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Bijective Lattice Path Proof of the Equality of the Dual Jacobi-Trudy Determinants

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Abstract. We give a bijective lattice path proof of the equality of the dual Jacobi-Trudy determinant formulas for Schur polynomials. Related ideas have appeared in [1, pp. 304–306] and [2, p. 24]. We remark that the same bijection works for the case of flagged skew Schur polynomials [2, 8] and that a determinant for q-counting restricted lattice paths [7] follows from the bijection.

1. Let $A=(a_1, \cdots, a_m)$ and $B=(b_1, \cdots, b_m)$ be partitions, i.e. sequences of *increasing* positive integers, $0 \leq a_1 \leq \cdots \leq a_m$, $0 \leq b_1 \leq \cdots \leq b_m$, and suppose that $a_i \geq b_i$ $(i=1, \cdots, m)$. Then the classical *Jacobi-Trudy identities* for Schur polynomials read:

$$S_{A/B}=\det(h_{a_i+i-b_j-j})_{1 \leq i,j \leq m}\,, \tag{1}$$

$$S_{A/B}=\det(e_{a'_i+i-b'_j-j})_{1 \leq i,j \leq n}\,. \tag{2}$$

Here $S_{A/B}$ is the Schur polynomial for the skew diagram A/B, h's are the complete homogeneous symmetric polynomials, e's are the elementary symmetric polynomials, and $A'=(a'_1, \cdots, a'_n)$ is the conjugate partition of A; for the terminology see [3, 4]. Note that we use the French notation following [3]. Proofs of (1) and (2) using the Gessel-Viennot method are known [5, 6]. In this note we give a straightforward combinatorial proof of the equality of the right-hand sides of (1) and (2).

As shown in [5, 6], the right-hand side of (1) is interpreted in terms of weighted lattice paths as follows. We consider lattice paths of N^2 taking horizontal and vertical steps. Let $\mathrm{NP}(B; A)$ be the set of m-tuples of nonintersecting paths from $(b_i+i, 1)$ to (a_i+i, p) $(i=1, \cdots, m)$, where p is the number of indeterminates. Then we have

$$\det(h_{a_i+i-b_j-j}(u_1, \cdots, u_p))_{1 \leq i,j \leq m}=\sum_{s \in \mathrm{NP}(B;A)} \mathrm{wt}(s)\,, \tag{3}$$

where, for $s=(s_1, \cdots, s_m)$ with s_i a path from $(b_i+i, 1)$ to (a_i+i, p), we put $\mathrm{wt}(s):=\mathrm{wt}(s_1)\cdots\mathrm{wt}(s_m)$ and $\mathrm{wt}(s_i)$ is the product of the weights of the horizontal steps that s_i takes; a horizontal step of height k carries indeterminate weight u_k.

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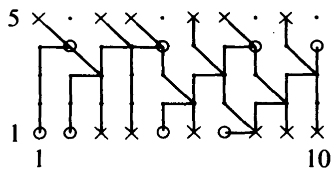
We now consider lattice paths of N^2 taking *north-west* and vertical steps. Let $\mathrm{NP}_c(B'; A')$ be the set of n-tuples of nonintersecting paths from $(n+m+1-i-b'_i, 1)$ to $(n+m+1-i-a'_i, p+1)$ $(i=1, \cdots, n)$. To $t=(t_1, \cdots, t_n) \in \mathrm{NP}_c(B', A')$ with t_i a path from $(n+m+1-i-b'_i, 1)$ to $(n+m+1-i-a'_i, p+1)$, we assign $\mathrm{wt}_c(t)$ in the same way as in the case of $\mathrm{NP}(B, A)$ except that a north-west step starting from height k carries weight u_k.

We construct a weight-preserving bijection between $\mathrm{NP}(B; A)$ and $\mathrm{NP}_c(B'; A')$ as follows: Take $s=(s_1, \cdots, s_m) \in \mathrm{NP}(B; A)$ and pick up all the horizontal steps appearing in s_i $(i=1, \cdots, m)$. Replace the horizontal step from (c, k) to $(c+1, k)$ by a north-west step from $(c+1, k)$ to $(c, k+1)$. Fill out with necessary vertical steps to obtain $t \in \mathrm{NP}_c(B'; A')$ corresponding to s. For example, consider the skew tableau below with $m=4$, $n=6$, and $p=4$:

$$\begin{array}{cccccc}
4 & & & & & \\
3 & 4 & 4 & & & \\
 & & 2 & 3 & 4 & \\
 & & & 1 & 2 & 3
\end{array}$$

The corresponding $s \in \mathrm{NP}(B; A)$ and $t \in \mathrm{NP}_c(B'; A')$ are:



where s connects endpoints marked $\circ$ with *horizontal* steps and vertical ones, and t connects endpoints marked $\times$ with *north-west* steps and vertical ones. The above procedure of obtaining t from s is reversible. Actually $s \in \mathrm{NP}(B; A)$ can be obtained by reading the tableau from left to right and $t \in \mathrm{NP}_c(B'; A')$ by reading it from bottom to top. Note that the set of integers $\{a_i+i \ (i=1, \cdots, m),\ n+m+1-j-a'_j \ (j=1, \cdots, n)\}$ is equal to $\{1, 2, \cdots, n+m\}$; see [4, p. 3, (1.7)]. Clearly this bijection between $\mathrm{NP}(B; A)$ and $\mathrm{NP}_c(B'; A')$ is weight-preserving. Thus we have

$$\sum_{s \in \mathrm{NP}(B; A)} \mathrm{wt}(s) = \sum_{t \in \mathrm{NP}_c(B'; A')} \mathrm{wt}_c(t) . \tag{4}$$

As the counterpart to (3), we have by using the Gessel-Viennot method that

$$\det(e_{a'_i+i-b'_j-j}(u_1, \cdots, u_p))_{1 \le i, j \le n} = \sum_{t \in \mathrm{NP}_c(B'; A')} \mathrm{wt}_c(t) . \tag{5}$$

For details we refer to [5]; note that $e_{a'_i+i-b'_j-j}(u_1, \cdots, u_p)$ is the sum of the weights of all the paths from $(n+m+1-j-b'_j, 1)$ to $(n+m+1-i-a'_i, p+1)$ and [5] uses north-east steps instead of north-west steps. Combining (3), (4), and (5) gives the desired proof.

Note that the number p of indeterminates can be taken to be countable infinity; we simply let the second coordinates of the upper endpoints tend to countable infinity.

2. Remarks about the flagged skew Schur polynomials [2, 8]. We can apply the above bijection construction to the flagged case by adjusting the second coordinates of the endpoints according to the row resp. column flags. Given row flags (β_i, α_i) $(i=1, \cdots, m)$, i.e. the integers in the ith row being greater than or equal to β_i and less than or equal to α_i, we take (b_i+i, β_i) and (a_i+i, α_i) $(i=1, \cdots, m)$ as endpoints. Similarly, given column flags (δ_i, γ_i) $(i=1, \cdots, n)$, i.e. the integers in the ith column (numbered from right to left) being greater than or equal to δ_i and less than or equal to γ_i, we take $(n+m+1-i-b'_i, \delta_i)$ and $(n+m+1-i-a'_i, \gamma_i)$ $(i=1, \cdots, n)$ as endpoints. Then we easily see that to a skew tableau with given row and column flags correspond the dual pair of paths that are obtained by the same procedure as in the proof of the equality of the right-hand sides of (1) and (2).

In the case of a one row partition with column flags, where $m=1, a_1=n$, $b_1=0, a'_i=1$ $(i=1, \cdots, n)$, $b'_i=0$ $(i=1, \cdots, n)$, we obtain the expression

$$\det(e_{1+i-j}(\delta_i, \gamma_j; u))_{1\le i,j\le n} \tag{6}$$

for the generating polynomial; $e_d(f, g; u)$ is the dth elementary symmetric polynomial in $u_f, u_{f+1}, \cdots, u_g$. In particular (6) with specialization $u_i := q^i$ $(i \in \mathbf{N})$ gives a determinant for q-counting restricted lattice paths [7, p. 136]:

$$\det\left(q^{(1+i-j)(i-j+2\delta_i)/2}\cdot\begin{bmatrix}\gamma_j-\delta_i+1\\ 1+i-j\end{bmatrix}_q\right)_{1\le i,j\le n}$$

where $[\cdots]_q$ denotes the q-binomial coefficient.

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