TOKYO J. MATH.
VOL. 14, NO. 2, 1991

On Morimoto Algorithm in Diophantine Approximation

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Introduction.

Let us denote the continued fraction expansion of an irrational number α $(0<\alpha<1)$ by

$$\alpha=[0: e_1, e_2, \cdots],$$

and its n-th convergent by p_n/q_n. We call the sequence of partial quotients $\{e_i : i=1, 2, \cdots\}$ the name of α associated with the simple continued fraction algorithm. The following theorems are well known.

THEOREM A. (1) (Galois) *α is a reduced quadratic irrational, that is, a quadratic irrational whose algebraic conjugate $\bar{\alpha}$ satisfies $\bar{\alpha}<-1$, iff the name of α is purely periodic.*

(2) (Lagrange) *α is a quadratic irrational iff the name of α is eventually periodic.*

(3) (Klein) *Let $\Gamma_{(\pm)}$ be a polygon jointing the lattice points (q_{2n-1}, p_{2n-1}), $n=1, 2, \cdots$ $((q_{2n}, p_{2n}), n=0, 1, \cdots$ for $\Gamma_-)$ in this order, then the polygons are approximating polygons of the line L: $\alpha x-y=0$, that is, $\Gamma_{(\pm)}$ satisfies the following properties*:

(i) *$\Gamma_{(\pm)}$ is a convex (concave) polygon, and*

(ii) *The domain D enclosed by Γ_+ and Γ_- in the first quadrant includes the half line $\alpha x-y=0$, $x\geqq 0$, and the domain D does not contain any lattice point.*

(4) (Lévy) *For almost all α, we have*

1) $\displaystyle\lim_{n\to\infty}\frac{1}{n}\log q_n=\frac{\pi^2}{12\log 2}$ *and*

2) $\displaystyle\lim_{n\to\infty}\left(-\frac{1}{n}\right)\log|q_n\alpha-p_n|=\frac{\pi^2}{12\log 2}.$

The purpose of this paper is to give an extension of above theorems to inhomogeneous linear forms $\alpha x+\beta-y$. Morimoto ([4]) presented a generalized algorithm of the simple continued fraction expansion, which induces vertex points (q_n, p_n)

Received December 3, 1990

of the approximating polygon of an inhomogeneous line L: $\alpha x+\beta-y=0$. We call this algorithm Morimoto algorithm. The first aim of this paper is to give the definition of Morimoto algorithm in terms of a transformation (X, T). Because, the original algorithm was given geometrically as an analogy of Klein's construction.

The algorithm is given as follows: Let X, X_1 and X_2 be subsets of $\boldsymbol{R}^2$ such that

$$X:=\{(\alpha, \beta):\ 0\leqq\alpha+\beta\leqq 1, 0\leqq\alpha\}-\{(0, 1)\}\,,$$

$$X_1:=\{(\alpha, \beta)\in X: 0\leqq\beta\} \quad \text{and}$$

$$X_2:=\{(\alpha, \beta)\in X: 0\geqq\beta\}\,.$$

For a positive integer $a\in \boldsymbol{N}$ and $\varepsilon\in\{-1, 1\}$, let us define a partition $\Delta(a)$ and $\Delta(a, \varepsilon)$ of X_1 and X_2 by

$$\Delta(a):=\{(\alpha, \beta)\in X_1:\ a\alpha+\beta\leqq 1, (a+1)\alpha+\beta>1\}\,,$$

$$\Delta(\mathrm{a}, -1):=\{(\alpha, \beta)\in X_2:\ a\alpha+\beta\leqq 1, \alpha>1/a\} \quad \text{and}$$

$$\Delta(a, 1):=\{(\alpha, \beta)\in X_2:\ (a+1)\alpha+\beta>1, \alpha\leqq 1/a\}\,. \quad \text{(See Fig. 1.)}$$

Let us define a transformation T on X by

$$T(\alpha, \beta):=\begin{cases}\left(\dfrac{1}{\alpha}-a, -\dfrac{\beta}{\alpha}\right) & \text{if} \quad (\alpha, \beta)\in\Delta(a)\cup\Delta(a, 1)\,,\\ \left(a-\dfrac{1}{\alpha}, 1+\dfrac{\beta}{\alpha}\right) & \text{if} \quad (\alpha, \beta)\in\Delta(a, -1)\,,\\ (\alpha, \beta) & \text{if} \quad \alpha=0\,.\end{cases} \tag{0.1}$$

Using integer valued functions $a(\alpha, \beta)$ and $\varepsilon(\alpha, \beta)$ by

$$\begin{aligned} a(\alpha, \beta)&:=\left[\frac{1-\beta}{\alpha}\right],\\ \varepsilon(\alpha, \beta)&:=\begin{cases}1 & \text{if} \quad (\alpha, \beta)\in\Delta(a)\cup\Delta(a, 1)\,,\\ -1 & \text{if} \quad (\alpha, \beta)\in\Delta(a, -1)\,,\end{cases}\end{aligned} \tag{0.2}$$

we define the sequence of integer vectors $\{(a_n, \varepsilon_n): n=1, 2, \cdots\}$ by

$$\begin{aligned} a_n&:=a(T^{n-1}(\alpha, \beta))\,,\\ \varepsilon_n&:=\varepsilon(T^{n-1}(\alpha, \beta))\,.\end{aligned} \tag{0.3}$$

This is essentially the same as 'die Folge von den charakteristischen Zahlentripel' in Morimoto [4] and is called a *name* of (α, β) with respect to the transformation (X, T) in this paper.

By Morimoto algorithm (X, T), we have the following main theorem as a generalization of Theorem A.

THEOREM. *For each* (α, β) $(0<\alpha, 0\leqq\beta, \alpha+\beta\leqq 1)$,

(1) (α, β) *has a finite name iff* $\alpha\in\boldsymbol{Q}$.

(2) $\beta\in\boldsymbol{Z}\alpha+\boldsymbol{Z}$ *iff there exists an* n *such that* $\alpha_{2n}=\beta_{2n}$ *or* $\beta_{2n}=0$ *where* $(\alpha_n, \beta_n)=T^n(\alpha, \beta)$.

(3) *The name of* (α, β) *is purely periodic iff* (α, β) *is reduced, that is,*

(i) α *is a quadratic irrational,*

(ii) $\beta\in\boldsymbol{Q}(\alpha)$ *where* $\boldsymbol{Q}(\alpha)$ *is the quadratic field generated by* α, *and*

(iii) *the pair of algebraic conjugates* $(\bar{\alpha}, \bar{\beta})$ *satisfies the relation* $1\leqq\bar{\beta}\leqq\bar{\alpha}$ *or* $\bar{\alpha}+1\leqq\bar{\beta}\leqq 0$.

(4) *The name of* (α, β) *is eventually periodic iff* α *is a quadratic irrational and* $\beta\in\boldsymbol{Q}(\alpha)$.

The main idea to prove the theorem is to determine the notion that (α, β) is reduced, and it is equivalent to determine the domain of a bijective lifting of the transformation (X, T) which is called the natural extension in the ergodic theory.

By means of the natural extension and ergodic theorems, we have also the following metrical theorem.

THEOREM. *For almost all* $(\alpha, \beta)\in X_1$, *we have*

$$\lim_{n\to\infty}\left(-\frac{1}{n}\right)\log|\alpha q_n+\beta-p_n|=\frac{\pi^2}{12\log 2} \quad \textit{and}$$

$$\lim_{n\to\infty}\frac{1}{n}\log q_n=\frac{\pi^2}{12\log 2},$$

where (q_n, p_n) *are vertices of the approximating polygons defined in* (1.7).

Finally, we remark that analogous discussions on the relations between the algebraic property of (α, β) and periodicity of inhomogeneous diophantine algorithms are found in Hara-Ito [1] and [2].

§ 1. Definition of Morimoto algorithm and its fundamental properties.

Let X, X_1, X_2, $\Delta(a)$, $\Delta(a, 1)$, $\Delta(a, -1)$, $T(\alpha, \beta)$, $a(\alpha, \beta)$, $\varepsilon(\alpha, \beta)$ and (a_n, ε_n) be as in the introduction. (See Fig. 1).

We see

$$X=X_1\cup X_2, \quad X_1\cap X_2=[0, 1]\times\{0\},$$

$$X_1=\bigcup_{a=1}^{\infty}\Delta(a)\cup I_0 \quad \text{(disjoint sum)} \quad \text{and}$$

$$X_2=\bigcup_{\varepsilon\in\{-1,1\}}\bigcup_{a=1}^{\infty}\Delta(a, \varepsilon)\cup\{(0, 0)\} \quad \text{(disjoint sum)},$$

where $I_0:=\{(\alpha, \beta)\in X: \alpha=0\}$.

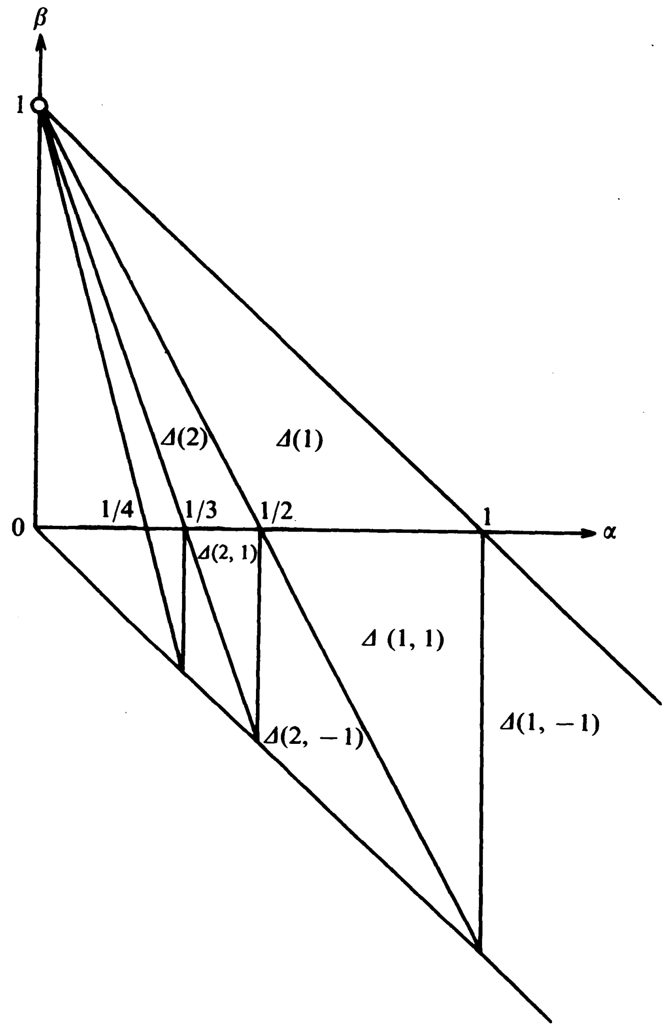


FIGURE 1.

The transformation T satisfies the following relations:

$$T(\Delta(a)) = X_2 \cap \{\alpha+\beta \neq 1\},$$

$$T(\Delta(a, 1)) = X_1 \cap \{\alpha+\beta \neq 1\}, \quad \text{and}$$

$$T(\Delta(\mathrm{a}, -1)) = X_1 \cap \{\alpha \neq 0\}.$$

REMARK 1.1. Let us observe the behavior of the transformation T on boundaries of X, X_1 and X_2 and on invariant sets. Denote the pieces of boundaries by

$$I := [0, 1] \times \{0\} = X_1 \cap X_2,$$

$$J_0 := \{(\alpha, \beta) \in X_2 : \alpha+\beta = 0\},$$

$$J_1 := \{(\alpha, \beta) \in X_1 : \alpha+\beta = 1\}, \quad \text{and}$$

$$J_2 := \{(\alpha, \beta) \in X_2 : \alpha+\beta = 1\}.$$

Then we see the following properties hold:

(1) $T(\partial\Delta(a)) \subset \partial X_2$, $T(\partial\Delta(a, \varepsilon)) \subset \partial X_1$, where ∂A means the boundary of a set A. (See Fig. 2).

(2) $T(J_2 - \{(1, 0)\}) = J_1 - \{(1, 0)\}$, $T(J_1) = J_0$ and $T(J_0) = I$.

(3) $T(I) = I - \{(1, 0)\}$ and the restriction $T|_I$ of T on I coincides with the simple continued fraction transformation S:

$$S(\alpha) := \begin{cases} \dfrac{1}{\alpha} - \left[\dfrac{1}{\alpha}\right] & \text{if } 0 < \alpha \leqq 1\,, \\ 0 & \text{if } \alpha = 0\,. \end{cases}$$

(4) The set $K = \{(\alpha, \beta) \in X : \alpha = \beta\}$ is T-invariant and the restriction $T|_K$ is also isomorphic to the simple continued fraction transformation S by the isomorphism $\phi : (\alpha, \alpha) \mapsto \alpha/(1-\alpha)$.

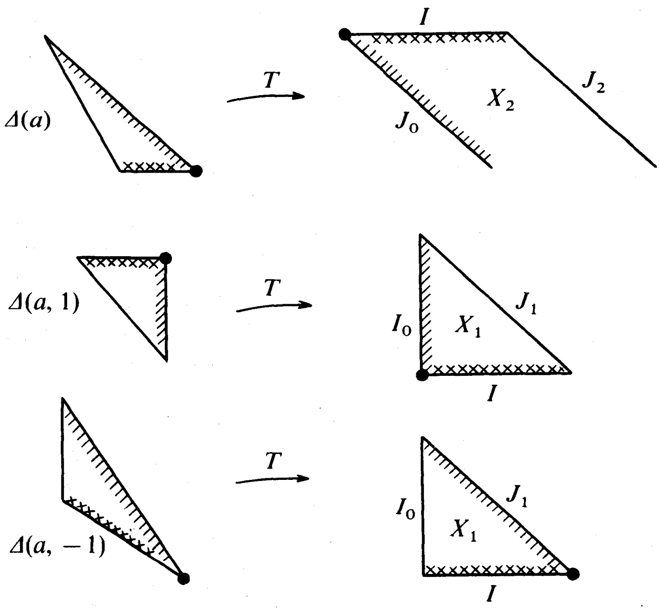


FIGURE 2.

For each $(\alpha, \beta) \in X - I_0$, we defined a finite or infinite sequence of integer vectors $\{(a_n, \varepsilon_n) : n = 1, 2, \cdots\}$ in (0.3), so called a *name* of (α, β) with respect to the transformation (X, T). In particular, for $(\alpha, \beta) \in X_1 - I_0$, the name of (α, β) is given by

$$T^{2k}(\alpha, \beta) \in \Delta(a_{2k+1}) \quad \text{and}$$
$$T^{2k+1}(\alpha, \beta) \in \Delta(a_{2k+2}, \varepsilon_{2k+2}) \qquad (k = 0, 1, 2, \cdots)\,.$$

We say (α, β) has a *finite name* if there exists j such that $T^j(\alpha, \beta) \in I_0$.

Let us denote

$$(\alpha_n, \beta_n) = T^n(\alpha, \beta) .$$

REMARK 1.2. We see that $(\alpha, \beta) \in X$ has a finite name iff the number α is a rational. In fact, it is easy to see that $\alpha \in \boldsymbol{Q}$ if $\alpha_n = 0$ for some n. Conversely, we assume α is a rational. If $\alpha = 1$, then we have $\alpha_1 = 0$. So we put $\alpha = p/q$, $(p, q) = 1$, $0 < p < q$. Then, from the definition of T, we see $\alpha_1 = (q - ap)/p$ or $(ap - q)/p$ and $0 \leqq |q - ap| \leqq p$. Therefore α_1 is denoted by $\alpha_1 = p_1/q_1$, $(p_1, q_1) = 1$ and $q_1 < q$. Continuing this procedure, we obtain the conclusion.

From now on, we assume that α is irrational. Let us define the affine transformation $\phi_{(a_k, \varepsilon_k)}$ (or simply we write it by ϕ_{a_k}), which is a map from (x_k, y_k)-plane to (x_{k-1}, y_{k-1})-plane, associated with the name $\{(a_k, \varepsilon_k) : k = 1, 2, \cdots\}$ of (α, β) by

$$\phi_{(a_k, \varepsilon_k)} : \begin{pmatrix} x_{k-1} \\ y_{k-1} \end{pmatrix} = \begin{pmatrix} a_k & \varepsilon_k \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix} + \begin{pmatrix} v_k \\ 0 \end{pmatrix} \tag{1.1}$$

where

$$v_k = \begin{cases} 0 & \text{if } \varepsilon_k = 1 , \\ 1 & \text{if } \varepsilon_k = -1 , \end{cases} \tag{1.2}$$

and $(x_0, y_0) = (x, y)$.

We put

$$\sigma_n = (-1)^n \varepsilon_1 \varepsilon_2 \cdots \varepsilon_n . \tag{1.3}$$

Let us denote lines associated with (α_n, β_n) by $L_n : \alpha_n x_n + \beta_n - y_n = 0$ $(L_0 = L)$. Then we have propositions:

PROPOSITION 1.1. *For each* $(\alpha, \beta) \in X$ $(\alpha \notin \boldsymbol{Q})$,

$$\alpha x + \beta - y = \sigma_n \alpha \alpha_1 \cdots \alpha_{n-1} (\alpha_n x_n + \beta_n - y_n) .$$

In particular, we have

$$\phi_{a_1} \circ \cdots \circ \phi_{a_n}(L_n) = L .$$

PROOF. From the definition of ϕ_{a_1} and T, we see that

$$\begin{aligned} \alpha x + \beta - y &= \alpha(a_1 x_1 + \varepsilon_1 y_1 + v_1) + \beta - x_1 \\ &= (-1)\alpha \left\{ \left(\frac{1}{\alpha} - a_1 \right) x_1 + \left(-v_1 - \frac{\beta}{\alpha} \right) - \varepsilon_1 y_1 \right\} \\ &= (-1)\alpha \varepsilon_1 (\alpha_1 x_1 + \beta_1 - y_1) . \end{aligned}$$

Therefore, we obtain the conclusion by induction. q.e.d.

Let us introduce 2×2 matrices as follows:

$$(1.4)\qquad \begin{pmatrix} r_n & s_n \\ t_n & u_n \end{pmatrix} = \begin{pmatrix} a_1 & \varepsilon_1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & \varepsilon_n \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} r_0 & s_0 \\ t_0 & u_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then we have formulae:

PROPOSITION 1.2. *For each* $(\alpha, \beta) \in X$ $(\alpha \notin \boldsymbol{Q})$,

(1) $\displaystyle \alpha\alpha_1 \cdots \alpha_{n-1} = \frac{1}{r_n + s_n \alpha_n}$,

(2) $\displaystyle \phi_{a_1} \circ \cdots \circ \phi_{a_n} : \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r_n & s_n \\ t_n & u_n \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} \sum_{k=0}^{n-1} v_{k+1} r_k \\ \sum_{k=0}^{n-1} v_{k+1} t_k \end{pmatrix}$.

PROOF. Let us assume the formula (1) holds for $n-1$, then by $\alpha_n = 1/(a_{n+1} + \varepsilon_{n+1}\alpha_{n+1})$, we have

$$\alpha\alpha_1 \cdots \alpha_{n-1}\alpha_n = \frac{1/(a_{n+1} + \varepsilon_{n+1}\alpha_{n+1})}{r_n + s_n(1/(a_{n+1} + \varepsilon_{n+1}\alpha_{n+1}))} = \frac{1}{r_{n+1} + s_{n+1}\alpha_{n+1}}.$$

The statement (2) is also obtained by induction. q.e.d.

LEMMA 1.3. *Let us assume* $(\alpha, \beta) \in X_1$, *then* r_n, s_n, t_n *satisfy the following inequalities*:

(1) $r_{2n+1} > 0$, $s_{2n+1} > 0$ *and* $r_{2n+3} > r_{2n-1}$. *If* $\varepsilon_{2n} = 1$, *then* $r_{2n+1} > r_{2n-1}$.

(2) $s_{2n+1} > s_{2n-1}$.

(3) $r_{2n} > r_{2n-1} \geqq s_{2n-3}$.

(4) $t_n > 0$ $(n \geqq 4)$, $u_{2n+1} > 0$.

PROOF. From the assumption $(\alpha, \beta) \in X_1$, we know $\varepsilon_{2n+1} = 1$ for all n. The proof is obtained by induction. By the definition of r_n and s_n, we have $r_1 = a_1 \geqq 1$, $s_1 = \varepsilon_1 = 1$,

$$r_{2n+1} = a_{2n+1} r_{2n} + s_{2n} = (a_{2n+1} a_{2n} + \varepsilon_{2n}) r_{2n-1} + a_{2n+1} s_{2n-1}, \quad \text{and}$$

$$s_{2n+1} = r_{2n} = a_{2n} r_{2n-1} + s_{2n-1}.$$

Therefore we see

$$r_{2n+1} > 0, \quad s_{2n+1} > 0, \quad r_{2n+1} \geqq s_{2n-1}, \quad s_{2n+1} > s_{2n-1} \quad \text{and} \quad s_{2n+1} = r_{2n} > r_{2n-1}.$$

Also we have $r_{2n+1} > r_{2n-1}$ if $\varepsilon_{2n} = 1$. Thus we obtain (1), (2) and (3) from these inequalities. We obtain (4) similarly. q.e.d.

Define matrices $A_{(a_n, \varepsilon_n)}$ associated with the name of (α, β) by

(1.5)
$$A_{(a_n,\varepsilon_n)} := \begin{pmatrix} a_n & \varepsilon_n & 0 \\ 1 & 0 & 0 \\ -v_n & 0 & -\varepsilon_n \end{pmatrix}$$

and define the product of $A_{(a_k,\varepsilon_k)}$ by

(1.6)
$$\begin{pmatrix} r_n & s_n & 0 \\ t_n & u_n & 0 \\ v_n & w_n & \sigma_n \end{pmatrix} := A_{(a_1,\varepsilon_1)} \cdots A_{(a_n,\varepsilon_n)}$$

LEMMA 1.4. *The following formulae hold*:

(1)
$$\begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix} = \alpha\alpha_1 \cdots \alpha_{n-1} \begin{pmatrix} r_n & s_n & 0 \\ t_n & u_n & 0 \\ v_n & w_n & \sigma_n \end{pmatrix} \begin{pmatrix} 1 \\ \alpha_n \\ \beta_n \end{pmatrix},$$

and in particular

(2)
$$\alpha = \frac{t_n + u_n\alpha_n}{r_n + s_n\alpha_n}, \qquad \beta = \frac{v_n + w_n\alpha_n + \sigma_n\beta_n}{r_n + s_n\alpha_n},$$

and

$$\alpha_n = \frac{-t_n + r_n\alpha}{u_n - s_n\alpha}, \qquad \beta_n = \frac{\sigma_n(t_n w_n - u_n v_n) - \sigma_n(r_n w_n - s_n v_n)\alpha + \beta}{u_n - s_n\alpha}.$$

PROOF. From the definition of T, (1) is obtained by induction. (2) follows from

$$\begin{pmatrix} r_n & s_n & 0 \\ t_n & u_n & 0 \\ v_n & w_n & \sigma_n \end{pmatrix}^{-1} = \begin{pmatrix} u_n\sigma_n & -s_n\sigma_n & 0 \\ -t_n\sigma_n & r_n\sigma_n & 0 \\ t_n w_n - u_n v_n & -r_n w_n + s_n v_n & \sigma_n \end{pmatrix}.$$

q.e.d.

For $(\alpha, \beta) \in X_1$, let us define pairs of integers (q_n, p_n) as follows:

(1.7)
$$\begin{pmatrix} q_n \\ p_n \end{pmatrix} := \phi_{a_1} \circ \cdots \circ \phi_{a_{2n-1}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then we have the following proposition:

PROPOSITION 1.5. *For* $(\alpha, \beta) \in X_1$ $(\alpha \notin \mathbf{Q})$ *and* (q_n, p_n) *defined by* (1.7), *we have*

(1) q_n *monotonically tends to* ∞ *as* $n \to \infty$.

(2) $\alpha q_n + \beta - p_n$ *tends to* 0 *as* $n \to \infty$. *Furthermore, if* $n < m$ *and* $\sigma_{2n-2} = \sigma_{2m-2}$, *then* $|\alpha q_n + \beta - p_n| > |\alpha q_m + \beta - p_m|$ *holds.*

(3) p_n/q_n *tends to* α *as* $n \to \infty$.

PROOF. From Proposition 1.2 (2) and Lemma 1.3 (1), we see

$$q_{n+1}-q_n=r_{2n+1}+v_{2n+1}r_{2n}+v_{2n}r_{2n-1}-r_{2n-1}$$
$$=r_{2n+1}+v_{2n}r_{2n-1}-r_{2n-1}>0\,.$$

Therefore we obtain (1).

From Proposition 1.1 and Proposition 1.2 (1), we have

$$\alpha q_n+\beta-p_n=\sigma_{2n-1}\alpha\alpha_1\cdots\alpha_{2n-2}(\alpha_{2n-1}+\beta_{2n-1})$$
$$=\sigma_{2n-1}\frac{\alpha_{2n-1}+\beta_{2n-1}}{r_{2n-1}+s_{2n-1}\alpha_{2n-1}}\,.$$

From Lemma 1.3 and $0\leqq\alpha_k+\beta_k\leqq1$ for all k, we see $\alpha q_n+\beta-p_n\to0$ as $n\to\infty$.

Let us assume $n<m$ and $\sigma_{2n-2}\neq\sigma_{2n}=\sigma_{2n+2}=\cdots=\sigma_{2m-4}\neq\sigma_{2m-2}$. Then we see $\varepsilon_{2n}=-1$, $\varepsilon_{2n+1}=\cdots=\varepsilon_{2m-3}=1$, $\varepsilon_{2m-2}=-1$ and $\varepsilon_{2m-1}=1$. By Lemma 1.4, we have

$$\begin{pmatrix}1\\ \alpha_{2n-1}\\ \beta_{2n-1}\end{pmatrix}=\alpha_{2n-1}\cdots\alpha_{2m-2}\begin{pmatrix}r & s & 0\\ t & u & 0\\ v & w & \sigma\end{pmatrix}\begin{pmatrix}1\\ \alpha_{2m-1}\\ \beta_{2m-1}\end{pmatrix},$$

where

$$\begin{pmatrix}r & s & 0\\ t & u & 0\\ v & w & \sigma\end{pmatrix}=\begin{pmatrix}a_{2n} & -1 & 0\\ 1 & 0 & 0\\ -1 & 0 & 1\end{pmatrix}\begin{pmatrix}a_{2n+1} & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}\cdots$$
$$\cdots\begin{pmatrix}a_{2m-3} & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}a_{2m-2} & -1 & 1\\ 1 & 0 & 0\\ -1 & 0 & 1\end{pmatrix}\begin{pmatrix}a_{2m-1} & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$$
$$=\begin{pmatrix}a_{2n} & -1 & 0\\ 1 & 0 & 0\\ -1 & 0 & 1\end{pmatrix}\begin{pmatrix}* & * & 0\\ * & * & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}a_{2m-2} & -1 & 1\\ 1 & 0 & 0\\ -1 & 0 & 1\end{pmatrix}\begin{pmatrix}a_{2m-1} & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$$
$$=\begin{pmatrix}* & * & 0\\ A & B & 0\\ -A+a_{2m-1} & -B+1 & 1\end{pmatrix}.$$

Therefore, we see

$$\alpha_{2n-1}+\beta_{2n-1}=\alpha_{2n-1}\alpha_{2n}\cdots\alpha_{2m-2}(t+u\alpha_{2m-1}+v+w\alpha_{2m-1}+\sigma\beta_{2m-1})$$
$$=\alpha_{2n-1}\alpha_{2n}\cdots\alpha_{2m-2}(a_{2m-1}+\alpha_{2m-1}+\beta_{2m-1})$$
$$>\alpha_{2n-1}\alpha_{2n}\cdots\alpha_{2m-2}(\alpha_{2m-1}+\beta_{2m-1})\,,$$

and so

$$\begin{aligned}|\alpha q_n+\beta-p_n| &= \alpha\alpha_1\cdots\alpha_{2n-2}(\alpha_{2n-1}+\beta_{2n-1})\\ &> \alpha\alpha_1\cdots\alpha_{2m-2}(\alpha_{2m-1}+\beta_{2m-1})\\ &= |\alpha q_m+\beta-p_m| .\end{aligned}$$

In the case of $\sigma_{2n-2}=\sigma_{2n}$, we can show $|\alpha q_n+\beta-p_n|>|\alpha q_{n+1}+\beta-p_{n+1}|$ in the same way.

We obtain (3) immediately from

$$\frac{p_n}{q_n}=-\frac{\alpha q_n+\beta-p_n}{q_n}+\alpha+\frac{\beta}{q_n} \qquad \text{q.e.d.}$$

COROLLARY 1.6. *If* (α, β) *and* (α', β') *in* X_1 *have the same infinite name* $\{(a_n, \varepsilon_n): n=1, 2, \cdots\}$, *then* $(\alpha, \beta)=(\alpha', \beta')$.

PROOF. Using the name we obtain (q_n, p_n), and by Proposition 1.5 (3), α is determined. Then, from Proposition 1.5 (2), β is also determined. q.e.d.

In the next section, we observe that the points (q_n, p_n) coincide with the vertices of the approximating polygon of the line L: $\alpha x+\beta-y=0$. Therefore, we call the algorithm (X, T) or (X_1, T^2) *Morimoto algorithm.*

§ 2. Geometry of Morimoto algorithm.*)

In this section we give a geometrical characterization of points (q_n, p_n).

For geometrical discussions, let us introduce some notations. For each $(\alpha, \beta)\in X_1$ $(\alpha\notin \boldsymbol{Q})$, we put

$$P_n:=\begin{pmatrix} q_n\\ p_n\end{pmatrix}=\phi_{a_1}\circ\cdots\circ\phi_{a_{2n-1}}\begin{pmatrix}1\\0\end{pmatrix}, \qquad P_0:=\begin{pmatrix}0\\0\end{pmatrix}, \quad P_{-1}:=\begin{pmatrix}0\\1\end{pmatrix},$$

$$P'_n:=\phi_{a_1}\circ\cdots\circ\phi_{a_{2n-1}}\begin{pmatrix}1\\1\end{pmatrix},$$

$$P''_n:=\begin{cases}\phi_{a_1}\circ\cdots\circ\phi_{a_{2n}}\begin{pmatrix}1\\0\end{pmatrix} & \text{if } \varepsilon_{2n}=1 ,\\[2ex] \phi_{a_1}\circ\cdots\circ\phi_{a_{2n}}\begin{pmatrix}1\\1\end{pmatrix} & \text{if } \varepsilon_{2n}=-1 ,\end{cases}$$

*) The reader who is not interested in geometrical discussions may prefer to skip this section except Propositions 2.4 and 2.5.

$$P_n''' := \begin{cases} \phi_{a_1} \circ \cdots \circ \phi_{a_{2n}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \text{if } \varepsilon_{2n} = 1 , \\ \phi_{a_1} \circ \cdots \circ \phi_{a_{2n}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{if } \varepsilon_{2n} = -1 , \end{cases}$$

$$P_{n,k} := \phi_{a_1} \circ \cdots \circ \phi_{a_{2n}} \begin{pmatrix} k \\ 1 \end{pmatrix} \quad (1 \leqq k \leqq a_{2n+1} - 1) , \qquad (P_{n,a_{2n+1}} = P_{n+1}) ,$$

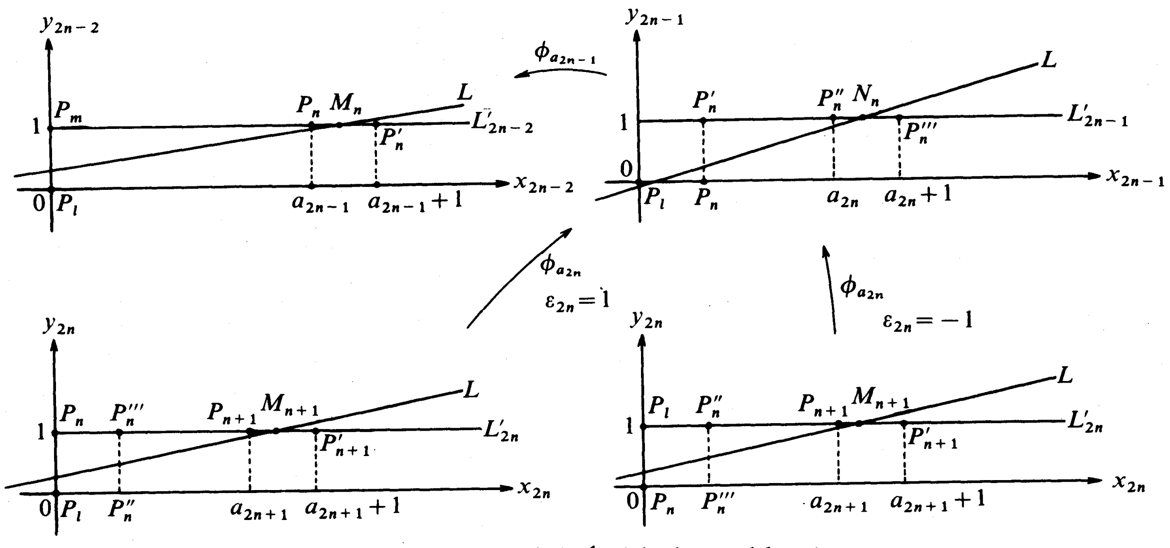
$$M_{n+1} := \phi_{a_1} \circ \cdots \circ \phi_{a_{2n}} \begin{pmatrix} \dfrac{1-\beta_{2n}}{\alpha_{2n}} \\ 1 \end{pmatrix},$$

$$N_{n+1} := \phi_{a_1} \circ \cdots \circ \phi_{a_{2n+1}} \begin{pmatrix} \dfrac{1-\beta_{2n+1}}{\alpha_{2n+1}} \\ 1 \end{pmatrix},$$

$$\Gamma_{n+1} := \phi_{a_1} \circ \cdots \circ \phi_{a_{2n}} \left(\left\{ \begin{pmatrix} x_{2n} \\ 1 \end{pmatrix} : 0 \leqq x_{2n} \leqq a_{2n+1} \right\} \right) , \quad \Gamma_1 := \left\{ \begin{pmatrix} x \\ 1 \end{pmatrix} : 0 \leqq x \leqq a_1 \right\} ,$$

$$\Pi_+ := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x \geqq 0, \alpha x + \beta - y < 0 \right\} , \quad \text{and}$$

$$\Pi_- := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x \geqq 0, \alpha x + \beta - y > 0 \right\} .$$



Here, $(\phi_{a_1} \circ \cdots \circ \phi_{a_n})^{-1}(A)$ is denoted by A.

FIGURE 3.

LEMMA 2.1. *For each n,*

(1) *the gradient of the image of the line* $y_n=1$ *by* $\phi_{a_1}\circ\cdots\circ\phi_{a_n}$ *is equal to* t_n/r_n.

(2) *For any* x_n, x'_n *such that* $x_n<x'_n$, *the images* (x, y) *and* (x', y') *of* $(x_n, 1)$ *and* $(x'_n, 1)$ *by* $\phi_{a_1}\circ\cdots\circ\phi_{a_n}$ *satisfy the inequality* $x<x'$.

PROOF. The image of the line $y_n=1$ by $\phi_{a_1}\circ\cdots\circ\phi_{a_n}$ is denoted by

$$x=r_nx_n+s_n+\sum_{k=0}^{n-1}v_kr_k\,,$$

$$y=t_nx_n+u_n+\sum_{k=0}^{n-1}v_kt_k\,.$$

Therefore, we obtain (1) and (2) by $r_n>0$. q.e.d.

LEMMA 2.2. *We have*

$$\frac{t_{2n}}{r_{2n}}<\frac{t_{2n+2}}{r_{2n+2}} \qquad \textit{if} \quad \sigma_{2n}=1\,, \quad \textit{and}$$

$$\frac{t_{2n}}{r_{2n}}>\frac{t_{2n+2}}{r_{2n+2}} \qquad \textit{if} \quad \sigma_{2n}=-1\,.$$

PROOF. From (1.4), we have

$$\frac{t_{2n+2}}{r_{2n+2}}-\frac{t_{2n}}{r_{2n}}=\frac{a_{2n+2}(r_{2n}u_{2n}-t_{2n}s_{2n})}{r_{2n+2}r_{2n}}=\frac{a_{2n+2}\sigma_{2n}}{r_{2n+2}r_{2n}}\,.$$

By $r_n>0$ and $a_n\geqq 1$ for all n, we obtain the conclusion. q.e.d.

LEMMA 2.3. *The points* P_n, P'_n, M_n $(n\geqq 1)$ *are rearranged with respect to their x-coordinates as follows*:

$$P_1, M_1, P'_1, P_2, M_2, P'_2, \cdots, P_n, M_n, P'_n, P_{n+1}, M_{n+1}, P'_{n+1}, \cdots.$$

And $P'_n=P_{n+1}$ *iff* $a_{2n}=a_{2n+1}=1$ *and* $\varepsilon_{2n}=-1$.

PROOF. Let us consider the (x_{2n-1}, y_{2n-1})-plane and the line L_{2n-1}: $\alpha_{2n-1}x_{2n-1}+\beta_{2n-1}-y_{2n-1}=0$. The line L_{2n-1} and the line $y_{2n-1}=1$ intersect at a point $(\phi_{a_1}\circ\cdots\circ\phi_{a_{2n-1}})^{-1}(N_n)$. From the definition of ε_{2n} and $\phi_{a_{2n}}$ we see the two lattice points $([(1-\beta_{2n-1})/\alpha_{2n-1}], 1)$ and $([(1-\beta_{2n-1})/\alpha_{2n-1}]+1, 1)$ are given by

$$\left(\left(\left[\frac{1-\beta_{2n-1}}{\alpha_{2n-1}}\right], 1\right), \left(\left[\frac{1-\beta_{2n-1}}{\alpha_{2n-1}}\right]+1, 1\right)\right)=\begin{cases}\left(\phi_{a_{2n}}\begin{pmatrix}1\\0\end{pmatrix}, \phi_{a_{2n}}\begin{pmatrix}1\\1\end{pmatrix}\right) & \text{if} \quad \varepsilon_{2n}=1\\ \left(\phi_{a_{2n}}\begin{pmatrix}1\\1\end{pmatrix}, \phi_{a_{2n}}\begin{pmatrix}1\\0\end{pmatrix}\right) & \text{if} \quad \varepsilon_{2n}=-1\end{cases}$$

$$=((\phi_{a_1}\circ\cdots\circ\phi_{a_{2n-1}})^{-1}(P''_n),(\phi_{a_1}\circ\cdots\circ\phi_{a_{2n-1}})^{-1}(P'''_n)). \quad \text{(See Fig. 3)}.$$

The point $(\phi_{a_1}\circ\cdots\circ\phi_{a_{2n-2}})^{-1}(M_n)=((1-\beta_{2n-2})/\alpha_{2n-2},1)$ is a cross point with the line L_{2n-2} and the line $y_{2n-2}=1$. Two lattice points $([(1-\beta_{2n-2})/\alpha_{2n-2}],1)$ and $([(1-\beta_{2n-2})/\alpha_{2n-2}]+1,1)$ are given by

$$\left(\left(\left[\frac{1-\beta_{2n-2}}{\alpha_{2n-2}}\right],1\right),\left(\left[\frac{1-\beta_{2n-2}}{\alpha_{2n-2}}\right]+1,1\right)\right)=\left(\phi_{a_{2n-1}}\begin{pmatrix}1\\0\end{pmatrix},\phi_{a_{2n-1}}\begin{pmatrix}1\\1\end{pmatrix}\right),$$
$$=((\phi_{a_1}\circ\cdots\circ\phi_{a_{2n-2}})^{-1}(P_n),(\phi_{a_1}\circ\cdots\circ\phi_{a_{2n-2}})^{-1}(P'_n)).$$

Therefore, by Lemma 2.1 (2), the order of points is given by

$$P_n, M_n, P'_n, P''_n, N_n, P'''_n, P_{n+1}, M_{n+1}, P'_{n+1} \qquad \text{if} \quad \varepsilon_{2n}=1,$$
$$P_n, M_n, P'_n, P''_n, N_n, P'''_n;\quad P''_n, P_{n+1}, M_{n+1}, P'_{n+1} \qquad \text{if} \quad \varepsilon_{2n}=-1,$$

where we see

$$a_{2n}=1 \qquad \text{iff} \quad P'_n \doteq P''_n,$$
$$\varepsilon_{2n}=1 \ \text{and}\ a_{2n+1}=1 \qquad \text{iff} \quad P'''_n=P_{n+1},$$
$$\varepsilon_{2n}=-1 \ \text{and}\ a_{2n+1}=1 \qquad \text{iff} \quad P''_n=P_{n+1}.$$

q.e.d.

Let us denote the ordered set of points P_n $(n\geqq -1)$ included in Π_+ (Π_-) by $\{P_{u(i)}: i=0,1,2,\cdots\}$ $(\{P_{v(i)}: i=0,1,2,\cdots\})$. Then we see $P_{u(0)}=P_{-1}$, $P_{u(1)}=P_1$ and $P_{v(0)}=P_0$. Let us define segments $\Gamma_{u(i)}$ $(\Gamma_{v(i)})$ and polygons Γ_+ (Γ_-) by

$$\Gamma_{u(i)}=\overline{P_{u(i-1)}P_{u(i)}}, \qquad \Gamma_{v(i)}=\overline{P_{v(i-1)}P_{v(i)}},$$
$$\Gamma_+=\bigcup_{i=1}^{\infty}\Gamma_{u(i)} \quad \text{and} \quad \Gamma_-=\bigcup_{i=1}^{\infty}\Gamma_{v(i)}.$$

Then we have the following geometrical theorem.

THEOREM 2.1 (Morimoto). *Let us assume α is an irrational and the line L: $\alpha x+\beta-y=0$ does not pass through any lattice points $\boldsymbol{Z}^2$. Then the points P_n have following properties*:

(1) $P_n\in\Pi_+$ *iff* $\sigma_{2n-2}=1$,
$P_n\in\Pi_-$ *iff* $\sigma_{2n-2}=-1$.

(2) *The vertices of the polygon Γ_+ coincide with the set $\{P_{u(n)}: n=0,1,2,\cdots\}$ and the polygon Γ_+ is convex. The vertices of the polygon Γ_- coincide with the set $\{P_{v(n)}: n=0,1,2,\cdots\}$ and the polygon Γ_- is concave.*

(3) *The domain D enclosed by Γ_+ and Γ_- in the first quadrant does not include any lattice point. All lattice points on the boundary are P_n's and $P_{n,k}$'s $(1\leqq k\leqq a_{2n+1}-1$, $n\geqq 0)$.*

PROOF. The proof is given by induction and each step corresponds with the step of the geometrical construction of polygons.

0th step (setting): We know that $M_1=((1-\beta)/\alpha, 1)$ is given as the cross point of L with $y=1$. Since $a_1=[(1-\beta)/\alpha]$, points $P_1=(a_1, 1)$ and $P'_1=(a_1+1, 1)$ are given as nearest lattice points of M_1 on the line $y=1$. And $\Gamma_{u(1)}$ is constructed as a segment $\overline{P_{-1}P_1}$ in Π_+ and P_0 is in Π_-. We see that the triangle $\triangle P_0P_{-1}P_1$ does not include any lattice point except on the boundary. (See Fig. 4).

1st step (1st construction): The point N_1 is given as a cross point with L and a line L'_1 which passes through the point P'_1 and is parallel to $\overline{P_0P_1}$, and the points P''_1, P'''_1 are given as nearest lattice points of N_1 on L'_1.

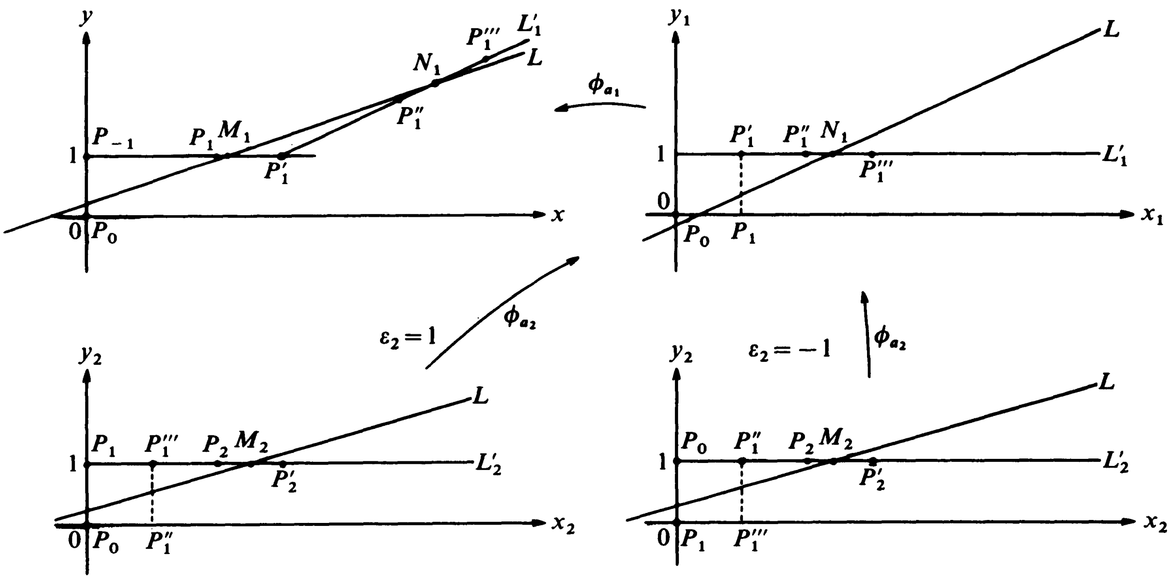


FIGURE 4.

2nd step (2nd construction): We know the segment $P_0P''_1$ and $P_1P'''_1$ are parallel. Therefore one of the prolongation of $P_0P''_1$ and $P_1P'''_1$ (toward P''_1 or P'''_1) intersects L, but which one intersects L is decided by the sign of ε_2. M_2 is given as a cross point of the prolongation L'_2 and L, and points P_2, P'_2 are given as nearest lattice points of M_2 on L'_2. (See Fig. 4). We have $P_2\in\Pi_+$ if $\sigma_2=\varepsilon_2=1$ and $P_2\in\Pi_-$ if $\sigma_2=\varepsilon_2=-1$. Γ_2 is constructed as a segment $\overline{P_1P_2}$ if $\varepsilon_2=1$ and as a segment $\overline{P_0P_2}$ if $\varepsilon_2=-1$. We see that $\triangle P_0P_1P_2$ does not include any lattice point except on the boundary.

$(2n-2)$th step (assumption of induction): Let us assume that the point P_n on Γ_n satisfies the following properties:

(i) $\Gamma_{u(1)}\cup\cdots\cup\Gamma_{u(k)}$ is convex and $\Gamma_{v(1)}\cup\cdots\cup\Gamma_{v(j)}$ is concave $(k+j=n)$ and P_k $(k\leq n)$ are vertices of above polygons.

(ii) $\triangle P_lP_mP_n$ does not include any lattice point except on boundary, where

$$l = \max\{i<n : P_i \text{ is on the opposite side of } P_n \text{ with respect to } L\}\,,$$

$$m = \max\{i<n : P_i \text{ is on the same side of } P_n \text{ with respect to } L\}\,.$$

Furthermore we assume

$$P_l = \phi_{a_1} \circ \cdots \circ \phi_{a_{2n-2}}\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad P_m = \phi_{a_1} \circ \cdots \circ \phi_{a_{2n-2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

These assumptions are satisfied for $n=1, 2$. (See Fig. 4).

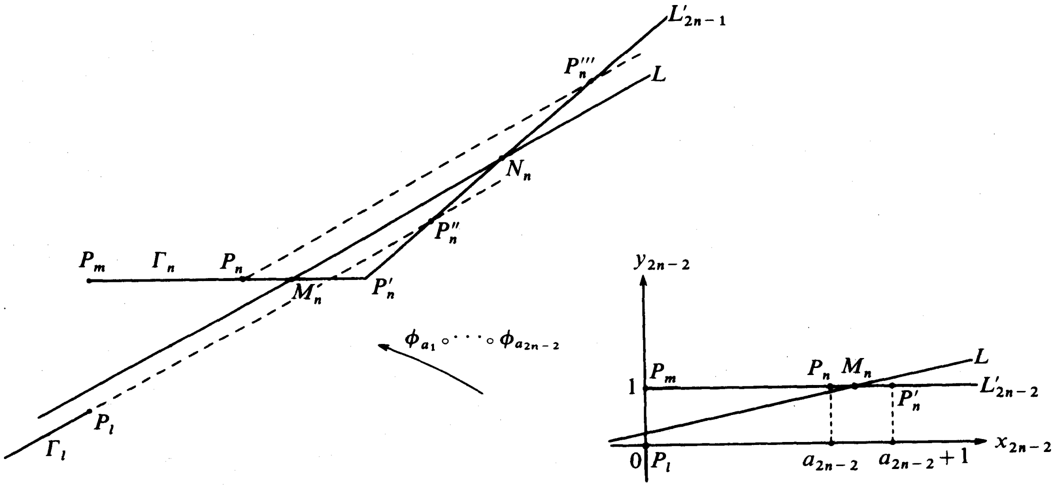


FIGURE 5.

2nth step: Let us assume $P_n \in \Pi_+$, that is $\sigma_{2n-2}=1$ and $\Gamma_n \subset \Pi_+$. (In the case of $P_n \in \Pi_-$, we can discuss in the same way.) Then we have $u(k)=n$, $u(k-1)=m$ and $v(j)=l$. The point N_n is given as a cross point with L and a line L'_{2n-1} which passes through the point P'_n and is parallel to P_lP_n, and the points P''_n and P'''_n are given as nearest lattice points of N_n on L'_{2n-1}. (See Fig. 5 and Fig. 3).

We know the segment $P_nP'''_n$ and $P_lP''_n$ are parallel. Therefore one of the prolongation of one of $P_nP'''_n$ or $P_lP''_n$ intersects L and which one intersects L is decided by the sign of ε_{2n}. The point M_{n+1} is given as a cross point of the prolongation L'_{2n} and L, and points P_{n+1} and P'_{n+1} are given as nearest lattice points of M_{n+1} on L'_{2n}. We have $P_{n+1} \in \Pi_+$ if $\varepsilon_{2n}=1$ (that is, $\sigma_{2n}=\varepsilon_{2n}\sigma_{2n-2}=1$) and $P_{n+1} \in \Pi_-$ if $\varepsilon_{2n}=-1$. Thus we obtain (1).

Γ_{n+1} is constructed as a segment $\overline{P_nP_{n+1}}$ if $\varepsilon_{2n}=1$ and as a segment $\overline{P_lP_{n+1}}$ if $\varepsilon_{2n}=-1$. From Lemma 2.1 the gradient of Γ_{n+1} is given by t_{2n}/r_{2n}, because $\Gamma_{n+1} \subset (\phi_{a_1} \circ \cdots \circ \phi_{a_{2n}})^{-1}(y_{2n}=1)$. We see that the new polygon added Γ_{n+1} is also convex (concave). In fact, in the case of $\Gamma_{n+1} \subset \Pi_+$, from Lemma 2.2 and $\sigma_{2n-2}=1$, we see $t_{2n-2}/r_{2n-2} < t_{2n}/r_{2n}$, and so the gradient of Γ_n is smaller than that of Γ_{n+1}.

In the case of $\Gamma_{n+1} \subset \Pi_-$, M_l is given as a cross point of the prolongation of Γ_l

and L, and M_{n+1} is given as a cross point of prolongation of Γ_{n+1} and L. From $l<n<n+1$ and Lemma 2.1, we know the x-coordinate of M_l is smaller than that of M_{n+1}. This means the gradient of Γ_l is greater than that of Γ_{n+1}, and we have the conclusion.

The statement (3) is a consequence of the fact that $\triangle P_l P_n P_{n+1}$ does not include any lattice point except on the boundary. The lattice points on the boundary are given by

$$\{P_n, P_l\}=\left\{\phi_{a_1}\circ\cdots\circ\phi_{a_{2n}}\begin{pmatrix}0\\1\end{pmatrix}, \phi_{a_1}\circ\cdots\circ\phi_{a_{2n}}\begin{pmatrix}0\\0\end{pmatrix}\right\},\quad P_{n+1}=\phi_{a_1}\circ\cdots\circ\phi_{a_{2n}}\begin{pmatrix}a_{2n+1}\\1\end{pmatrix},$$

and $P_{n,k}=\phi_{a_1}\circ\cdots\circ\phi_{a_{2n}}\begin{pmatrix}k\\1\end{pmatrix}\quad (k=1, 2, \cdots, a_{2n+1}-1)$. q.e.d.

We call $\Gamma_\pm$ approximating polygons of L: $\alpha x+\beta-y=0$, and call P_n principally approximate points and $P_{n,k}$ $(1\leqq k\leqq a_{2n+1}-1)$ intermediately approximate points of L.

We discuss the necessary and sufficient conditions of $T^n(\alpha, \beta)\in I$ for some n.

PROPOSITION 2.4. *Let us assume that* $\alpha\notin \boldsymbol{Q}$ *and* $(\alpha, \beta)\in X_1$. *Then following conditions are equivalent*:

(1) *there exists* n_0 *such that* $T^{2n_0}(\alpha, \beta)\in I$, *that is*, $\beta_{2n}=0$ *for* $n\geqq n_0$,

(2) *there exists* n_1 *such that* $\varepsilon_{2n}=1$ *for all* $n\geqq n_1$,

(3) *there exists a lattice point* $(k, l)\in \boldsymbol{Z}^2$ *such that* $\alpha k+\beta=l$ *and* $k\geqq 0$.

REMARK 2.1. In case that (α, β) satisfies the above conditions, we see that one of approximating polygons $\Gamma_\pm$ consists of finite segments and ends at (k, l), which is a principally approximate point P_n (for some n).

PROOF. Since $I-\{(0, 0)\}\subset\bigcup_{a=1}^{\infty}\Delta(a, 1)$ and the set I is T-invariant, we obtain (2) from (1). Conversely, let us assume (2). From the assumption $\varepsilon_{2n+2}=1$, the point $(\alpha_{2n+2}, \beta_{2n+2})=T^2(\alpha_{2n}, \beta_{2n})$ is written as

$$(\alpha_{2n+2}, \beta_{2n+2})=\left(\frac{\alpha_{2n}}{1-a_{2n+1}\alpha_{2n}}-a_{2n+2}, \frac{\beta_{2n}}{1-a_{2n+1}\alpha_{2n}}\right).$$

Therefore, considering $(\alpha_{2n}, \beta_{2n})\in\Delta(a_{2n+1})$, that is, $\beta_{2n}\leqq 1-a_{2n+1}\alpha_{2n}<\alpha_{2n}+\beta_{2n}\leqq 1$, we see that

$$\beta_{2n+2}=\frac{\beta_{2n_1}}{(1-a_{2n+1}\alpha_{2n})\cdots(1-a_{2n_1+1}\alpha_{2n_1})}.$$

Suppose $\beta_{2n_1}\neq 0$, the sequence $\{\beta_{2n+2}\}$ is monotonically increasing and bounded. Therefore we have $\prod_{n=n_1}^{\infty}(1-a_{2n+1}\alpha_{2n})<\infty$ and $\sum_{n=n_1}^{\infty}a_{2n+1}\alpha_{2n}<\infty$. Thus we see α_{2n} tends to 0, and so $\beta_{2n+1}=-\beta_{2n}/\alpha_{2n}$ tends to $-\infty$. Therefore $(\alpha_{2n+1}, \beta_{2n+1})\in\Delta(1, -1)$ for sufficiently large n. This contradicts the assumption $\varepsilon_{2n+2}=1$ for all $n\geqq n_1$.

Next, assume that $\beta_{2n_0-2}\neq 0$ and $\beta_{2n_0}=0$. From the relation

$$\beta_{2n_0}=\begin{cases}\dfrac{\beta_{2n_0-2}}{1-a_{2n_0-1}\alpha_{2n_0-2}} & \text{if } \varepsilon_{2n_0}=1\,,\\[2ex] 1-\dfrac{\beta_{2n_0-2}}{1-a_{2n_0-1}\alpha_{2n_0-2}} & \text{if } \varepsilon_{2n_0}=-1\,,\end{cases}$$

we see that $\varepsilon_{2n_0}=-1$ and $\beta_{2n_0-2}=1-a_{2n_0-1}\alpha_{2n_0-2}$. Therefore a lattice point $(a_{2n_0-1}, 1)$ belongs to the line L_{2n_0-2}: $\alpha_{2n_0-2}x_{2n_0-2}+\beta_{2n_0-2}-y_{2n_0-2}=0$, and so the lattice point

$$\phi_{a_1}\circ\cdots\circ\phi_{a_{2n_0-2}}\begin{pmatrix}a_{2n_0-1}\\ 1\end{pmatrix}=\phi_{a_1}\circ\cdots\circ\phi_{a_{2n_0-1}}\begin{pmatrix}1\\ 0\end{pmatrix}=\begin{pmatrix}q_{n_0}\\ p_{n_0}\end{pmatrix}$$

belongs to L: $\alpha x+\beta-y=0$ and by Lemma 1.3 and (1.7) we see $q_{n_0}>0$.

Conversely we assume (3). Let us denote $\begin{pmatrix}k_n\\ l_n\end{pmatrix}$ by

$$\begin{pmatrix}k\\ l\end{pmatrix}=\phi_{a_1}\circ\cdots\circ\phi_{a_n}\begin{pmatrix}k_n\\ l_n\end{pmatrix},$$

then the lattice points (k_n, l_n) satisfy $\alpha_n k_n+\beta_n=l_n$. From the definition of ϕ_{a_n} we have $k_{2n+2}=k_{2n}-a_{2n+1}l_{2n}$. Assume $k_{2n}>0$. Then $\beta_{2n}\neq 0$ holds because $\alpha_{2n}\notin \boldsymbol{Q}$. We see $l_{2n}=\alpha_{2n}k_{2n}+\beta_{2n}>0$ and $k_{2n+2}<k_{2n}$. If $l_{2n}>1$, we have

$$k_{2n+2}=k_{2n}-a_{2n+1}l_{2n}\geqq k_{2n}-l_{2n}\frac{1-\beta_{2n}}{\alpha_{2n}}=\beta_{2n}\frac{l_{2n}-1}{\alpha_{2n}}>0$$

and so $k_{2n}>k_{2n+2}\geqq 1$. Thus there exists an m such that $k_{2m}\geqq l_{2m}=1$. We obtain $k_{2m}=(1-\beta_{2m})/\alpha_{2m}=a_{2m+1}$, and so $\alpha_{2m+1}+\beta_{2m+1}=0$. Hence, by Remark 1.1 (2) we have $\beta_{2m+2}=0$. q.e.d.

Next, we discuss the necessary and sufficient conditions of $T^{2n}(\alpha, \beta)\in K$ $(=\{(\alpha, \beta)\in X: \alpha=\beta\})$ for some n.

PROPOSITION 2.5. *Let us assume* $\alpha\notin \boldsymbol{Q}$ *and* $(\alpha, \beta)\in X_1$. *Then the following conditions are equivalent*:

(1) *there exists* n_0 *such that* $\alpha_{2n_0}=\beta_{2n_0}$, *that is,* $\alpha_{2n}=\beta_{2n}$ *for all* $n\geqq n_0$,
(2) *there exists* n_1 *such that* $a_{2n}=1$, $\varepsilon_{2n}=-1$ *for all* $n\geqq n_1$,
(3) *there exists a lattice point* $(k, l)\in \boldsymbol{Z}^2$ *such that* $\alpha k+\beta=l$ *and* $k<0$.

PROOF. Assume that $\alpha_{2n}=\beta_{2n}$. Then, from the definition of T, we have $\beta_{2n+1}=-1$ and so $(\alpha_{2n+1}, \beta_{2n+1})\in \Delta(1, -1)$. Therefore, $\varepsilon_{2n+2}=-1$ and $a_{2n+2}=1$. Moreover, from $\varepsilon_{2n+2}=-1$, we have

$$T^2(\alpha_{2n}, \beta_{2n})=\left(a_{2n+2}-\frac{\alpha_{2n}}{1-a_{2n+1}\alpha_{2n}},\ 1-\frac{\beta_{2n}}{1-a_{2n+1}\alpha_{2n}}\right),$$

and we see from $a_{2n+2}=1$ that $\alpha_{2n+2}=\beta_{2n+2}$.

Conversely, let us assume $a_{2n}=1$, $\varepsilon_{2n}=-1$ for all $n\geqq n_1$. Then from the definition of T we have

$$T^2(\alpha_{2n}, \beta_{2n})=\left(1-\frac{\alpha_{2n}}{1-a_{2n+1}\alpha_{2n}}, 1-\frac{\beta_{2n}}{1-a_{2n+1}\alpha_{2n}}\right),$$

and $\alpha_{2n+2}-\beta_{2n+2}=-(\alpha_{2n}-\beta_{2n})/(1-a_{2n+1}\alpha_{2n})$.

Suppose that $\alpha_{2n_1}-\beta_{2n_1}\neq 0$. Then the sequence $\{|\alpha_{2n}-\beta_{2n}|\}$ is monotonously increasing and tends to some non-zero constant c. The sign of the sequence $\alpha_{2n}-\beta_{2n}$ is alternative. With a similar discussion in the proof of Proposition 2.4, we see α_{2n} converges to 0. Therefore, β_{2n} tends to c or $-c$, alternatively. This contradicts $\beta_{2n}\geqq 0$.

Let us assume there exists n such that $\alpha_{2n}\neq\beta_{2n}$ and $\alpha_{2n+2}=\beta_{2n+2}$. Therefore we see that

$$\begin{cases} \dfrac{\alpha_{2n}}{1-a_{2n+1}\alpha_{2n}}-a_{2n+2}=\dfrac{\beta_{2n}}{1-a_{2n+1}\alpha_{2n}} & \text{if} \quad \varepsilon_{2n+2}=1\,, \\ a_{2n+2}-\dfrac{\alpha_{2n}}{1-a_{2n+1}\alpha_{2n}}=1-\dfrac{\beta_{2n}}{1-a_{2n+1}\alpha_{2n}} & \text{if} \quad \varepsilon_{2n+2}=-1\,, \end{cases}$$

that is,

$$\begin{cases} -(1+a_{2n+1}a_{2n+2})\alpha_{2n}+\beta_{2n}=-a_{2n+2} & \text{if} \quad \varepsilon_{2n+2}=1\,, \\ (-a_{2n+1}(a_{2n+2}-1)-1)\alpha_{2n}+\beta_{2n}=-a_{2n+2}+1 & \text{if} \quad \varepsilon_{2n+2}=-1\,. \end{cases}$$

This means the line L_{2n} passes through a lattice point and so L passes through a lattice point. By Proposition 2.4 the first coordinate of the lattice point is negative.

Let us assume (3) and denote (k_n, l_n) similarly in the proof of Proposition 2.4. Then we have $\alpha_n k_n+\beta_n=l_n$ and $k_{2n+2}=k_{2n}-a_{2n+1}l_{2n}$. If $k_{2n}\leqq -1$, then $\beta_{2n}\neq 0$, $k_{2n}\leqq l_{2n}\leqq 0$ and $k_{2n}\leqq k_{2n+2}\leqq -1$. If $l_{2n}<0$, we have $k_{2n}<k_{2n+2}\leqq l_{2n+2}\leqq 0$. Therefore there exists an n_1 such that $l_{2n}=0$ for all $n\geqq n_1$. For this n, we see that $\alpha_{2n}k_{2n}+\beta_{2n}=0$. Thus we have $k_{2n}=-\beta_{2n}/\alpha_{2n}=\beta_{2n+1}\in \boldsymbol{Z}-\{0\}$, and so $(\alpha_{2n+1}, \beta_{2n+1})\in\Delta(1, -1)$. This means $a_{2n+2}=1$ and $\varepsilon_{2n+2}=-1$. q.e.d.

§ 3. Natural extension of Morimoto algorithm.

Let us consider a so called natural extension of Morimoto algorithm for the sake of later discussions. Put

$$\text{(3.1)} \qquad \begin{aligned} X_1^* &:=\{(\gamma, \delta): (1\leqq\delta\leqq\gamma) \text{ or } (\gamma+1\leqq\delta\leqq 0)\}-\{(1, 1)\} \quad \text{and} \\ X_2^* &:=\{(\gamma, \delta): -1\leqq\delta\leqq 0, \gamma+\delta\leqq -1\}-\{(0, -1)\}\,, \end{aligned}$$

and define a domain $\bar{X}$ by

$$\bar{X} := (X_1 \times X_1^*) \cup (X_2 \times X_2^*)$$

and define a transformation $\bar{T}$ on $\bar{X}$ by

$$(3.2)\qquad \bar{T}(\alpha, \beta, \gamma, \delta) = \begin{cases} \left(\dfrac{1}{\alpha} - a, \; -\dfrac{\beta}{\alpha}, \dfrac{1}{\gamma} - a, \; -\dfrac{\delta}{\gamma}\right) \\ \qquad\qquad \text{if} \quad (\alpha, \beta, \gamma, \delta) \in \Delta(a) \times X_1^* \cup \Delta(a, 1) \times X_2^*, \\ \left(a - \dfrac{1}{\alpha}, 1 + \dfrac{\beta}{\alpha}, a - \dfrac{1}{\gamma}, 1 + \dfrac{\delta}{\gamma}\right) \\ \qquad\qquad \text{if} \quad (\alpha, \beta, \gamma, \delta) \in \Delta(a, -1) \times X_2^*. \end{cases}$$

It is easy to see that

$$\bar{T}(\Delta(a) \times X_1^*) = (X_2 - J_2) \times \Delta^*(a), \quad \bar{T}(\Delta(a, 1) \times X_2^*) = (X_1 - J_1) \times \Delta^*(a, 1) \quad \text{and}$$

$$\bar{T}(\Delta(a, -1) \times X_2^*) = (X_1 - I_0) \times \Delta^*(a, -1),$$

where

$$\Delta^*(a) = \Delta_2^* - (a, 0),$$

$$\Delta^*(a, 1) = \Delta_1^* - (a, 0),$$

$$\Delta^*(a, -1) = -\Delta_1^* + (a, 1),$$

with

$$\Delta_2^* = \{(\gamma, \delta) : -1 \leqq \delta \leqq 0, -1 \leqq \gamma + \delta \leqq 0, \gamma \neq 0, \gamma \neq 1\},$$

$$\Delta_1^* = \{(\gamma, \delta) : \delta \leqq 0, 0 \leqq \delta - \gamma \leqq 1, \gamma \neq 0\}. \qquad \text{(see Fig. 6).}$$

We call $(\bar{X}, \bar{T})$ a *natural extension* of Morimoto algorithm.

REMARK 3.1. We have

$$\bigcup_{a=1}^{\infty} \Delta^*(a) = X_2^* \cap \{\gamma \neq -1, -2, \cdots\} \quad \text{and} \quad \bigcup_{\varepsilon \in \{1, -1\}} \bigcup_{a=1}^{\infty} \Delta^*(a, \varepsilon) = X_1^* \cap \{\gamma \neq -1\}.$$

REMARK 3.2. Let us denote $(\alpha_n, \beta_n, \gamma_n, \delta_n) := \bar{T}^n(\alpha, \beta, \gamma, \delta)$. Then we have the following formulae similar to those in Lemma 1.4:

$$\begin{pmatrix} 1 \\ \gamma \\ \delta \end{pmatrix} = \gamma\gamma_1 \cdots \gamma_{n-1} \begin{pmatrix} r_n & s_n & 0 \\ t_n & u_n & 0 \\ v_n & w_n & \sigma_n \end{pmatrix} \begin{pmatrix} 1 \\ \gamma_n \\ \delta_n \end{pmatrix}$$

and

$$\gamma_n = \frac{-t_n + r_n\gamma}{u_n - s_n\gamma}, \quad \delta_n = \frac{\sigma_n(t_n w_n - u_n v_n) - \sigma_n(r_n w_n - s_n v_n)\gamma + \delta}{u_n - s_n\gamma}.$$

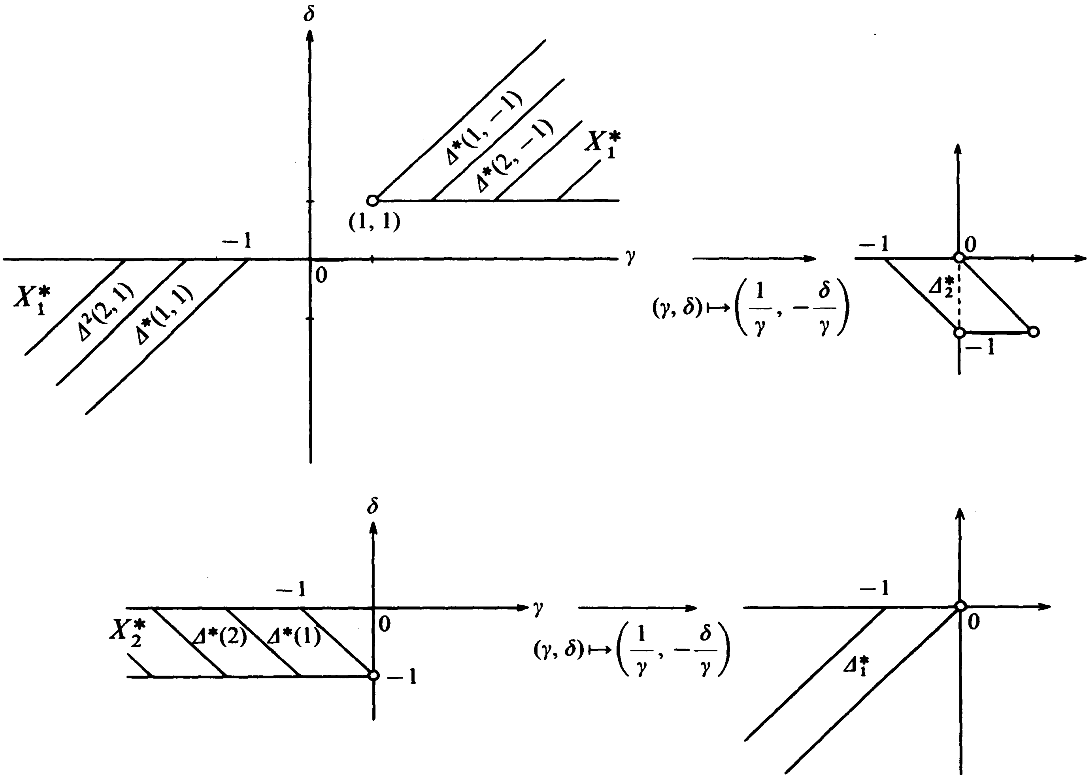


FIGURE 6.

SUBLEMMA. *The following relation holds*:

$$\sigma_n(r_n w_n - s_n v_n) = -\sum_{k=0}^{n-1} v_{k+1} r_k .$$

PROOF. The proof is obtained by induction. The case $n=1$ is an easy consequence from the definitions. Assume the case it is true for n. Then we see

$$\begin{aligned}
&\sigma_{n+1}(r_{n+1} w_{n+1} - s_{n+1} v_{n+1}) \\
&= -\varepsilon_{n+1}\sigma_n\{(a_{n+1} r_n + s_n)\varepsilon_{n+1} v_n - \varepsilon_{n+1} r_n(a_{n+1} v_n + w_n - \sigma_n v_{n+1})\} \\
&= -\sigma_n(s_n v_n - r_n w_n) - r_n v_{n+1} \\
&= -\sum_{k=0}^{n-1} v_{k+1} r_k - v_{n+1} r_n .
\end{aligned}$$

q.e.d.

FUNDAMENTAL LEMMA. *For each* $(\alpha, \beta) \in X_1$, *we have*

$$\bar{T}^n(\alpha, \beta, -\infty, 0) = \left(\alpha_n, \beta_n, -\frac{r_n}{s_n}, -\frac{\sum_{k=0}^{n-1} v_{k+1} r_k}{s_n}\right) \in \bar{X} .$$

PROOF. From Remark 3.2, $\bar{T}^n(\alpha, \beta, \gamma, \delta)$ is denoted by

$$\bar{T}^n(\alpha, \beta, \gamma, \delta) = \left(\alpha_n, \beta_n, \frac{-t_n + r_n\gamma}{u_n - s_n\gamma}, \frac{\sigma_n(t_n w_n - u_n v_n) - \sigma_n(r_n w_n - s_n v_n)\gamma + \delta}{u_n - s_n\gamma}\right).$$

Take $(\gamma, \delta) \to (-\infty, 0)$. Since $\bar{T}(\alpha, \beta, -\infty, 0) = (\alpha_1, \beta_1, -a_1, 0) \in X_2 \times X_2^*$, we have $\bar{T}^n(\alpha, \beta, -\infty, 0) \in \bar{X}$. From the Sublemma we have the conclusion. q.e.d.

Using the idea of natural extension and the Fundamental lemma, we have a proposition.

PROPOSITION 3.1. *For each* $(\alpha, \beta) \in X_1$ *and its principally approximate points* $P_n = (q_n, p_n)$, *we have*

(1) $q_n | q_n\alpha + \beta - p_n | < \lambda$ *iff* $\bar{T}^{2n-1}(\alpha, \beta, -\infty, 0) \in D_\lambda$

$$\textit{where} \quad D_\lambda := \left\{(\alpha, \beta, \gamma, \delta) \in X_2 \times X_2^* : \frac{-(\gamma+\delta)(\alpha+\beta)}{-\gamma+\alpha} < \lambda\right\}.$$

(2) *Let us denote principally and intermediately approximate points by* $P_{n,k} = (u_{n,k}, v_{n,k})$ $(0 \leqq k \leqq a_{2n+1} - 1)$. *We have*

$$u_{n,k} | u_{n,k}\alpha + \beta - v_{n,k} | < \lambda \quad \textit{iff} \quad \bar{T}^{2n}(\alpha, \beta, -\infty, 0) \in D_\lambda^{(k)},$$

$$\textit{where} \quad D_\lambda^{(k)} = \left\{(\alpha, \beta, \gamma, \delta) \in X_1 \times X_1^* : \left|\frac{(-k\gamma + 1 - \delta)(1 - k\alpha - \beta)}{-\gamma + \alpha}\right| < \lambda\right\}.$$

PROOF. By the definition of the nth principally approximate point (q_n, p_n) and Propositions 1.1 and 1.2, we have

$$q_n | q_n\alpha + \beta - p_n | = \frac{\left(r_{2n-1} + \sum_{k=0}^{2n-2} v_{k+1} r_k\right)(\alpha_{2n-1} + \beta_{2n-1})}{r_{2n-1} + s_{2n-1}\alpha_{2n-1}}$$

$$= \left(\frac{r_{2n-1}}{s_{2n-1}} + \frac{\sum_{k=0}^{2n-2} v_{k+1} r_k}{s_{2n-1}}\right)(\alpha_{2n-1} + \beta_{2n-1}) \Big/ \left(\frac{r_{2n-1}}{s_{2n-1}} + \alpha_{2n-1}\right).$$

Therefore from $(\alpha_{2n-1}, \beta_{2n-1}, -r_{2n-1}/s_{2n-1}, -(\sum_{k=0}^{2n-2} v_{k+1} r_k)/s_{2n-1}) \in X_2 \times X_2^*$ and the definition of D_λ, we have (1). Similarly we can obtain (2). q.e.d.

COROLLARY 3.2. *For each* $(\alpha, \beta) \in X_1$, *we have*

$$q_n | q_n\alpha + \beta - p_n | < 1 \qquad \textit{for all} \quad n.$$

PROOF. For any $(\alpha, \beta, \gamma, \delta) \in X_2 \times X_2^*$, we see

$$(-\gamma + \alpha) - (-\gamma - \delta)(\alpha + \beta) = -\gamma(1 - \alpha - \beta) + \alpha(1 + \delta) + \beta\delta > 0,$$

which shows

$$0 \leqq -\frac{(\gamma+\delta)(\alpha+\beta)}{\alpha-\gamma} < 1 .$$

q.e.d.

§4. Quadratic fields and periodic points of T.

Let us consider the map T^2 on X_1. We define a map $\overline{T^2}$ on $X_1 \times \boldsymbol{R}^2$ by

$$(4.1) \qquad (\overline{T^2})(\alpha, \beta, \gamma, \delta) = \begin{cases} \left(\dfrac{-a_2+(a_1a_2+1)\alpha}{1-a_1\alpha}, \dfrac{\beta}{1-a_1\alpha}, \dfrac{-a_2+(a_1a_2+1)\gamma}{1-a_1\gamma}, \dfrac{\delta}{1-a_1\gamma}\right), \\ \qquad \text{if} \quad (\alpha, \beta) \in \delta_{a_1}(a_2, 1) , \\ \left(\dfrac{a_2-(a_1a_2+1)\alpha}{1-a_1\alpha}, \dfrac{1-a_1\alpha-\beta}{1-a_1\alpha}, \dfrac{a_2-(a_1a_2+1)\gamma}{1-a_1\gamma}, \dfrac{1-a_1\gamma-\delta}{1-a_1\gamma}\right), \\ \qquad \text{if} \quad (\alpha, \beta) \in \delta_{a_1}(a_2, -1) , \end{cases}$$

where $\delta_a(k, \pm 1)$ is a refinement of $\Delta(a)$ given by

$$\delta_a(1, -1) := \left\{(\alpha, \beta) \in \Delta(a) : \alpha < \frac{1}{a+1}\right\} ,$$

$$\delta_a(k, 1) := \left\{(\alpha, \beta) \in \Delta(a) : \alpha \geqq \frac{1}{a+1/k}, \ (a(k+1)+1)\alpha + \beta < k+1\right\} ,$$

$$\delta_a(\mathrm{k}, -1) := \left\{(\alpha, \beta) \in \Delta(a) : \alpha < \frac{1}{a+1/k}, \ (ak+1)\alpha + \beta \geqq k\right\} ,$$

and therefore the following relations hold:

$$T(\delta_a(k, \varepsilon)) = \Delta(k, \varepsilon) \quad \text{and} \quad T^2(\delta_a(k, \varepsilon)) = X_1 \quad \text{(except on boundaries)} .$$

The restriction $\overline{T^2}|_{\bar{X}_1}$ on $\bar{X}_1 = X_1 \times X_1^*$ is the natural extension on $\bar{X}_1$ of (X_1, T^2) and coincides with $\bar{T}^2|_{X_1}$, and so we denote it by $\bar{T}^2$.

Let us assume in this section that α is quadratic irrational and $(\alpha, \beta) \in X_1$. We denote the simple continued fraction expansion of α by

$$\alpha = [0: e_1, e_2, \cdots, e_N, \overline{e_{N+1}, \cdots, e_{N+k}}]$$

where $N+1$ is the first index of the periodicity of digits $\{e_i\}$ and k is the length of the period. We introduce a set of numbers $\Xi(\alpha)$ associated with α as follows:

Let us denote

$$\alpha^{(0)} := \alpha , \qquad \alpha^{(i+1)} := \frac{1}{\alpha^{(i)}} - e_{i+1} \ (= [0: e_{i+2}, e_{i+3}, \cdots]) \quad (i \geqq 0) ,$$

$$\alpha^{(i,1)}:=1-\alpha^{(i)}=(e_i+1)-\frac{1}{\alpha^{(i-1)}} \qquad (i\geqq 1),$$

$$\alpha^{(i,j)}:=2-\frac{1}{\alpha^{(i,j-1)}} \qquad (2\leqq j\leqq e_{i+1}).$$

A set of numbers $\Xi(\alpha)$ associated with a quadratic number α is defined by

$$\Xi(\alpha):=\{\alpha^{(i)},\ \alpha^{(i,j)}\ (i\geqq 0,\ 1\leqq j\leqq e_{i+1})\}.$$

Then $\Xi(\alpha)$ is a finite set because α is quadratic.

LEMMA 4.1. *We have a following property*:

$$1-\frac{1}{\alpha^{(i+1)}+j}=\alpha^{(i,e_{i+1}-j+1)} \qquad (1\leqq j\leqq e_{i+1}).$$

PROOF. Let us use $w=\begin{pmatrix} a & b \\ c & d\end{pmatrix}z$ to denote the linear transformation $w=(c+dz)/(a+bz)$. We know that

$$\begin{aligned}\alpha^{(i+1)}&=\begin{pmatrix} 0 & 1 \\ 1 & -e_{i+1}\end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & -e_i\end{pmatrix}\alpha^{(i-1)} \qquad (i\geqq 1)\\ &=\begin{pmatrix} 1 & -1 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 2\end{pmatrix}^{e_{i+1}-1}\begin{pmatrix} 0 & 1 \\ -1 & e_i+1\end{pmatrix}\alpha^{(i-1)}\\ &=\begin{pmatrix} 1 & -1 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 2\end{pmatrix}^{e_{i+1}-1}\alpha^{(i,1)}\\ &=\begin{pmatrix} 1 & -1 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 2\end{pmatrix}^{j-1}\alpha^{(i,e_{i+1}-j+1)}.\end{aligned}$$

Therefore we have

$$\begin{pmatrix} j & 1 \\ j-1 & 1\end{pmatrix}\alpha^{(i+1)}=\alpha^{(i,e_{i+1}-j+1)} \qquad \text{for} \quad 1\leqq j\leqq e_{i+1}.$$

Thus we have the conclusion, that is,

$$\frac{(j-1)+\alpha^{(i+1)}}{j+\alpha^{(i+1)}}=1-\frac{1}{j+\alpha^{(i+1)}}=\alpha^{(i,e_{i+1}-j+1)} \qquad (1\leqq j\leqq e_{i+1}).$$

q.e.d.

We sometimes use the following formulae which are equivalent to Lemma 4.1.

COROLLARY 4.2.

(1) $1-\alpha^{(i,j)}=\dfrac{1}{\alpha^{(i+1)}+e_{i+1}-j+1} \qquad (1\leqq j\leqq e_{i+1})$,

(2) $\dfrac{1}{1-\alpha^{(i,j)}}-e_{i+1}-1+j=\alpha^{(i+1)} \qquad (1\leqq j\leqq e_{i+1})$.

LEMMA 4.3. *For $\alpha'\in\Xi(\alpha)$ and $(\alpha',\beta)\in X_1$, T^2 is denoted by*

$$T^2(\alpha',\beta)=\begin{cases}\left(\alpha^{(i+2)},\dfrac{\beta}{\alpha^{(i)}\alpha^{(i+1)}}\right) \\ \qquad \text{if } \alpha'=\alpha^{(i)} \text{ and } (\alpha',\beta)\in\delta_{e_{i+1}}(e_{i+2},1), \\ \left(\alpha^{(i+2,1)},1-\dfrac{\beta}{\alpha^{(i)}\alpha^{(i+1)}}\right) \\ \qquad \text{if } \alpha'=\alpha^{(i)} \text{ and } (\alpha',\beta)\in\delta_{e_{i+1}}(e_{i+2}+1,-1), \\ \left(\alpha^{(i,e_{i+1}-j+1)},1-\dfrac{\beta}{\alpha^{(i)}(\alpha^{(i+1)}+j)}\right) \\ \qquad \text{if } \alpha'=\alpha^{(i)} \text{ and } (\alpha',\beta)\in\delta_{e_{i+1}-j}(1,-1)\ (1\leqq j\leqq e_{i+1}-1), \\ \left(\alpha^{(i+1)},\dfrac{\beta}{1-\alpha^{(i,j)}}\right) \\ \qquad \text{if } \alpha'=\alpha^{(i,j)} \text{ and } (\alpha',\beta)\in\delta_1(e_{i+1}-j,1)\ (1\leqq j\leqq e_{i+1}-1), \\ \left(\alpha^{(i+1,1)},1-\dfrac{\beta}{1-\alpha^{(i,j)}}\right) \\ \qquad \text{if } \alpha'=\alpha^{(i,j)} \text{ and } (\alpha',\beta)\in\delta_1(e_{i+1}-j+1,-1)\ (1\leqq j\leqq e_{i+1}-1), \\ \left(\alpha^{(i+3)},\dfrac{\beta}{\alpha^{(i,e_{i+1})}\alpha^{(i+2)}}\right) \\ \qquad \text{if } \alpha'=\alpha^{(i,e_{i+1})} \text{ and } (\alpha',\beta)\in\delta_{e_{i+2}+1}(e_{i+3},1), \\ \left(\alpha^{(i+3,1)},1-\dfrac{\beta}{\alpha^{(i,e_{i+1})}\alpha^{(i+2)}}\right) \\ \qquad \text{if } \alpha'=\alpha^{(i,e_{i+1})} \text{ and } (\alpha',\beta)\in\delta_{e_{i+2}+1}(e_{i+3}+1,-1), \\ \left(\alpha^{(i+1,e_{i+2}-j+1)},1-\dfrac{\beta}{\alpha^{(i,e_{i+1})}(\alpha^{(i+2)}+j)}\right) \\ \qquad \text{if } \alpha'=\alpha^{(i,e_{i+1})} \text{ and } (\alpha',\beta)\in\delta_{e_{i+2}-j+1}(1,-1)\ (1\leqq j\leqq e_{i+2}).\end{cases}$$

PROOF. (1) Assume that $\alpha'=\alpha^{(i)}$. Since

$$\frac{1}{e_{i+1}+1}<\alpha^{(i)}<\frac{1}{e_{i+1}},$$

$(\alpha^{(i)}, \beta)$ belongs to

$$\delta_{e_{i+1}}(e_{i+2}, 1), \quad \delta_{e_{i+1}}(e_{i+2}+1, -1) \quad \text{or} \quad \bigcup_{j=1}^{e_{i+1}-1} \delta_{e_{i+1}-j}(1, -1).$$

Thus, we see

$$\begin{array}{ll}
(\alpha^{(i)}, \beta) \xrightarrow{T} \left(\alpha^{(i+1)}, -\dfrac{\beta}{\alpha^{(i)}}\right) \xrightarrow{T} \left(\alpha^{(i+2)}, \dfrac{\beta}{\alpha^{(i)}\alpha^{(i+1)}}\right) & \text{on } \delta_{e_{i+1}}(e_{i+2}, 1), \\
(\alpha^{(i)}, \beta) \xrightarrow{T} \left(\alpha^{(i+1)}, -\dfrac{\beta}{\alpha^{(i)}}\right) \longrightarrow \left(1-\alpha^{(i+2)}, 1-\dfrac{\beta}{\alpha^{(i)}\alpha^{(i+1)}}\right) & \text{on } \delta_{e_{i+1}}(e_{i+2}+1, -1), \\
(\alpha^{(i)}, \beta) \longrightarrow \left(\alpha^{(i+1)}+j, -\dfrac{\beta}{\alpha^{(i)}}\right) \longrightarrow \left(1-\dfrac{1}{\alpha^{(i+1)}+j}, 1-\dfrac{\beta}{\alpha^{(i)}(\alpha^{(i+1)}+j)}\right) & \text{on } \delta_{e_{i+1}-j}(1, -1),
\end{array}$$

and so from Lemma 4.1 we have the conclusion in the case of $\alpha'=\alpha^{(i)}$.

(2) Assume that $\alpha'=\alpha^{(i,j)}$ $(1 \leqq j \leqq e_{i+1}-1)$. Then from the definition of $\alpha^{(i)}$ and $\alpha^{(i,j)}$, we know

$$\frac{1}{e_{i+1}+1}<\alpha^{(i)}<\frac{1}{e_{i+1}} \quad \text{and} \quad \frac{e_{i+1}-1}{e_{i+1}}<\alpha^{(i,1)}<\frac{e_{i+1}}{e_{i+1}+1}.$$

By induction, we have

$$\frac{1}{1+\dfrac{1}{e_{i+1}-j}}<\alpha^{(i,j)}<\frac{1}{1+\dfrac{1}{e_{i+1}-j+1}}.$$

Therefore (α', β) belongs to $\delta_1(e_{i+1}-j, 1)$ or $\delta_1(e_{i+1}-j+1, -1)$. Thus, from Corollary 4.2, we see

$$(\alpha^{(i,j)},\beta)\xrightarrow{T}\left(\frac{1}{\alpha^{(i,j)}}-1,\ -\frac{\beta}{\alpha^{(i,j)}}\right)$$

$$\xrightarrow{T}\left(\frac{1}{\dfrac{1}{\alpha^{(i,j)}}-1}-e_{i+1}+j,\ \frac{\beta}{\alpha^{(i,j)}\left(\dfrac{1}{\alpha^{(i,j)}}-1\right)}\right)=\left(\alpha^{(i+1)},\ \frac{\beta}{1-\alpha^{(i,j)}}\right)$$

on $\delta_1(e_{i+1}-j,1)$,

$$\searrow\left(e_{i+1}-j+1-\frac{1}{\dfrac{1}{\alpha^{(i,j)}}-1},\ 1-\frac{\beta}{\alpha^{(i,j)}\left(\dfrac{1}{\alpha^{(i,j)}}-1\right)}\right)=\left(\alpha^{(i+1,1)},\ 1-\frac{\beta}{1-\alpha^{(i,j)}}\right)$$

on $\delta_1(e_{i+1}-j+1,-1)$.

(3) Assume that $\alpha'=\alpha^{(i,e_{i+1})}$. From Corollary 4.2, we know

$$\alpha^{(i+1)}=\frac{\alpha^{(i,e_{i+1})}}{1-\alpha^{(i,e_{i+1})}}.$$

Therefore, we see $1/\alpha^{(i,e_{i+1})}=1+1/\alpha^{(i+1)}$, and so α' belongs to

$$\delta_{e_{i+2}+1}(e_{i+3},1),\quad \delta_{e_{i+2}+1}(e_{i+3}+1,-1)\quad\text{or}\quad \delta_{e_{i+2}-j+1}(1,-1)\ (1\leqq j\leqq e_{i+2}).$$

Thus, we see

$$(\alpha',\beta)\xrightarrow{T}\left(\alpha^{(i+2)},\ -\frac{\beta}{\alpha^{(i,e_{i+1})}}\right)\nearrow\left(\alpha^{(i+3)},\ \frac{\beta}{\alpha^{(i,e_{i+1})}\alpha^{(i+2)}}\right)$$

on $\delta_{e_{i+2}+1}(e_{i+3},1)$,

$$\left(\alpha^{(i+2)},\ -\frac{\beta}{\alpha^{(i,e_{i+1})}}\right)\longrightarrow\left(\alpha^{(i+3,1)},\ 1-\frac{\beta}{\alpha^{(i,e_{i+1})}\alpha^{(i+2)}}\right)$$

on $\delta_{e_{i+2}+1}(e_{i+3}+1,-1)$,

$$(\alpha',\beta)\searrow\left(\alpha^{(i+2)}+j,\ -\frac{\beta}{\alpha^{(i,e_{i+1})}}\right)\longrightarrow\left(1-\frac{1}{\alpha^{(i+2)}+j},\ 1-\frac{\beta}{\alpha^{(i,e_{i+1})}(\alpha^{(i+2)}+j)}\right)$$

on $\delta_{e_{i+2}-j+1}(1,-1)$.

q.e.d.

REMARK 4.1. Let us define $\mathrm{ind}(\alpha')=i$ if $\alpha'=\alpha^{(i)}$ or $\alpha^{(i,j)}$, then we have $\mathrm{ind}(\alpha')<\mathrm{ind}(\alpha'')$ if $T^4(\alpha',\beta)=(\alpha'',\beta'')$.

COROLLARY 4.4. *Under the same assumptions and notations, the set* $\Xi(\alpha)$ *is T-invariant, that is, for any* $(\alpha',\beta)\in X$ *and* $\alpha'\in\Xi(\alpha)$, *the first component of* $T^2(\alpha',\beta)$

belongs to $\Xi(\alpha)$.

We say (α, β) is *reduced* if it has the following properties:

(i) $(\alpha, \beta)\in X_1$,

(ii) α is a quadratic irrational,

(iii) $\beta\in \boldsymbol{Q}(\alpha)$, where $\boldsymbol{Q}(\alpha)$ is the quadratic field generated by α, and

(iv) $(\alpha, \beta, \bar{\alpha}, \bar{\beta})\in \bar{X}_1 := X_1\times X_1^*$, where $\bar{\alpha}$ means the algebraic conjugate of α.

PROPOSITION 4.5. *If* (α, β) *is reduced, then*

(1) (α_2, β_2) $(=T^2(\alpha, \beta))$ *is reduced, and*

(2) *there exists unique* (α^*, β^*) *such that* (α^*, β^*) *is reduced and* $T^2(\alpha^*, \beta^*)=(\alpha, \beta)$.

PROOF. From the definition of T and the concept "reduced", we have $\boldsymbol{Q}(\alpha, \beta)=\boldsymbol{Q}(\alpha)=\boldsymbol{Q}(\alpha_2)$ and so $\beta_2\in \boldsymbol{Q}(\alpha_2)$. From the definition of the natural extension $\bar{T}$, we see $\bar{T}^2(\alpha, \beta, \bar{\alpha}, \bar{\beta})\in X_1\times X_1^*$. If we put $\bar{T}^2(\alpha, \beta, \bar{\alpha}, \bar{\beta})=(\alpha_2, \beta_2, \gamma_2, \delta_2)$ and use (4.1), we know $(\gamma_2, \delta_2)=(\bar{\alpha}_2, \bar{\beta}_2)$. Thus, we obtain (1).

From the definition of $\bar{T}$ and Remark 3.1, there exists $(\alpha^*, \beta^*, \gamma^*, \delta^*)\in X_1\times X_1^*$ such that $\bar{T}^2(\alpha^*, \beta^*, \gamma^*, \delta^*)=(\alpha, \beta, \bar{\alpha}, \bar{\beta})$ because $\alpha\notin \boldsymbol{Q}$.

Suppose that there exist $(\alpha', \beta', \gamma', \delta')\neq(\alpha'', \beta'', \gamma'', \delta'')$ in $X_2\times X_2^*$ such that $\bar{T}(\alpha', \beta', \gamma', \delta')=\bar{T}(\alpha'', \beta'', \gamma'', \delta'')=(\alpha, \beta, \bar{\alpha}, \bar{\beta})$. Let us assume $(\alpha', \beta')\in \Delta(a, \varepsilon)$ and $(\alpha'', \beta'')\in \Delta(a_1, \varepsilon_1)$, then from the definition of $\bar{T}$ we know $\varepsilon=\varepsilon_1$.

In case that $\varepsilon=\varepsilon_1=1$, we see $1/\gamma'=(1/\gamma'')\pm 1$, $-\delta'/\gamma'=-\delta''/\gamma''$ and $a=a_1\pm 1$. Let us assume $1/\gamma'=(1/\gamma'')-1$, then we have $a=a_1-1$, $\gamma'+\delta'=-1$ and $\delta''=-1$. We see $\bar{\alpha}=(1/\gamma'')-a-1$ and $\bar{\beta}=1/\gamma''$. We obtain $\bar{\alpha}=\bar{\beta}-a-1$, that is, $\alpha=\beta-a-1$. This contradicts $(\alpha, \beta)\in X_1$.

In case that $\varepsilon=\varepsilon_1=-1$, we can discuss similarly. Thus, we succeeded in showing there exists unique $(\alpha', \beta', \gamma', \delta')\in X_2\times X_2^*$ such that $\bar{T}(\alpha', \beta', \gamma', \delta')=(\alpha, \beta, \bar{\alpha}, \bar{\beta})$. Furthermore, from this equality, we can show $(\gamma', \delta')=(\bar{\alpha}', \bar{\beta}')$ easily.

Suppose that there exist $(\alpha^*, \beta^*, \gamma^*, \delta^*)\neq(\alpha^{*\prime}, \beta^{*\prime}, \gamma^{*\prime}, \delta^{*\prime})$ in $X_1\times X_1^*$ such that $\bar{T}(\alpha^*, \beta^*, \gamma^*, \delta^*)=\bar{T}(\alpha^{*\prime}, \beta^{*\prime}, \gamma^{*\prime}, \delta^{*\prime})=(\alpha', \beta', \bar{\alpha}', \bar{\beta}')$. Then we can assume that $(\alpha^*, \beta^*)\in \Delta(a)$, $(\alpha^{*\prime}, \beta^{*\prime})\in \Delta(a+1)$, $\delta^*=\gamma^*+1$ and $\delta^{*\prime}=1$. Hence we have $\bar{\alpha}'=(1/\gamma^{*\prime})-a-1$, $\bar{\beta}'=-1/\gamma^{*\prime}$ and so $\bar{\alpha}'+\bar{\beta}'=-a-1$, that is, $\alpha'+\beta'=-a-1$. This contradicts $(\alpha', \beta')\in X_2$. Thus, we showed there exists unique $(\alpha^*, \beta^*, \gamma^*, \delta^*)\in X_1\times X_1^*$ such that $\bar{T}(\alpha^*, \beta^*, \gamma^*, \delta^*)=(\alpha', \beta', \bar{\alpha}', \bar{\beta}')$. Furthermore, from this equality, we can show $(\gamma^*, \delta^*)=(\bar{\alpha}^*, \bar{\beta}^*)$ easily. q.e.d.

LEMMA 4.6. *If* (α, β) *is reduced, then* $\{(\alpha_{2n}, \beta_{2n}) : n=0, 1, 2, \cdots\}$ *is a finite set.*

PROOF. Since $\Xi(\alpha)$ is finite, the set $\{\alpha_{2n}\}$ is finite. From the definition of T, for $(\alpha, \beta)\in X$ we have

$$\beta_n=-\varepsilon_n\frac{\beta_{n-1}}{\alpha_{n-1}}+\nu_n\,,$$

and so β_n is denoted by

$$\beta_n=(-1)^n\varepsilon_n\varepsilon_{n-1}\cdots\varepsilon_1\frac{\beta}{\alpha_{n-1}\alpha_{n-2}\cdots\alpha}+(-1)^{n-1}\varepsilon_n\varepsilon_{n-1}\cdots\varepsilon_2\frac{\nu_1}{\alpha_{n-1}\cdots\alpha_1}$$

$$+\cdots+(-1)\varepsilon_n\frac{\nu_{n-1}}{\alpha_{n-1}}+\nu_n\,.$$

Using Proposition 1.2 (1) for α_i instead of α, we see that there exist integers $m(i)$ and $n(i)$ such that

$$\frac{1}{\alpha_{n-1}\alpha_{n-2}\cdots\alpha_i}=m(i)+n(i)\alpha_n\,.$$

Therefore, there exist some integers p, q, r and s such that β_n is denoted by

$$\beta_n=(p+q\alpha_n)\beta+(r+s\alpha_n)\,. \tag{$*$}$$

Thus, if we denote $\beta=(r+s\sqrt{D})/t$ and $\beta_n=(b_n+c_n\sqrt{D})/d_n$ $(r, s, t, b_n, c_n, d_n\in \boldsymbol{Z})$, where D is the discriminant of the quadratic number α, then denominators d_n of β_n are bounded because from Corollary 4.4 the number of α_n's is finite and the form $(*)$ holds.

By Proposition 4.5, we see $(\alpha_{2n}, \beta_{2n}, \bar{\alpha}_{2n}, \bar{\beta}_{2n})\in\bar{X}_1$. Therefore we have

$$0\leqq\beta_{2n}\leqq 1-\alpha_{2n}\,,\quad\text{and}$$

$$1\leqq\bar{\beta}_{2n}\leqq\bar{\alpha}_{2n}\quad\text{or}\quad 1+\bar{\alpha}_{2n}\leqq\bar{\beta}_{2n}\leqq 0\,.$$

Thus, b_{2n} and c_{2n} are estimated by d_{2n}, α_{2n} and $\bar{\alpha}_{2n}$, and accordingly the set $\{(b_{2n}, c_{2n}, d_{2n}) : n=1, 2, \cdots\}$ is finite. q.e.d.

REMARK 4.2. From Corollary 1.6, for $(\alpha, \beta)\in X_1$ the sequence $\{(\alpha_n, \beta_n)\}$ is (purely) periodic iff its name is so.

PROPOSITION 4.7. *If (α, β) is reduced, then the name (a_n, ε_n) of (α, β) is purely periodic.*

PROOF. From finiteness of $\{(\alpha_{2n}, \beta_{2n}) : n=0, 1, 2, \cdots\}$, there exist k and N such that

$$(\bar{T}^2)^k(\alpha_{2N}, \beta_{2N}, \bar{\alpha}_{2N}, \bar{\beta}_{2N})=(\alpha_{2N}, \beta_{2N}, \bar{\alpha}_{2N}, \bar{\beta}_{2N})\,.$$

Therefore, by Proposition 4.5 (2), we have

$$(\bar{T}^2)^k(\alpha, \beta, \bar{\alpha}, \bar{\beta})=(\alpha, \beta, \bar{\alpha}, \bar{\beta})\,.$$

This means the name of (α, β) is purely periodic. q.e.d.

A quadratic number α is called *reduced* if $0<\alpha<1$ and $\bar{\alpha}<-1$. Then this is well-known that α is quadratic and reduced iff its continued fraction expansion is purely periodic. Hence we have

LEMMA 4.8. *Let α be a reduced quadratic irrational number and its continued fraction expansion be denoted by*

$$\alpha=[0:\overline{e_1,\cdots,e_k}].$$

Then, for all $i\geqq 1$, we have the following:

(1) $\overline{\alpha^{(i)}}+j<-1 \qquad (0\leqq j\leqq e_i-1)$,

(2) $1<\overline{\alpha^{(i,j)}}<2 \qquad (2\leqq j\leqq e_{i+1})$,

(3) $2<\overline{\alpha^{(i,1)}}$,

(4) $1<(1-\overline{\alpha^{(i,j)}})\overline{\alpha^{(i+1)}} \qquad (1\leqq j\leqq e_{i+1}-1)$.

PROOF. Since $\alpha^{(i)}$ is also reduced, we know $\overline{\alpha^{(i)}}<-1$. Then, we see $-1<1/\overline{\alpha^{(i-1)}}=\overline{\alpha^{(i)}}+e_i<0$, and so we obtain (1). From the definition, we see

$$\overline{\alpha^{(i,1)}}=e_i+1-\frac{1}{\overline{\alpha^{(i-1)}}}>e_i+1\geqq 2$$

and inductively

$$2>\overline{\alpha^{(i,j+1)}}=2-\frac{1}{\overline{\alpha^{(i,j)}}}>1 \qquad (1\leqq j\leqq e_{i+1}-1).$$

From (1) and (2), we remark $1-\overline{\alpha^{(i,j)}}<0$ and $\overline{\alpha^{(i+1)}}<0$. By Corollary 4.2 (1), we have

$$(1-\overline{\alpha^{(i,j)}})\overline{\alpha^{(i+1)}}=\frac{-\overline{\alpha^{(i+1)}}}{-\overline{\alpha^{(i+1)}}-e_{i+1}+j-1}>\frac{-\overline{\alpha^{(i+1)}}}{-\overline{\alpha^{(i+1)}}}=1 \qquad \text{for} \quad 1\leqq j\leqq e_{i+1}-1.$$

q.e.d.

LEMMA 4.9. *Let us assume that $(\alpha,\beta,\bar{\alpha},\delta),(\alpha,\beta,\bar{\alpha},\delta')\in X_1\times \boldsymbol{R}^2$ and α is a reduced quadratic irrational. Then, there exists a constant c_1 $(0<c_1<1)$ such that the inequality either*

$$|\delta_{2n+2}-\delta'_{2n+2}|<c_1|\delta_{2n}-\delta'_{2n}| \quad or \quad |\delta_{2n+4}-\delta'_{2n+4}|<c_1|\delta_{2n}-\delta'_{2n}|$$

holds for all n, where $(\alpha_{2n},\beta_{2n},\bar{\alpha}_{2n},\delta_{2n})$ means $(\bar{T}^2)^n(\alpha,\beta,\bar{\alpha},\delta)$.

PROOF. We put $|\delta_{2n+2}-\delta'_{2n+2}|=A|\delta_{2n}-\delta'_{2n}|$ and $|\delta_{2n+4}-\delta'_{2n+4}|=B|\delta_{2n}-\delta'_{2n}|$. By (4.1) and Lemma 4.3, in case that $\alpha_{2n}=\alpha^{(i)}$, we have

$$A=\frac{1}{\overline{\alpha^{(i)}}\,\overline{\alpha^{(i+1)}}} \quad \text{or} \quad A=\frac{1}{\overline{\alpha^{(i)}}\,(\overline{\alpha^{(i+1)}}+j)} \quad (1\leqq j\leqq e_{i+1}-1).$$

From Lemma 4.8 (1), we see $A<1$. In case that $\alpha_{2n}=\alpha^{(i,e_{i+1})}$, we have

$$A=-\frac{1}{\overline{\alpha^{(i,e_{i+1})}}\,\overline{\alpha^{(i+2)}}} \quad \text{or} \quad A=-\frac{1}{\overline{\alpha^{(i,e_{i+1})}}\,(\overline{\alpha^{(i+2)}}+j)} \quad (1\leqq j\leqq e_{i+2}).$$

Also we see $A<1$ except the case $j=e_{i+2}$.

If $\alpha_{2n}=\alpha^{(i,j)}$ and $\alpha_{2n+2}=\alpha^{(i+1)}$, we have

$$B=-\frac{1}{\overline{\alpha^{(i+1)}}\,\overline{\alpha^{(i+2)}}(1-\overline{\alpha^{(i,j)}})}\qquad\text{or}$$

$$B=-\frac{1}{\overline{\alpha^{(i+1)}}(\overline{\alpha^{(i+2)}}+k)(1-\overline{\alpha^{(i,j)}})}\quad(1\leqq j\leqq e_{i+1}-1,\ 1\leqq k\leqq e_{i+2}-1)\,.$$

From Lemma 4.8 (1) (4), we see $B<1$.

If $\alpha_{2n}=\alpha^{(i,j)}$ and $\alpha_{2n+2}=\alpha^{(i+1,1)}$, we have from Lemma 4.8 (4)

$$B=\frac{1}{(1-\overline{\alpha^{(i+1,1)}})(1-\overline{\alpha^{(i,j)}})}=\frac{1}{\overline{\alpha^{(i+1)}}(1-\overline{\alpha^{(i,j)}})}<1\,.$$

In case that $\alpha_{2n}=\alpha^{(i,e_{i+1})}$ and $(\alpha_{2n},\beta_{2n})\in\delta_1(1,-1)$, we have

$$\frac{1}{B}=(1-\overline{\alpha^{(i+1,1)}})(\overline{\alpha^{(i+2)}}+e_{i+2})\overline{\alpha^{(i,e_{i+1})}}=\overline{\alpha^{(i+1)}}\frac{1}{\overline{\alpha^{(i+1)}}}\overline{\alpha^{(i,e_{i+1})}}>1\,.$$

Since $\Xi(\alpha)$ is a finite set, we have the conclusion. q.e.d.

Let us consider the boundary of X_1^*. We put

$$\sigma_1:=\partial X_1^*\cap\{\delta=\gamma\}\,,\quad \sigma_2=\partial X_1^*\cap\{\delta=1\}\,,\quad \sigma_3:=\partial X_1^*\cap\{\delta=\gamma+1\}$$
$$\text{and}\qquad \sigma_4:=\partial X_1^*\cap\{\delta=0\}\,.$$

Then we have the following lemma.

LEMMA 4.10. *Let us assume $0<\alpha<1$ and $\bar{\alpha}<-1$ or $\bar{\alpha}>1$. Then there exists a constant c_2 which satisfies the following*:

For any β, γ and δ such that $(\alpha,\beta,\bar{\alpha},\gamma)\notin\bar{X}_1$, $(\alpha,\beta,\bar{\alpha},\delta)\in\bar{X}_1$ and $|\gamma-\delta|<c_2$, we have (i) $(\bar{T}^2)^2(\alpha,\beta,\bar{\alpha},\gamma)\in\bar{X}_1$, (ii) $a_2=a_4=1$, $\varepsilon_2=\varepsilon_4=-1$ *or* (iii) $\varepsilon_2=\varepsilon_4=1$.

Furthermore, if the case (ii) *happens, then $(\alpha,\beta,\bar{\alpha},\gamma)$, $(\alpha,\beta,\bar{\alpha},\delta)$ and their images by $\bar{T}^2$ and $(\bar{T}^2)^2$ are very near σ_1. If the case* (iii) *happens, then they are very near σ_4.*

PROOF. From the definition of $\bar{T}^2$ (see Fig. 6), we see

$$\bar{T}^2((\alpha,\beta)\times\sigma_1)\subset\begin{cases}(\alpha_2,\beta_2)\times\sigma_1 & \text{if}\quad a_2=1,\varepsilon_2=-1\,,\\ (\alpha_2,\beta_2)\times\sigma_3 & \text{if}\quad a_2=1,\varepsilon_2=1\,,\\ (\alpha_2,\beta_2)\times(X_1^*)^\circ & \text{otherwise}\,,\end{cases}$$

$$\bar{T}^2((\alpha,\beta)\times\sigma_4)\subset\begin{cases}(\alpha_2,\beta_2)\times\sigma_4 & \text{if}\quad \varepsilon_2=1\,,\\ (\alpha_2,\beta_2)\times\sigma_2 & \text{if}\quad \varepsilon_2=-1\,,\end{cases}$$

$$\bar{T}^2((\alpha,\beta)\times\sigma_2)\subset(\alpha_2,\beta_2)\times(X_1^*)^\circ\quad\text{and}\quad\bar{T}^2((\alpha,\beta)\times\sigma_3)\subset(\alpha_2,\beta_2)\times(X_1^*)^\circ\,,$$

where A° means the interior of a set A. If $\bar{T}^2((\alpha, \beta)\times\sigma_i)$ is contained in $(\alpha_2, \beta_2)\times(X_1^*)^\circ$, then $(\bar{T}^2)^2((\alpha, \beta)\times\sigma_i)$ is contained in $(\alpha_4, \beta_4)\times(X_1^*)^\circ$ also.

If $(\bar{\alpha}, \gamma)$ and $(\bar{\alpha}, \delta)$ are near σ_2 or σ_3 and $|\gamma-\delta|<c$ for small c, we see $\bar{T}^2(\alpha, \beta, \bar{\alpha}, \gamma)\in\bar{X}_1$. We can discuss other cases similarly, and we have the conclusion. q.e.d.

PROPOSITION 4.11. *For $(\alpha, \beta)\in X_1$, let α be a quadratic irrational and $\beta\in\boldsymbol{Q}(\alpha)$. Then the name of (α, β) is periodic.*

PROOF. Let us denote the continued fraction expansion of α by $\alpha=[0: e_1, \cdots, \overline{e_N, e_{N+1}, \cdots, e_{N+k}}]$. From Remark 4.1, for large n we see $\alpha_n\in\Xi':=\{\alpha^{(i)}, \alpha^{(i,j)}: N+1\leqq i\leqq N+k,\ 1\leqq j\leqq e_{i+1}\}\subset\Xi(\alpha)$. Therefore, for simplicity, we may assume $\alpha\in\Xi'$, and then from Lemma 4.8 we see $\bar{\alpha}_n<-1$ or $\bar{\alpha}_n>1$ for all n. Then we can choose δ such that $(\alpha, \beta, \bar{\alpha}, \delta)\in\bar{X}_1$, and we put $(\alpha_{2n}, \beta_{2n}, \bar{\alpha}_{2n}, \delta_{2n}):=(\bar{T}^2)^n(\alpha, \beta, \bar{\alpha}, \delta)\in\bar{X}_1$. If there exists some n such that $(\alpha_{2n}, \beta_{2n}, \bar{\alpha}_{2n}, \bar{\beta}_{2n})\in\bar{X}_1$ then by Proposition 4.7 we have the conclusion.

Let us assume $(\alpha_{2n}, \beta_{2n}, \bar{\alpha}_{2n}, \bar{\beta}_{2n})\notin\bar{X}_1$ for all n, then the distance d_n between $(\alpha_{2n}, \beta_{2n}, \bar{\alpha}_{2n}, \bar{\beta}_{2n})$ and $(\alpha_{2n}, \beta_{2n}, \bar{\alpha}_{2n}, \delta_{2n})$ is equal to $|\bar{\beta}_{2n}-\delta_{2n}|$. From Lemma 4.9, the subsequence $\{d_{n_i}\}$ $(n_{i+1}=n_i+1$ or $n_i+2)$ tends to 0. Since $\Xi(\alpha)$ is finite, by Lemma 4.10 there exists n_0 such that $a_{2n}=1$, $\varepsilon_{2n}=-1$, $\bar{\alpha}_{2n}>1$ for all $n\geqq n_0$ or $\varepsilon_{2n}=1$, $\bar{\alpha}_{2n}<-1$ for all $n\geqq n_0$ holds. From Proposition 2.5 or 2.4, we have $\bar{\alpha}_{2n}=\bar{\beta}_{2n}$ or $\bar{\beta}_{2n}=0$, and so $(\alpha_{2n}, \beta_{2n}, \bar{\alpha}_{2n}, \bar{\beta}_{2n})\in\bar{X}_1$. Thus, $(\alpha_{2n}, \beta_{2n})$ is also reduced for large n and we have the conclusion. q.e.d.

Now, we have the following theorem.

THEOREM 4.1. *Let us consider $(\alpha, \beta)\in X_1$ and Morimoto algorithm (X, T) or (X_1, T^2). Then, we have the following:*

(1) *(α, β) has a finite name iff $\alpha\in\boldsymbol{Q}$,*

(2) *the name of (α, β) is purely periodic iff (α, β) is reduced,*

(3) *the name of (α, β) is eventually periodic iff α is a quadratic irrational and $\beta\in\boldsymbol{Q}(\alpha)$,*

(4) *there exists a $(k, l)\in\boldsymbol{Z}^2$ satisfying $\beta=k\alpha+l$ iff there exists an n_0 such that $\varepsilon_{2n}=1$ hold for all $n\geqq n_0$ or there exists an n_1 such that $\varepsilon_{2n}=-1$ and $a_{2n}=1$ hold for all $n\geqq n_1$.*

PROOF. Let us assume that the name is eventually periodic. Then, by Remark 4.2 there exist n and m such that $(\alpha_n, \beta_n)=(\alpha_m, \beta_m)$ and $n<m$. From Lemma 1.4, we have

$$\begin{pmatrix}1\\ \alpha_n\\ \beta_n\end{pmatrix}=\alpha_n\alpha_{n+1}\cdots\alpha_{m-1}\begin{pmatrix}r & s & 0\\ t & u & 0\\ v & w & \sigma\end{pmatrix}\begin{pmatrix}1\\ \alpha_m\\ \beta_m\end{pmatrix},$$

and

$$\begin{pmatrix}1\\ \alpha\\ \beta\end{pmatrix}=\alpha\alpha_1\cdots\alpha_{n-1}\begin{pmatrix}r' & s' & 0\\ t' & u' & 0\\ v' & w' & \sigma'\end{pmatrix}\begin{pmatrix}1\\ \alpha_n\\ \beta_n\end{pmatrix},$$

where $r, s, \cdots, r', s', \cdots$ are all in $\boldsymbol{Z}$.

Hence, we see

$$\alpha_n=\frac{t+u\alpha_m}{r+s\alpha_m}, \qquad \beta_n=\frac{v+w\alpha_m+\sigma\beta_m}{r+s\alpha_m},$$

$$\alpha=\frac{t'+u'\alpha_n}{r'+s'\alpha_n}, \quad \text{and} \quad \beta=\frac{v'+w'\alpha_n+\sigma'\beta_n}{r'+s'\alpha_n}.$$

From these equalities, we see that α_n is a quadratic irrational, and so is α. (Since the name is infinite, $\alpha\notin\boldsymbol{Q}$.) And we have $\beta\in\boldsymbol{Q}(\alpha)$ easily.

Next, let us assume the name is purely periodic. We know α is a quadratic irrational and $\beta\in\boldsymbol{Q}(\alpha)$. Then we showed $(\alpha_{2n}, \beta_{2n})$ were reduced for some n in the proof of Proposition 4.11. From the pure periodicity, there exists an m such that $(T^2)^m(\alpha_{2n}, \beta_{2n})=(\alpha, \beta)$. Then, by Proposition 4.5 (α, β) is reduced.

The other conclusions have been shown in earlier discussions. q.e.d.

REMARK 4.3. We can apply Morimoto algorithm for any (α, β) in $\boldsymbol{R}^2$. Taking $\alpha-[\alpha]$, $\beta-[\beta]$ instead of α, β, we can assume $0\leqq\alpha<1$ and $0\leqq\beta<1$. If $\alpha+\beta>1$, we take $a_1=0$ and we put $\alpha_1=1/\alpha$ and $\beta_1=-\beta/\alpha$. Then, we see $(\alpha_1, \beta_1)\in X_2$ and we can continue the algorithm.

§5. Ergodicity and metrical theorems.

Let us define a function $K(\alpha, \beta, \gamma, \delta)$ on $\bar{X}$ by

$$K(\alpha, \beta, \gamma, \delta):=\frac{1}{|\alpha-\gamma|^3}.$$

Then we have the following lemmas.

LEMMA 5.1. *The function K satisfies an equality except on boundary*:

$$K(\bar{T}(\alpha, \beta, \gamma, \delta))J(\bar{T})(\alpha, \beta, \gamma, \delta)=K(\alpha, \beta, \gamma, \delta),$$

where $J(\bar{T})$ is the Jacobian of $\bar{T}$.

The proof follows from the fact that the Jacobian $J(\bar{T})$ is calculated by

$$J(\bar{T})=\frac{1}{\alpha^3\gamma^3}.$$

LEMMA 5.2. *The function K is integrable and*

$$\iiiint_{\bar{X}} K(\alpha, \beta, \gamma, \delta)d\alpha d\beta d\gamma d\delta=2\log 2.$$

From the above lemmas, we have the following theorem.

THEOREM 5.1. *Let us define a measure* $\bar{\mu}$ *on* $\bar{X}$ *by*

$$d\bar{\mu}=\frac{d\alpha d\beta d\gamma d\delta}{(2\log 2)\,|\,\alpha-\gamma\,|^3},$$

then the measure $\bar{\mu}$ *is invariant with respect to* $\bar{T}$ *and the dynamical system* $(\bar{X}, \bar{T}, \bar{\mu})$ *is ergodic.*

COROLLARY 5.2. (1) *Let us define a measure* μ *on* X *by*

$$d\mu=\begin{cases}\dfrac{d\alpha d\beta}{2\log 2}\displaystyle\iint_{X_1^*}\frac{d\gamma d\delta}{|\,\alpha-\gamma\,|^3}=\frac{1}{2\log 2}\cdot\frac{d\alpha d\beta}{1-\alpha^2} & \text{if } (\alpha, \beta)\in X_1\,,\\[2ex] \dfrac{d\alpha d\beta}{2\log 2}\displaystyle\iint_{X_2^*}\frac{d\gamma d\delta}{|\,\alpha-\gamma\,|^3}=\frac{1}{2\log 2}\cdot\frac{d\alpha d\beta}{2\alpha(1+\alpha)} & \text{if } (\alpha, \beta)\in X_2\,,\end{cases}$$

then the measure μ *is invariant with respect to* T *and the dynamical system* (X, T, μ) *is ergodic.*

(2) *Let us define a measure* μ_i *on* X_i *by*

$$d\mu_1=\frac{1}{\log 2}\cdot\frac{d\alpha d\beta}{1-\alpha^2}\,,\qquad d\mu_2=\frac{1}{\log 2}\cdot\frac{d\alpha d\beta}{2\alpha(1+\alpha)}\,,$$

then the measure μ_i *is invariant with respect to* T^2 *and the dynamical system* (X_i, T^2, μ_i) *is weak Bernoulli, respectively.*

PROOF of the theorem and corollaries. From Lemma 5.1 and 5.2, the measure $\bar{\mu}$ on $\bar{X}$ is an invariant measure with respect to $\bar{T}$. From the commutative relation

$$\begin{array}{ccc}\bar{X} & \xrightarrow{\bar{T}} & \bar{X}\\ \downarrow\pi & & \downarrow\pi\\ X & \xrightarrow{T} & X\end{array}$$

where π is a projection such that $\pi(\alpha, \beta, \gamma, \delta)=(\alpha, \beta)$, we see that $\mu=\pi_*(\bar{\mu})$ and that μ is invariant with respect to T. From $T^2(X_i)=X_i$, the measure μ_i is invariant with respect to T^2.

On the other hand, we see the dynamical system (X_i, T^2, μ_i) satisfies Schweiger's condition (see Schweiger [6], Ito-Yuri [3], Yuri [7]). Therefore, the dynamical system (X_i, T^2, μ_i) satisfies weak Bernoulli condition. Hence, the dynamical system (X, T, μ) is ergodic, and so is the natural extension $(\bar{X}, \bar{T}, \bar{\mu})$ (Rohlin [5]). q.e.d.

We obtain some metrical theorems by using the individual ergodic theorem.

THEOREM 5.3. *For almost all* $(\alpha, \beta) \in X_1$, *we have*

$$\lim_{n\to\infty}\left(-\frac{1}{n}\right)\log|\alpha q_n+\beta-p_n|=\frac{\pi^2}{12\log 2}.$$

PROOF. From Proposition 1.1 and the definition (1.7) of (q_n, p_n), we know

$$|\alpha q_n+\beta-p_n|=\alpha\alpha_1\cdots\alpha_{2n-2}(\alpha_{2n-1}+\beta_{2n-1}).$$

Therefore, we have

$$\frac{1}{n}\log|\alpha q_n+\beta-p_n|=\frac{1}{n}\sum_{k=0}^{2n-2}\log\alpha_k+\frac{1}{n}\log|\alpha_{2n-1}+\beta_{2n-1}|.$$

We show that

$$\lim_{n\to\infty}\frac{1}{n}\log|\alpha_{2n-1}+\beta_{2n-1}|=0 \qquad \text{for almost all } (\alpha, \beta).$$

Since

$$\mu_2(\alpha_{2n-1}+\beta_{2n-1}<\eta)=\mu_2(\alpha+\beta<\eta)=\frac{1}{2\log 2}\left(\log(\eta+1)+\eta\log\left(1+\frac{1}{\eta}\right)\right),$$

we see

$$\sum_{n=1}^{\infty}\mu_2(\alpha_{2n-1}+\beta_{2n-1}<e^{-n\varepsilon})<\infty.$$

Thus, by the Borel-Cantelli lemma, we obtain

$$\#\left\{n:-\frac{1}{n}\log|\alpha_{2n-1}+\beta_{2n-1}|>\varepsilon\right\}<\infty \qquad \text{for almost all } (\alpha, \beta).$$

Therefore, by ergodic theorem, we have

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\log|\alpha q_n+\beta-p_n|&=2\int_X\log\alpha\,d\mu\\&=2\left(\int_{X_1}\log\alpha\,d\mu+\int_{X_2}\log\alpha\,d\mu\right)\\&=\frac{1}{\log 2}\left(\int_0^1\frac{\log\alpha}{1+\alpha}d\alpha+\int_0^1\frac{\log\alpha}{2(1+\alpha)}d\alpha+\int_1^{+\infty}\frac{\log\alpha}{2\alpha(1+\alpha)}d\alpha\right)\\&=-\frac{\pi^2}{12\log 2}.\end{aligned}$$

q.e.d.

LEMMA 5.3. *There exists a constant λ such that $\lambda>1$ and $s_{2n+1}>\lambda^n$ for $n\geqq 3$.*

PROOF. From (1.4) we have

$$\begin{aligned} r_{2n+3} &= a_{2n+3}r_{2n+2}+s_{2n+2} \\ &= a_{2n+3}(a_{2n+2}r_{2n+1}+s_{2n+1})+\varepsilon_{2n+2}r_{2n+1} \\ &= (a_{2n+3}a_{2n+2}+\varepsilon_{2n+2})r_{2n+1}+a_{2n+3}s_{2n+1} \\ &= (a_{2n+3}a_{2n+2}+\varepsilon_{2n+2})r_{2n+1}+a_{2n+3}(a_{2n}r_{2n-1}+s_{2n-1}) \\ &\geqq a_{2n+3}a_{2n}r_{2n-1}+a_{2n+3}s_{2n-1} \\ &= a_{2n+3}a_{2n}r_{2n-1}+a_{2n+3}r_{2n-2} \\ &> a_{2n+3}a_{2n}r_{2n-1}+a_{2n+3}r_{2n-3} \end{aligned}$$

and so $r_{2n+3}>r_{2n-1}+r_{2n-3}$.

We choose λ such that $\lambda>1$, $\lambda^3-\lambda-1<0$, $\lambda^5<2$ and $\lambda^6<3$. Let us assume $r_{2k-1}>\lambda^k$ for all $k\leqq n$, then we have

$$r_{2n+1}>r_{2n-3}+r_{2n-5}>\lambda^{n-1}+\lambda^{n-2}=\lambda^{n-2}(\lambda+1)>\lambda^{n-2}\lambda^3=\lambda^{n+1}\,.$$

From Lemma 1.3, we see

$$s_{2n+1}=r_{2n}>r_{2n-1}>\lambda^n\,.$$

q.e.d.

LEMMA 5.4. *If* $\lim_{n\to\infty}\frac{1}{n}\log s_{2n+1}=A$, *then* $\lim_{n\to\infty}\frac{1}{n}\log q_n=A$.

PROOF. From Lemma 1.3, we see

$$\begin{aligned} s_{2n-3}<q_n &= r_{2n-1}+\sum_{k=0}^{2n-2}v_{k+1}r_k \\ &< r_{2n-1}+r_1+r_3+\cdots+r_{2n-3} \\ &< s_{2n+1}+s_3+s_5+\cdots+s_{2n-1} \\ &< ns_{2n+1}\,, \end{aligned}$$

and so

$$\frac{1}{n}\log s_{2n-3}<\frac{1}{n}\log q_n<\frac{1}{n}\log n+\frac{1}{n}\log s_{2n+1}\,.$$

Thus we have the conclusion. q.e.d.

THEOREM 5.4. *For almost all $(\alpha,\beta)\in X_1$, we have*

$$\lim_{n\to\infty}\frac{1}{n}\log q_n=\frac{\pi^2}{12\log 2}\,.$$

PROOF. From Lemma 1.2, we see

$$\alpha\alpha_1\cdots\alpha_{2n}=\frac{1}{r_{2n+1}+s_{2n+1}\alpha_{2n+1}}=\frac{1}{s_{2n+1}}\cdot\frac{1}{\left(\frac{r_{2n+1}}{s_{2n+1}}+\alpha_{2n+1}\right)},$$

and we have

$$\frac{1}{n}\log s_{2n+1}=-\frac{1}{n}\sum_{k=0}^{2n}\log\alpha_k-\frac{1}{n}\log\left(\frac{r_{2n+1}}{s_{2n+1}}+\alpha_{2n+1}\right).$$

In the proof of Theorem 5.3, we showed

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{2n}\log\alpha_k=-\frac{\pi^2}{12\log 2}.$$

Hence we prove

$$\lim_{n\to\infty}\frac{1}{n}\log\left(\frac{r_{2n+1}}{s_{2n+1}}+\alpha_{2n+1}\right)=0 \qquad \text{for almost all } (\alpha,\beta).$$

We take $(\gamma,\delta)\in X_1^*$ such that $\gamma<0$ and put

$$\bar{T}^{2n+1}(\alpha,\beta,\gamma,\delta)=(\alpha_{2n+1},\beta_{2n+1},\gamma_{2n+1},\delta_{2n+1}).$$

Then we know

$$\gamma_{2n+1}=\frac{-t_{2n+1}+r_{2n+1}\gamma}{u_{2n+1}-s_{2n+1}\gamma},$$

and so we have

$$\left|\frac{r_{2n+1}}{s_{2n+1}}+\gamma_{2n+1}\right|=\frac{1}{s_{2n+1}|u_{2n+1}-s_{2n+1}\gamma|}<\frac{1}{s_{2n+1}}<\lambda^{-n}.$$

Let us assume $|r_{2n+1}/s_{2n+1}+\alpha_{2n+1}|<e^{-n\varepsilon}$ for small $\varepsilon>0$, then we have $|\alpha_{2n+1}-\gamma_{2n+1}|<2e^{-n\varepsilon}$. In fact, we see

$$|\alpha_{2n+1}-\gamma_{2n+1}|\leqq\left|\alpha_{2n+1}+\frac{r_{2n+1}}{s_{2n+1}}\right|+\left|\frac{r_{2n+1}}{s_{2n+1}}+\gamma_{2n+1}\right|\leqq e^{-n\varepsilon}+\lambda^{-n}\leqq 2e^{-n\varepsilon}.$$

We have easily $\bar{\mu}_2(\alpha-\gamma<c)=c/(6\log 2)$ for small $c>0$ and we see

$$\sum_{n=1}^{\infty}\bar{\mu}_2\left(\frac{r_{2n+1}}{s_{2n+1}}+\alpha_{2n+1}<e^{-n\varepsilon}\right)\leqq\sum_{n=1}^{\infty}\bar{\mu}_2(\alpha-\gamma<2e^{-n\varepsilon})<+\infty.$$

Thus, by Borel-Cantelli lemma, we obtain

$$\#\left\{n : -\frac{1}{n}\log\left(\frac{r_{2n+1}}{s_{2n+1}}+\alpha_{2n+1}\right)>\varepsilon\right\}<+\infty$$

for almost all (α, β). q.e.d.

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