z(t))\,, \qquad P_x\otimes P_y\text{-a.s.}$$

In fact G is the indicator of the event

$$\mathscr{E}=\{\text{there exists } t_n\uparrow\infty \text{ such that } w(t_n)=z(t_n)\}\,.$$

For $\alpha\in\mathscr{H}$ and $x\in S$, $\{\alpha(w(t)), t\geq 0, P_x\}$ is a bounded martingale, so there exists $\lim_{t\to\infty}\alpha(w(t))$ P_x-a.s. According to [2], define

$$\mathscr{H}^*=\{\alpha\in\mathscr{H} : \lim_{t\to\infty}\alpha(w(t))=0 \text{ or } 1\ P_x\otimes P_y\text{-a.s. on } \mathscr{E} \text{ for any pair } (x, y)\}\,.$$

Let

$$\overline{\mathscr{H}}=\left\{\boldsymbol{h}=(h_a)_{a=0}^{\infty} : h_a\in\mathscr{H} \text{ for all } a\in\boldsymbol{Z}_+ \text{ and } \sum_{a=0}^{\infty} h_a=1\right\}$$

and let

$$\overline{\mathscr{H}^*}=\left\{\boldsymbol{h}=(h_a)_{a=0}^{\infty} : h_a\in\mathscr{H}^* \text{ for all } a\in\boldsymbol{Z}_+ \text{ and } \sum_{a=0}^{\infty} h_a=1\right\}.$$

For $\boldsymbol{h}\in\overline{\mathscr{H}}$, define $\nu_{\boldsymbol{h}}\in\mathscr{P}(X)$ by

$$\nu_{\boldsymbol{h}}\{\eta : \eta(x)=a(x) \text{ for all } x\in A\}=\prod_{x\in A} h_{a(x)}(x)\,,$$

where A is a finite subset of S and $a(x)\in\boldsymbol{Z}_+$ for all $x\in A$.

THEOREM 2.2. (1) *$\mu_{\boldsymbol{h}}=\lim_{t\to\infty}\nu_{\boldsymbol{h}}S(t)$ exists for all $\boldsymbol{h}\in\overline{\mathscr{H}}$, and $\mu_{\boldsymbol{h}}\in\mathscr{I}$.*

(2) *$\mu_{\boldsymbol{h}}\{\eta : \eta(x)=a\}=h_a(x)$ for all $x\in S$ and $a\in\boldsymbol{Z}_+$.*

(3) *$\mathscr{I}_e=\{\mu_{\boldsymbol{h}} : \boldsymbol{h}\in\overline{\mathscr{H}^*}\}$.*

THEOREM 2.3. (1) *Let $\mu\in\mathscr{P}(X)$ and $\boldsymbol{h}\in\overline{\mathscr{H}^*}$. Then $\lim_{t\to\infty}\mu S(t)=\mu_{\boldsymbol{h}}$ if and only if*

$$\lim_{t\to\infty}\sum_{y\in S} p_t(x, y)\mu\{\eta : \eta(y)=a\}=h_a(x) \tag{2.3}$$

and

$$\lim_{t\to\infty}\sum_{u,v\in S} p_t(x, u)p_t(x, v)\mu\{\eta : \eta(u)=\eta(v)=a\}=h_a^2(x) \tag{2.4}$$

for all $x\in S$ and $a\in\boldsymbol{Z}_+$.

(2) *Suppose $g(x, y)=1$ for all $x, y\in S$. In this case $\mathscr{I}_e=\{\delta_a : a\in\boldsymbol{Z}_+\}$, where $\delta_a\in\mathscr{P}(X)$ is the δ-measure at η such that $\eta(x)=a$ for all $x\in S$, a being a point of $\boldsymbol{Z}_+$. Let $(\alpha_a)_{a=0}^{\infty}$ satisfy $\alpha_a\in[0, 1]$ for all $a\in\boldsymbol{Z}_+$ and $\sum_{a=0}^{\infty}\alpha_a=1$, and let $\mu\in\mathscr{P}(X)$. Then $\lim_{t\to\infty}\mu S(t)=\sum_{a=0}^{\infty}\alpha_a\delta_a$ if and only if*

$$\text{(2.5)} \qquad \lim_{t\to\infty} \sum_{y\in S} p_t(x, y)\mu\{\eta : \eta(y)=a\} = \alpha_a$$

for all $x\in S$ *and* $a\in \boldsymbol{Z}_+$.

In the special case where $S=\boldsymbol{Z}^d$ and $p(x, y)=p(0, y-x)$, we obtain the following corollary from Theorem 2.3. Let $\mathscr{S}$ be the set of all shift invariant probability measures on X and let $\mathscr{S}_e$ be the set of all extreme points for $\mathscr{S}$.

COROLLARY 2.1. (1) *Suppose* $\{w(t)-z(t), t\geq 0, P_x\otimes P_y\}$ *is recurrent. In this case* $\mathscr{I}_e=\{\delta_a : a\in \boldsymbol{Z}_+\}$. *If* $\mu\in\mathscr{S}$, *then* $\lim_{t\to\infty}\mu S(t)=\sum_{a=0}^{\infty} h_a\delta_a$, *where* $h_a=\mu\{\eta : \eta(x)=a\}$ *for* $a\in\boldsymbol{Z}_+$.

(2) *Suppose* $\{w(t)-z(t), t\geq 0, P_x\otimes P_y\}$ *is transient. In this case* $\mathscr{I}_e=\{\mu_{\boldsymbol{h}} : \boldsymbol{h}=(h_a)_{a=0}^{\infty}, h_a\in[0, 1]$ *for all* $a\in\boldsymbol{Z}_+$ *and* $\sum_{a=0}^{\infty} h_a=1\}$. *If* $\mu\in\mathscr{S}_e$, *then* $\lim_{t\to\infty}\mu S(t)=\mu_{\boldsymbol{h}}$, *where* $\boldsymbol{h}=(h_a)_{a=0}^{\infty}$ *and* $h_a=\mu\{\eta : \eta(x)=a\}$ *for* $a\in\boldsymbol{Z}_+$.

3. Proof of Theorem 2.1.

In this section, we prove Theorem 2.1. For $n\geq 0$, let

$$\begin{cases} \eta_t^{(0)}(x)=\eta(x)\,, \\ \eta_t^{(n)}(x)=\eta(x)+\displaystyle\int_{(0,t]\times S}\{\eta_{s-}^{(n-1)}(y)-\eta_{s-}^{(n-1)}(x)\}N_x(dsdy)\,, \qquad n\geq 1\,. \end{cases}$$

Then $\eta_t^{(n)}(x)\in\boldsymbol{Z}$ for all $n\geq 0$. Define a metric in $\boldsymbol{Z}$ by

$$d(a, b)=\begin{cases} 0 & \text{if} \quad a=b\,, \\ 1 & \text{if} \quad a\neq b\,. \end{cases}$$

For $n\geq 1$, by Itô's formula, we see

$$\begin{aligned} d(\eta_s^{(n+1)}(x), \eta_s^{(n)}(x)) &= \int_{(0,s]\times S}\{d(\eta_{u-}^{(n+1)}(x)+\eta_{u-}^{(n)}(y)-\eta_{u-}^{(n)}(x)\,, \\ &\qquad \eta_{u-}^{(n)}(x)+\eta_{u-}^{(n-1)}(y)-\eta_{u-}^{(n-1)}(x))-d(\eta_{u-}^{(n+1)}(x), \eta_{u-}^{(n)}(x))\}N_x(dudy) \\ &\leq \int_{(0,s]\times S}\{d(\eta_{u-}^{(n)}(y), \eta_{u-}^{(n-1)}(y))+d(\eta_{u-}^{(n)}(x), \eta_{u-}^{(n-1)}(x))\}N_x(dudy)\,. \end{aligned}$$

Therefore

$$\begin{aligned} &E\left[\sup_{0\leq s\leq t} d(\eta_s^{(n+1)}(x), \eta_s^{(n)}(x))\right] \\ &\leq \int_0^t \sum_{y\in S} E[d(\eta_u^{(n)}(y), \eta_u^{(n-1)}(y))+d(\eta_u^{(n)}(x), \eta_u^{(n-1)}(x))]p(x, y)du\,. \end{aligned}$$

Put

$$\psi_n(t) = \sup_{x \in S} E\left[\sup_{0 \le s \le t} d(\eta_s^{(n+1)}(x), \eta_s^{(n)}(x))\right], \qquad n \ge 0 .$$

Then, for $n \ge 1$,

$$\psi_n(t) \le 2 \int_0^t \psi_{n-1}(u) du .$$

Since $\psi_0(t) \le t$, we have

$$\psi_n(t) \le \frac{2^n t^{n+1}}{(n+1)!}$$

for all $n \ge 0$. Therefore

$$\sum_{n=1}^{\infty} P\left\{\sup_{0 \le s \le t} d(\eta_s^{(n+1)}(x), \eta_s^{(n)}(x)) > \frac{1}{n^2}\right\} \le \sum_{n=1}^{\infty} n^2 \frac{2^n t^{n+1}}{(n+1)!} < \infty .$$

Hence, by Borel-Cantelli's lemma, we see with probability one that $\eta_t^{(n)}(x)$ converges uniformly on each finite t-interval as $n \to \infty$. Put

$$\eta_t(x) = \lim_{n \to \infty} \eta_t^{(n)}(x) .$$

Then this is a solution of (2.2). Since $\eta \in X$, we see $\eta_t \in X$ from (2.2). The uniqueness follows from Gronwall's lemma.

4. Two-position voter model.

In this section, we summarize the results on the invariant measures for the two-position voter model obtained in [2] and [3].

Let

$$X_0 = \{0, 1\}^S = \{\eta^0 : S \to \{0, 1\}\}$$

with the product topology. Then X_0 is a compact metric space. Let $C(X_0)$ be the Banach space of all continuous functions on X_0 with the supremum norm. Denote by $\{S_0(t), t \ge 0\}$ the strongly continuous contraction semigroup on $C(X_0)$ which corresponds to the Markov process on X_0 describing the time evolution of the two-position voter model, and define $\mathscr{P}(X_0)$, $\mathscr{I}_0$, and $(\mathscr{I}_0)_e$ associated with $\{S_0(t), t \ge 0\}$ in the same way as in Section 2. Then $\mathscr{I}_0$ is a non-empty, compact and convex subset of $\mathscr{P}(X_0)$, and is the closed convex hull of $(\mathscr{I}_0)_e$. For $\alpha \in \mathscr{H}$, define $\nu_\alpha \in \mathscr{P}(X_0)$ to be the product measure with marginals

$$\nu_\alpha\{\eta^0 : \eta^0(x) = 1\} = \alpha(x) .$$

THEOREM 4.1 ([2]).
(1) $\mu_\alpha = \lim_{t\to\infty} \nu_\alpha S_0(t)$ *exists for all* $\alpha \in \mathscr{H}$, *and* $\mu_\alpha \in \mathscr{I}_0$.
(2) $\mu_\alpha\{\eta^0 : \eta^0(x)=1\} = \alpha(x)$ *for all* $x \in S$.
(3) $(\mathscr{I}_0)_e = \{\mu_\alpha : \alpha \in \mathscr{H}^*\}$.

THEOREM 4.2 ([2], [3]).
(1) *Let* $\mu \in \mathscr{P}(X_0)$ *and* $\alpha \in \mathscr{H}^*$. *Then* $\lim_{t\to\infty} \mu S_0(t) = \mu_\alpha$ *if and only if*

$$\lim_{t\to\infty} \sum_{y\in S} p_t(x, y)\mu\{\eta^0 : \eta^0(y)=1\} = \alpha(x)$$

and

$$\lim_{t\to\infty} \sum_{u,v\in S} p_t(x, u)p_t(x, v)\mu\{\eta^0 : \eta^0(u)=\eta^0(v)=1\} = \alpha^2(x)$$

for all $x \in S$.

(2) *Suppose* $g(x, y)=1$ *for all* $x, y \in S$. *In this case* $(\mathscr{I}_0)_e = \{\delta_i : i=0, 1\}$. *Let* $\alpha \in [0, 1]$ *and* $\mu \in \mathscr{P}(X_0)$. *Then* $\lim_{t\to\infty} \mu S_0(t) = \alpha\delta_1 + (1-\alpha)\delta_0$ *if and only if*

$$\lim_{t\to\infty} \sum_{y\in S} p_t(x, y)\mu\{\eta^0 : \eta^0(y)=1\} = \alpha$$

for all $x \in S$.

5. Proof of Theorem 2.2 and Theorem 2.3.

To prove Theorem 2.2 and Theorem 2.3, we make use of Shiga's method in [5] and [6]. So we only sketch the proof. To prove the convergence of a family of probability measures on X, we need to show the tightness of the family and the convergence of the finite dimensional distributions, since our state space is not compact.

First we introduce a dual process. Let

$$Y = \{\zeta \in X : \text{Supp}(\zeta) \text{ is a finite set } (\neq \varnothing)\},$$

where $\text{Supp}(\zeta) = \{x \in S : \zeta(x) \neq 0\}$ for $\zeta \in X$. Then Y is a countable set. Here we regard $x \in S$ as a site and interpret $\zeta \in Y$ as follows. The relation $\zeta(x)=a$ means the absence of particles at the site x if $a=0$ and the existence of one particle of type a at x if $a \geq 1$. So $\zeta \in Y$ is regarded as a configuration of finitely many particles each of which has a type. Let Δ be an extra point and set

$$\hat{Y} = Y \cup \{\Delta\}.$$

This will be the state space for the dual process we are going to define. For $\eta \in X$ and $\zeta \in \hat{Y}$, define

$$F_\zeta(\eta)=\begin{cases}1 & \text{if } \zeta\in Y \text{ and } \zeta(x)=\eta(x) \text{ for all } x\in \mathrm{Supp}(\zeta),\\ 0 & \text{otherwise}.\end{cases}$$

This will be a duality function. Note that $F_\zeta\in\mathscr{D}$ for all $\zeta\in\hat{Y}$. Let $\hat{\mathscr{D}}$ be the set of all bounded functions on $\hat{Y}$. Define a linear operator on $\hat{\mathscr{D}}$ by

$$\begin{cases}\hat{L}f(\zeta)=\sum_{x\in\mathrm{Supp}(\zeta)}\sum_{y\notin\mathrm{Supp}(\zeta)}p(x,y)\{f(\zeta-\zeta(x)e_x+\zeta(x)e_y)-f(\zeta)\}\\ \qquad+\sum_{x\in\mathrm{Supp}(\zeta)}\sum_{\substack{y\in\mathrm{Supp}(\zeta)\\ y\neq x}}p(x,y)\delta_{\zeta(x),\zeta(y)}\{f(\zeta-\zeta(x)e_x)-f(\zeta)\}\\ \qquad+\sum_{x\in\mathrm{Supp}(\zeta)}\sum_{y\in\mathrm{Supp}(\zeta)}p(x,y)(1-\delta_{\zeta(x),\zeta(y)})\{f(\Delta)-f(\zeta)\}, \qquad \zeta\in Y,\\ \hat{L}f(\Delta)=0,\end{cases}$$

where $e_x\in X$, $x\in S$, is defined by

$$e_x(z)=\begin{cases}1 & \text{if } z=x,\\ 0 & \text{if } z\neq x, \quad z\in S,\end{cases}$$

and $\delta_{a,b}$, $a, b\in\boldsymbol{Z}_+$, is the Kronecker's δ. Then we get a continuous time Markov chain $\{\zeta_t, t\geq 0, \boldsymbol{P}_\zeta\}$ on $\hat{Y}$ with generator $\hat{L}$. Here $\boldsymbol{P}_\zeta$, $\zeta\in\hat{Y}$, is a probability measure on $\boldsymbol{D}([0,\infty)\to\hat{Y})$ and $\boldsymbol{P}_\zeta\{\zeta_0=\zeta\}=1$. Denote by $\boldsymbol{E}_\zeta$ the expectation with respect to $\boldsymbol{P}_\zeta$. This process is as follows. There is a configuration of finitely many particles and a particle at site x waits an exponential time with parameter one and then it chooses a site y with probability $p(x, y)$. If y is vacant, then it goes to y; if y is occupied by a particle which has the same type as it, then it goes to y and coalesces with that particle; otherwise the configuration ζ_t itself goes to Δ and remains there forever. In the two-position voter model in [2] and [3], the coalescing Markov chain was used for the dual process. Our dual process coincides with this, as long as all particles have the same type or every particle does not choose a site which is occupied by a particle of a different type. We see $\{\eta_t, t\geq 0, P^\eta\}$ and $\{\zeta_t, t\geq 0, \boldsymbol{P}_\zeta\}$ are in the following duality relation.

LEMMA 5.1. *For any $\eta\in X$, $\zeta\in\hat{Y}$, and $t\geq 0$,*

$$E^\eta[F_\zeta(\eta_t)]=\boldsymbol{E}_\zeta[F_{\zeta_t}(\eta)].$$

Before proving Theorem 2.2, we prepare a lemma about the tightness.

LEMMA 5.2. *Let $\mu\in\mathscr{P}(X)$. Then $\{\mu S(t)\}_{t\geq 0}$ is a tight family in $\mathscr{P}(X)$ if and only if for any $\varepsilon>0$ and any $x\in S$, there exist $a_0\in\boldsymbol{Z}_+$ and $t_0\geq 0$ such that*

$$\sum_{a=a_0+1}^{\infty}\sum_{y\in S}p_t(x,y)\mu\{\eta:\eta(y)=a\}\leq\varepsilon$$

for all $t \geq t_0$.

PROOF. It is obvious since, by Lemma 5.1, we have

$$\mu S(t)\{\eta : \eta(x)=a\} = \sum_{y\in S} p_t(x, y)\mu\{\eta : \eta(y)=a\}$$

for all $t\geq 0$, $a\in \boldsymbol{Z}_+$, and $x\in S$.

PROOF OF THEOREM 2.2. To prove (1), let $\boldsymbol{h}\in\overline{\mathscr{H}}$. By Lemma 5.2, we easily see $\{\nu_{\boldsymbol{h}}S(t)\}_{t\geq 0}$ is a tight family in $\mathscr{P}(X)$. So it is enough to show $\lim_{t\to\infty}\langle \nu_{\boldsymbol{h}}S(t), F_\zeta\rangle$ exists for all $\zeta\in Y$. Let $\zeta\in Y$ and write the elements of Supp(ζ) by $x_1, x_2, \cdots, x_n$. By Lemma 5.1,

$$\langle \nu_{\boldsymbol{h}}S(t), F_\zeta\rangle = \boldsymbol{E}_\zeta\left[\int_X F_{\zeta_t}(\eta)\nu_{\boldsymbol{h}}(d\eta)\right]$$

for all $t\geq 0$. For $0\leq k\leq n$, define

$$\tau_k=\inf\{t\geq 0 : \zeta_t\in Y \text{ and } \#\mathrm{Supp}(\zeta_t)=n-k\}$$

and let

$$\tau=\inf\{t\geq 0 : \zeta_t=\Delta\} .$$

We adopt here the convention that the infimum over the empty set is ∞. These are the stopping times. Now we consider $\{(w_1(t), w_2(t), \cdots, w_n(t)), t\geq 0, P_{x_1}\otimes P_{x_2}\otimes\cdots\otimes P_{x_n}\}$, which is a continuous time Markov chain on S^n starting at $(x_1, x_2, \cdots, x_n)$. Denote by $E_{(x_1,\cdots,x_n)}$ the expectation with respect to $P_{x_1}\otimes P_{x_2}\otimes\cdots\otimes P_{x_n}$. Let

$$\sigma=\inf\{t\geq 0 : w_i(t)=w_j(t) \text{ for some } 1\leq i\neq j\leq n\} .$$

Then we easily see

$$\text{(5.1)}\qquad \{\mathrm{Supp}(\zeta_t), 0\leq t\leq \tau\wedge\tau_1, \boldsymbol{P}_\zeta\} \stackrel{d}{=} \{\{w_1(t), w_2(t), \cdots, w_n(t)\}, 0\leq t\leq\sigma, P_{x_1}\otimes P_{x_2}\otimes\cdots\otimes P_{x_n}\} .$$

By the strong Markov property,

$$\lim_{t\to\infty} \boldsymbol{E}_\zeta\left[\int_X F_{\zeta_t}(\eta)\nu_{\boldsymbol{h}}(d\eta), \tau<\infty\right]=0 .$$

Therefore

$$\text{(5.2)}\qquad \lim_{t\to\infty} \boldsymbol{E}_\zeta\left[\int_X F_{\zeta_t}(\eta)\nu_{\boldsymbol{h}}(d\eta)\right]=\lim_{t\to\infty} \boldsymbol{E}_\zeta\left[\int_X F_{\zeta_t}(\eta)\nu_{\boldsymbol{h}}(d\eta), \tau=\infty\right].$$

Define

$$H(t, \zeta)=\boldsymbol{E}_\zeta\left[\int_X F_{\zeta_t}(\eta)\nu_{\boldsymbol{h}}(d\eta), \tau\wedge\tau_1=\infty\right], \qquad t\geq 0 .$$

Then, by (5.1),

$$\lim_{t\to\infty} H(t,\zeta)=\lim_{t\to\infty} E_\zeta\left[\prod_{x\in\mathrm{Supp}(\zeta_t)} h_{\zeta_t(x)}(x),\ \tau\wedge\tau_1=\infty\right]$$
$$=\lim_{t\to\infty} E_{(x_1,\cdots,x_n)}\left[\prod_{i=1}^{n} h_{\zeta(x_i)}(w_i(t)),\ \sigma=\infty\right]. \tag{5.3}$$

Since

$$E_{(x_1,\cdots,x_n)}\left[\prod_{i=1}^{n} h_{\zeta(x_i)}(w_i(t))\right]=\prod_{i=1}^{n} h_{\zeta(x_i)}(x_i)$$

for all $t\geq 0$, (5.3) becomes, by the strong Markov property,

$$\lim_{t\to\infty} H(t,\zeta)=\prod_{i=1}^{n} h_{\zeta(x_i)}(x_i)-E_{(x_1,\cdots,x_n)}\left[\prod_{i=1}^{n} h_{\zeta(x_i)}(w_i(\sigma)),\ \sigma<\infty\right]. \tag{5.4}$$

Denote the right-hand side of (5.4) by $H(\zeta)$. Then (5.2) becomes

$$\lim_{t\to\infty} E_\zeta\left[\int_X F_{\zeta_t}(\eta)\nu_h(d\eta)\right]=\lim_{t\to\infty}\sum_{k=0}^{n-1} E_\zeta\left[\int_X F_{\zeta_t}(\eta)\nu_h(d\eta),\ \tau=\infty,\ \tau_k<\infty,\ \tau_{k+1}=\infty\right]$$
$$=\sum_{k=0}^{n-1} E_\zeta\left[\lim_{t\to\infty} H(t,\zeta_{\tau_k}),\ \tau_k<\infty\right]$$
$$=\sum_{k=0}^{n-1} E_\zeta[H(\zeta_{\tau_k}),\ \tau_k<\infty]\,.$$

Now we have shown that $\mu_h=\lim_{t\to\infty}\nu_h S(t)$ exists. We easily see $\mu_h\in\mathscr{I}$. (2) follows from the fact that $\nu_h S(t)\{\eta:\eta(x)=a\}=h_a(x)$ for all $t\geq 0$, $a\in\boldsymbol{Z}_+$, and $x\in S$.

We prove Theorem 2.3 before showing Theorem 2.2 (3). According to [6], define a mapping $\pi_a: X\to X_0$, $a\in\boldsymbol{Z}_+$, by

$$(\pi_a\eta)(x)=\begin{cases}1 & \text{if}\quad \eta(x)=a\,,\\ 0 & \text{if}\quad \eta(x)\neq a\,,\quad \eta\in X,\ x\in S\,.\end{cases}$$

Then we easily obtain the following lemma.

LEMMA 5.3. *For any $f_0\in C(X_0)$, $\eta\in X$, and $a\in\boldsymbol{Z}_+$,*

$$S(t)(f_0\circ\pi_a)(\eta)=(S_0(t)f_0)(\pi_a\eta)\,.$$

PROOF OF THEOREM 2.3. To prove (1), assume first that $\lim_{t\to\infty}\mu S(t)=\mu_h$. Since $\pi_a\nu_h=\nu_{h_a}$, we have $\pi_a\mu_h=\mu_{h_a}$ by Lemma 5.3, where $\pi_a\mu\in\mathscr{P}(X_0)$ represents the image measure of $\mu\in\mathscr{P}(X)$ under the map π_a. Therefore we have $\lim_{t\to\infty}(\pi_a\mu)S_0(t)=\mu_{h_a}$ by Lemma 5.3. Hence we obtain (2.3) and (2.4) from Theorem 4.2 (1). For the converse, assume that (2.3) and (2.4) hold for all $x\in S$ and $a\in\boldsymbol{Z}_+$. It is enough to show

$$(5.5) \qquad \lim_{t\to\infty} \langle \mu S(t), F_\zeta \rangle = \langle \mu_h, F_\zeta \rangle$$

for all $\zeta \in Y$. Let $\zeta \in Y$ and consider the Markov chain $\{(w_1(t), w_2(t), \cdots, w_n(t)), t \geq 0, P_{x_1} \otimes P_{x_2} \otimes \cdots \otimes P_{x_n}\}$ introduced in the proof of Theorem 2.2 (1). Define

$$F(t, \zeta) = \boldsymbol{E}_\zeta \left[\int_X F_{\zeta_t}(\eta)\mu(d\eta), \tau \wedge \tau_1 = \infty \right], \qquad t \geq 0 .$$

Then, by (5.1),

$$(5.6) \qquad \lim_{t\to\infty} F(t, \zeta) = \lim_{t\to\infty} \boldsymbol{E}_\zeta[\mu\{\eta : \eta(x) = \zeta_t(x) \text{ for all } x \in \mathrm{Supp}(\zeta_t)\}, \tau \wedge \tau_1 = \infty]$$

$$= \lim_{t\to\infty} E_{(x_1, \cdots, x_n)}[\mu\{\eta : \eta(w_i(t)) = \zeta(x_i), i = 1, 2, \cdots, n\}, \sigma = \infty] .$$

By (2.3) and (2.4), we have

$$\lim_{t\to\infty} E_{(x_1, \cdots, x_n)}[\mu\{\eta : \eta(w_i(t)) = \zeta(x_i), i = 1, 2, \cdots, n\}] = \prod_{i=1}^{n} h_{\zeta(x_i)}(x_i) .$$

Therefore (5.6) becomes, by the strong Markov property,

$$\lim_{t\to\infty} F(t, \zeta) = H(\zeta) .$$

Hence we obtain (5.5) in the same way as in the proof of Theorem 2.2 (1). Next we show (2). The necessity of (2.5) for $\lim_{t\to\infty} \mu S(t) = \sum_{a=0}^{\infty} \alpha_a \delta_a$ follows from Theorem 4.2 (2). To prove the sufficiency of (2.5), it is enough to show

$$(5.7) \qquad \lim_{t\to\infty} \langle \mu S(t), F_\zeta \rangle = \left\langle \sum_{a=0}^{\infty} \alpha_a \delta_a, F_\zeta \right\rangle$$

for all $\zeta \in Y$. Let $\zeta \in Y$ and assume first that there exist $x, y \in \mathrm{Supp}(\zeta)$ such that $\zeta(x) \neq \zeta(y)$. Then $\boldsymbol{P}_\zeta\{\tau < \infty\} = 1$. Therefore, by the strong Markov property,

$$\lim_{t\to\infty} \langle \mu S(t), F_\zeta \rangle = \lim_{t\to\infty} \boldsymbol{E}_\zeta \left[\int_X F_{\zeta_t}(\eta)\mu(d\eta), \tau < \infty \right] = 0 .$$

So we obtain (5.7) in this case. Secondly assume that $\zeta(x) = a$ for all $x \in \mathrm{Supp}(\zeta)$ for some $a \geq 1$, and let $\#\mathrm{Supp}(\zeta) = n$. Then $\boldsymbol{P}_\zeta\{\tau_{n-1} < \infty\} = 1$. Therefore, by the strong Markov property and (2.5),

$$\lim_{t\to\infty} \langle \mu S(t), F_\zeta \rangle = \lim_{t\to\infty} \boldsymbol{E}_\zeta \left[\int_X F_{\zeta_t}(\eta)\mu(d\eta), \tau_{n-1} < \infty \right] = \alpha_a .$$

So we obtain (5.7) in this case, too. This completes the proof of (2).

To prove Theorem 2.2 (3), we introduce the following lemma, in which the assertion

(1) is used to prove the assertion (2). These assertions are natural extensions of those in the two-position case and can be proved in a similar way (see [3]).

LEMMA 5.4. *If $\mu\in\mathcal{I}_e$, then the following assertions hold.*

(1) $\lim_{t\to\infty}\sum_{y\in S}p_t(x,y)\mu\{\eta:\eta(y)=\eta(z)=a\}=\mu\{\eta:\eta(x)=a\}\mu\{\eta:\eta(z)=a\}$

for all $x, z\in S$ and $a\in \boldsymbol{Z}_+$.

(2) $E_{(x,y)}[\mu\{\eta:\eta(w(t))=\eta(z(t))=a\}]\downarrow\mu\{\eta:\eta(x)=a\}\mu\{\eta:\eta(y)=a\}$

as $t\uparrow\infty$ for all $x, y\in S$ and $a\in\boldsymbol{Z}_+$, where $E_{(x,y)}$ represents the expectation with respect to $P_x\otimes P_y$.

Now we show Theorem 2.2 (3) by using the method in [5].

PROOF OF THEOREM 2.2 (3). Let $\boldsymbol{h}\in\overline{\mathcal{H}^*}$. Then we see $\mu_{\boldsymbol{h}}\in\mathcal{I}_e$ by Theorem 2.3 (1). Conversely let $\mu\in\mathcal{I}_e$ and define

$$\boldsymbol{h}=(h_a)_{a=0}^{\infty},$$

where $h_a(x)=\mu\{\eta:\eta(x)=a\}$ for $a\in\boldsymbol{Z}_+$ and $x\in S$. Then we see $\boldsymbol{h}\in\overline{\mathcal{H}^*}$ and $\mu=\mu_{\boldsymbol{h}}$ by Theorem 2.3 (1) and Lemma 5.4 (2) in the same way as in the two-position case (see [3]) where the proof was based on [5]. This completes the proof of Theorem 2.2.

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Present Address:
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND TECHNOLOGY, KEIO UNIVERSITY
HIYOSHI, KOHOKU-KU, YOKOHAMA 223, JAPAN