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On an Asymptotic Property of a Nonlinear Ordinary Differential Equation

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§1. Introduction.

In the study of the fifth Painlevé equation we treated an equation of the form

$$x(xu')' = \frac{\alpha}{2}\tanh u \cosh^{-2} u + \frac{\gamma}{4} x \sinh 2u + \frac{\delta}{8} x^2 \sinh 4u \tag{1}$$

($' = d/dx$), where $\alpha, \gamma \in \boldsymbol{R}$, $\delta < 0$. In [4] we studied an asymptotic behaviour of the solution $u = u_0(x) = u(x_0, u_0, u'_0; x)$ ($x_0 > 0$, $u_0, u'_0 \in \boldsymbol{R}$) as $x \to +\infty$ satisfying an initial condition

$$u_0(x_0) = u_0\,, \qquad u'_0(x_0) = u'_0\,. \tag{2}$$

In this paper we consider a more general nonlinear equation of the form

$$v'' + v\Phi(x, v) = 0\,. \tag{3}$$

Under some assumptions we prove that the solution $v = V(x)$ satisfying an initial condition as above can be prolonged over the interval $x_0 \leq x < +\infty$, and we give an asymptotic expression of $V(x)$ as $x \to +\infty$. Analogous problems are studied in [1], [2] and [3].

§2. Main result.

Let r and ε be positive constants. Consider an equation of the form

$$u'' + u(1 + x^{-1}p(u) + x^{-1-\varepsilon}f(x, u)) = 0 \tag{4}$$

satisfying the following conditions.

(A) *$p(u)$ is a polynomial of degree $2n$ (≥ 0)*

$$p(u) = \lambda_0 + \lambda_1 u + \cdots + \lambda_{2n} u^{2n}$$

where

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$$\lambda_0 \in \boldsymbol{R} \qquad\qquad if \quad n=0\,,$$
$$\lambda_0, \cdots, \lambda_{2n-1} \in \boldsymbol{R}\,, \quad \lambda_{2n}>0 \qquad if \quad n\geq 1\,.$$

(B) *$f(x, u)$ and $(\partial/\partial x)f(x, u)$ are real-valued continuous functions in the domain*

$$D(r)=\{(x, u)\in \boldsymbol{R}^2\,;\; x>r,\; -\infty<u<+\infty\}$$

and have the properties:

(B1) *For every positive constant r', there exists a positive constant $L(r')$ such that*

$$|f(x, u)|\leq L(r')\,, \qquad \left|\frac{\partial}{\partial x}f(x, u)\right|\leq L(r')$$

for $x>r$, $|u|<r'$.

(B2) *If we put*

$$F(x, u)=2\int_0^u vf(x, v)dv\,, \tag{5}$$

$$G(x, u)=2\int_0^u v\frac{\partial}{\partial x}f(x, v)dv\,, \tag{6}$$

then there exist real constants α_1, α'_1 and β_1 such that

$$\inf_{|s|\leq|u|} F(x, s)\geq -\alpha_1 u^2-\beta_1\,, \tag{7}$$

$$\inf_{|s|\leq|u|} (-G(x, s))\geq -\alpha'_1 u^2-\beta_1 \tag{8}$$

for $x>r$, $-\infty<u<+\infty$.

By condition (A), if we put

$$P(u)=2\int_0^u vp(v)dv\,, \tag{9}$$

then there exist real constants α_2 and β_2 such that

$$\inf_{|s|\leq|u|} P(s)\geq -\alpha_2 u^2-\beta_2 \tag{10}$$

for $-\infty<u<+\infty$.

Let u_0 and u'_0 be real constants and let x_0 be a positive constant satisfying $x_0>r$. We denote by $u=U(x)$ a solution of equation (4) satisfying

$$U(x_0)=u_0\,, \qquad U'(x_0)=u'_0\,. \tag{11}$$

Then we have

THEOREM 1. *Assume that* $x_0 > r$ *and that*

$$\alpha_2 x_0^{-1} + \alpha_1 x_0^{-1-\varepsilon} + (1/2)(|\alpha'_1| + \alpha'_1)\varepsilon^{-1} x_0^{-\varepsilon} < 1 . \tag{12}$$

Then the solution $U(x)$ *can be prolonged over* $x_0 \leq x < +\infty$, *and satisfies*

$$U(x) = R_0(\rho, x)\cos(x + \Lambda(\rho)\log x + \Theta_0(\theta, x)) , \tag{13}$$

$$U'(x) = -R_1(\rho, x)\sin(x + \Lambda(\rho)\log x + \Theta_1(\theta, x)) , \tag{14}$$

$$R_i(\rho, x) = \rho + O(x^{-m(\varepsilon)}) , \quad \Theta_i(\theta, x) = \theta + O(x^{-m(\varepsilon)}) \qquad (i = 0, 1)$$

as $x \to +\infty$. *Here* $\rho = \rho(x_0, u_0, u'_0)$ *and* $\theta = \theta(x_0, u_0, u'_0)$ *are integral constants satisfying* $\rho \geq 0$, $0 \leq \theta < 2\pi$, *and* $m(\varepsilon)$ *and* $\Lambda(\rho)$ *are constants defined by*

$$m(\varepsilon) = \min\{1, \varepsilon\} \quad (>0) , \tag{15}$$

$$\Lambda(\rho) = \sum_{j=0}^{n} \lambda_{2j} \binom{2j+1}{j} 2^{-2j-1} \rho^{2j} \quad (\in \boldsymbol{R}) . \tag{16}$$

§3. Properties of $U(x)$.

This and the next sections are devoted to the proof of Theorem 1. We start from the following.

PROPOSITION 2. *If the solution* $U(x)$ *can be prolonged over the interval* $x_0 \leq x < +\infty$, *then* $|U(x)|$ *is bounded for* $x_0 \leq x < +\infty$.

PROOF. For $x \geq x_0$,

$$U''(x) + U(x) + x^{-1}U(x)p(U(x)) + x^{-1-\varepsilon}U(x)f(x, U(x)) = 0 . \tag{17}$$

Multiplying (17) by $2U'(x)$ and integrating between y and x, we have

$$\begin{aligned} &U'(x)^2 - U'(y)^2 \\ &= -U(x)^2 - x^{-1}P(U(x)) - x^{-1-\varepsilon}F(x, U(x)) \\ &\quad + U(y)^2 + y^{-1}P(U(y)) + y^{-1-\varepsilon}F(y, U(y)) \\ &\quad - \int_y^x t^{-2}P(U(t))dt - (1+\varepsilon)\int_y^x t^{-2-\varepsilon}F(t, U(t))dt \\ &\quad + \int_y^x t^{-1-\varepsilon}G(t, U(t))dt \end{aligned} \tag{18}$$

for $x \geq y \geq x_0$ (cf. (5), (6) and (9)). Suppose that $|U(x)|$ is not bounded for $x \geq x_0$. Then there exists a sequence $\{x_n\}$ such that

$$(x_0 <)\ x_1 < \cdots < x_n < \cdots , \qquad x_n \to +\infty ; \tag{19}$$

$$|U(x_n)|\to+\infty\,; \tag{20}$$

$$|U(x)|\le|U(x_n)| \qquad \text{for} \quad x_0\le x\le x_n\,. \tag{21}$$

By (21), condition (B2) and (10), we have $P(U(t))\ge -\alpha_2 U(x_n)^2-\beta_2$, $F(t, U(t))\ge -\alpha_1 U(x_n)^2-\beta_1$, $-G(t, U(t))\ge -\alpha'_1 U(x_n)^2-\beta_1$ for $x_0\le t\le x_n$. Put $x=x_n$ $(>y)$ in (18) and observe that the integrands satisfy

$$\begin{aligned}&t^{-2}P(U(t))+(1+\varepsilon)t^{-2-\varepsilon}F(t, U(t))-t^{-1-\varepsilon}G(t, U(t))\\ &\ge -U(x_n)^2(\alpha_2 t^{-2}+\alpha_1(1+\varepsilon)t^{-2-\varepsilon}+\alpha'_1 t^{-1-\varepsilon})\\ &\quad -\beta(t^{-2}+(1+\varepsilon)t^{-2-\varepsilon}+t^{-1-\varepsilon})\end{aligned}$$

($\beta=\max\{\beta_1, \beta_2\}$) for $y\le t\le x_n$. We have

$$\begin{aligned}&U'(x_n)^2+U(x_n)^2(1-\alpha_2 y^{-1}-\alpha_1 y^{-1-\varepsilon}-\alpha'_1\varepsilon^{-1}(y^{-\varepsilon}-x_n^{-\varepsilon}))\\ &\le U'(y)^2+U(y)^2+y^{-1}P(U(y))+y^{-1-\varepsilon}F(y, U(y))\\ &\quad +|\beta|(y^{-1}+y^{-1-\varepsilon}+\varepsilon^{-1}y^{-\varepsilon})\end{aligned} \tag{22}$$

for $x_0\le y\le x_n$. If we take $y=y_0$ so large that

$$1-\alpha_2 y_0^{-1}-\alpha_1 y_0^{-1-\varepsilon}-|\alpha'_1|\varepsilon^{-1}y_0^{-\varepsilon}>1/2\,,$$

then

$$U'(x_n)^2+(1/2)U(x_n)^2\le C(y_0)$$

for n satisfying $x_n\ge y_0$, where $C(y_0)$ is a positive constant depending on y_0. This contradicts (20). Thus the boundedness of $U(x)$ is proved.

We put $y=x_0$ in (18). Then each term in (18) is evaluated as follows. By Proposition 2 and condition (B1), we have

$$P(U(x))=O(1)\,, \quad F(x, U(x))=O(1)\,, \quad G(x, U(x))=O(1)$$

for $x\ge x_0$, and hence

$$\int_{x_0}^{x} t^{-1-\varepsilon}G(t, U(t))dt=\int_{x_0}^{+\infty} t^{-1-\varepsilon}G(t, U(t))dt+O(x^{-\varepsilon})$$

and so on. Thus we have

PROPOSITION 3. *If the solution $U(x)$ can be prolonged over the interval $x_0\le x<+\infty$, then*

$$U'(x)^2+U(x)^2=c+O(x^{-m(\varepsilon)}) \tag{23}$$

for $x_0\le x<+\infty$, where $m(\varepsilon)$ is a positive constant given by (15) *and c is some nonnegative* *constant depending on* x_0, u_0 *and* u'_0.

Furthermore we have

PROPOSITION 4. *If* x_0 $(>r)$ *satisfies* (12), *then the solution* $U(x)$ *can be prolonged over* $x_0 \le x < +\infty$.

PROOF. Suppose that the prolongation of $U(x)$ is possible for $x_0 \le x < T$ $(< +\infty)$ but impossible for $x \ge T$. There exists a sequence $\{\xi_n\}$ such that

$$x_0 < \xi_1 < \cdots < \xi_n < \cdots < T, \quad \xi_n \to T; \tag{24}$$

$$|U(\xi_n)| + |U'(\xi_n)| \to +\infty. \tag{25}$$

Then the sequence $\{|U(\xi_n)|\}$ is unbounded as $\xi_n \to T$. In fact, if we suppose the boundedness of $\{|U(\xi_n)|\}$, then, by (18) (with $y = x_0$), the sequence $\{|U'(\xi_n)|\}$ is also bounded as $\xi_n \to T$, which contradicts (25). Hence there exists a subsequence $\{s_N = \xi_{n(N)}\,;\ N = 1, 2, \cdots\}$ such that

$$x_0 < s_1 < \cdots < s_N < \cdots < T, \qquad s_N \to T; \tag{26}$$

$$|U(s_N)| \to +\infty; \tag{27}$$

$$|U(x)| \le |U(s_N)| \qquad \text{for} \quad x_0 \le x \le s_N. \tag{28}$$

Putting $x = s_N$, $y = x_0$ in (18), we have

$$\begin{aligned} &U'(s_N)^2 + U(s_N)^2(1 - \alpha_2 x_0^{-1} - \alpha_1 x_0^{-1-\varepsilon} - \alpha'_1 \varepsilon^{-1}(x_0^{-\varepsilon} - s_N^{-\varepsilon})) \\ &\le u_0'^2 + u_0^2 + x_0^{-1} P(u_0) + x_0^{-1-\varepsilon} F(x_0, u_0) \\ &\quad + \beta(x_0^{-1} + x_0^{-1-\varepsilon} + \varepsilon^{-1} x_0^{-\varepsilon}) \end{aligned}$$

by the same argument as in the proof of Proposition 2. If x_0 satisfies (12), then $1 - \alpha_2 x_0^{-1} - \alpha_1 x_0^{-1-\varepsilon} - \alpha'_1 \varepsilon^{-1}(x_0^{-\varepsilon} - s_N^{-\varepsilon}) > \eta_0 > 0$ for some positive constant η_0. Hence

$$U'(s_N)^2 + \eta_0 U(s_N)^2 \le C_0$$

for $N \ge 1$, where C_0 is some positive constant. This contradicts (27). Therefore $U(x)$ is prolonged over $x_0 \le x < +\infty$.

§4. Asymptotic behaviour (Completion of the proof of Theorem 1).

The remainder part of the proof of Theorem 1 is divided into three steps.

4.1. By Propositions 3 and 4, the solution $u = U(x)$ can be prolonged over the interval $x_0 \le x < +\infty$ and satisfies

$$U(x)^2 + U'(x)^2 = c + O(x^{-m(\varepsilon)}) \tag{23}$$

on the same interval. Since $U(x)$ satisfies equation (4), using condition (B1), we have

$$U''(x)+(1+x^{-1}p(U(x)))U(x)=O(x^{-1-\varepsilon}) \tag{29}$$

for $x \geq x_0$. We put $x=it$ $(i=\sqrt{-1})$. Then $x \geq x_0$ implies $|t| \geq x_0$, $t \in i\boldsymbol{R}^-$, where $\boldsymbol{R}^-=\{s; s<0\}$.

PROPOSITION 5. *There exists a* 2×2 *matrix* $T(t)$ *such that*

i) $T(t)=I+O(t^{-1})$ *for* $|t| \geq x_0$, $t \in i\boldsymbol{R}^-$;

ii) *the functions* $y(t)$, $z(t)$ *defined by*

$$\begin{pmatrix} y(t) \\ z(t) \end{pmatrix} = T(t)\begin{pmatrix} U(it)+iU'(it) \\ U(it)-iU'(it) \end{pmatrix} \tag{30}$$

satisfy

$$\dot{y}(t)=(1-\gamma it^{-1})y(t)-it^{-1}(\gamma'+Q(y(t), z(t)))+O(t^{-1-m(\varepsilon)}), \tag{31}$$

$$\dot{z}(t)=-(1-\gamma it^{-1})z(t)+it^{-1}(\gamma'+Q(y(t), z(t)))+O(t^{-1-m(\varepsilon)}) \tag{32}$$

$(\dot{}=d/dt)$ *for* $|t| \geq x_0$, $t \in i\boldsymbol{R}^-$, *where*

$$\gamma=\gamma(c)=\sum_{j=0}^{n} \lambda_{2j}\binom{2j+1}{j}2^{-2j-1}c^j \in \boldsymbol{R}, \tag{33}$$

$$\gamma'=\gamma'(c)=\sum_{j=1}^{n} \lambda_{2j-1}\binom{2j}{j}2^{-2j}c^j \in \boldsymbol{R} \tag{34}$$

and $Q(y, z)$ *is a polynomial in* (y, z) *expressed as*

$$Q(y, z)=\sum_{k=2}^{2n+1} q_k(c)(y^k+z^k)\,. \tag{35}$$

PROOF. We put

$$x=it\,, \qquad 2U(it)=Y(t)+Z(t)\,, \qquad 2iU'(it)=Y(t)-Z(t)\,.$$

By (23), the functions $|Y(t)|$ and $|Z(t)|$ are bounded for $|t| \geq x_0$, $t \in i\boldsymbol{R}^-$. Then relation (23) becomes

$$U'(x)^2+U(x)^2=Y(t)Z(t)=c+O(t^{-m(\varepsilon)})\,. \tag{36}$$

Note that, by (36),

$$p\left(\frac{1}{2}(Y(t)+Z(t))\right)\cdot\frac{1}{2}(Y(t)+Z(t))=\sum_{k=0}^{2n} \lambda_k 2^{-k-1}(Y(t)+Z(t))^{k+1}$$

$$=\gamma(Y(t)+Z(t))+\gamma'+Q(Y(t), Z(t))+O(t^{-m(\varepsilon)})\,, \tag{37}$$

where γ and γ' are the constants given by (33) and (34), and $Q(y, z)$ is a polynomial satisfying (35). Using (37), we derive from (29) that

$$\begin{pmatrix}\dot{Y}(t)\\ \dot{Z}(t)\end{pmatrix}=M(t)\begin{pmatrix}Y(t)\\ Z(t)\end{pmatrix}-\begin{pmatrix} it^{-1}(\gamma'+Q(Y(t),Z(t)))+O(t^{-1-m(\varepsilon)})\\ -it^{-1}(\gamma'+Q(Y(t),Z(t)))+O(t^{-1-m(\varepsilon)})\end{pmatrix}, \tag{38}$$

where

$$M(t)=\begin{pmatrix}1-\gamma it^{-1} & -\gamma it^{-1}\\ \gamma it^{-1} & -(1-\gamma it^{-1})\end{pmatrix}.$$

The eigenvalues $\pm\chi(t)$ of $M(t)$ satisfy $\chi(t)=1-\gamma it^{-1}+O(t^{-2})$ as $|t|\to+\infty$. Choosing a 2×2 matrix $T(t)=I+O(t^{-1})$ so that $T(t)M(t)T(t)^{-1}=\mathrm{diag}[\chi(t),-\chi(t)]$ and putting ${}^{t}(y(t),z(t))=T(t){}^{t}(Y(t),Z(t))$, we have relations (31) and (32). Thus the proposition is proved.

4.2. For $t\in i\boldsymbol{R}^{-}$, we put

$$\Gamma(t)=\{\tau=is\,;\ -\infty<s\le -|t|\}.$$

PROPOSITION 6. *If $N=0$ or $N\ge 2$, then*

$$\int_{\Gamma(t)} e^{-\tau}\tau^{\gamma i-1}y(\tau)^{N}d\tau=O(t^{-1}),\qquad \int_{\Gamma(t)} e^{-\tau}\tau^{\gamma i-1}z(\tau)^{N}d\tau=O(t^{-1}),$$

$$\int_{\Gamma(t)} e^{\tau}\tau^{-\gamma i-1}y(\tau)^{N}d\tau=O(t^{-1}),\qquad \int_{\Gamma(t)} e^{\tau}\tau^{-\gamma i-1}z(\tau)^{N}d\tau=O(t^{-1})$$

for $|t|\ge x_0$, $t\in i\boldsymbol{R}^{-}$.

PROOF. If $N\ge 2$, then, by (31) and the boundedness of $y(\tau)$,

$$y(\tau)^{N}=y(\tau)^{N-1}\dot{y}(\tau)+O(\tau^{-1})$$

for $|\tau|\ge x_0$, $\tau\in i\boldsymbol{R}^{-}$. Using this relation and integrating by parts, we have

$$\int_{\Gamma(t)} e^{-\tau}\tau^{\gamma i-1}y(\tau)^{N}d\tau=\int_{\Gamma(t)} e^{-\tau}\tau^{\gamma i-1}\frac{d}{d\tau}\left(\frac{y(\tau)^{N}}{N}\right)d\tau+\int_{\Gamma(t)} O(\tau^{-2})d\tau$$

$$=\frac{1}{N}\int_{\Gamma(t)} e^{-\tau}\tau^{\gamma i-1}y(\tau)^{N}d\tau+O(t^{-1})$$

for $|t|\ge x_0$, $t\in i\boldsymbol{R}^{-}$, which implies the first estimate for $N\ge 2$. When $N=0$, we have

$$\int_{\Gamma(t)} e^{-\tau}\tau^{\gamma i-1}d\tau=\int_{\Gamma(t)}\left(-\frac{d}{d\tau}e^{-\tau}\right)\tau^{\gamma i-1}d\tau$$

$$=(\gamma i-1)\int_{\Gamma(t)} e^{-\tau}\tau^{\gamma i-2}d\tau+O(t^{-1})=O(t^{-1})$$

for $|t|\geq x_0$, $t\in i\boldsymbol{R}^-$. Other estimates can be obtained in a similar way.

4.3. Multiplying (31) by $e^{-t}t^{\gamma i}$, we have

$$\frac{d}{dt}(e^{-t}t^{\gamma i}y(t))=g(t)=-ie^{-t}t^{\gamma i-1}(\gamma'+Q(y(t),z(t)))+O(t^{-1-m(\varepsilon)}).$$

Integrating between $t_0=-x_0 i$ and t ($\in i\boldsymbol{R}^-$), and using Propositions 5 and 6, we have

$$\begin{aligned}e^{-t}t^{\gamma i}y(t)-c_1'&=\int_{t_0}^{t}g(\tau)d\tau=\int_{\Gamma(t_0)}g(\tau)d\tau-\int_{\Gamma(t)}g(\tau)d\tau\\&=c_1''+O(t^{-m(\varepsilon)}),\end{aligned}$$

namely

$$y(t)=(c_1+O(t^{-m(\varepsilon)}))e^{t}t^{-\gamma i} \tag{39}$$

for $|t|\geq x_0$, $t\in i\boldsymbol{R}^-$, where c_1', c_1'' and c_1 are some complex constants. Similarly

$$z(t)=(c_2+O(t^{-m(\varepsilon)}))e^{-t}t^{\gamma i} \tag{40}$$

for $|t|\geq x_0$, $t\in i\boldsymbol{R}^-$, where c_2 is some complex constant. Furthermore, by (36) and (30),

$$Y(t)Z(t)=y(t)z(t)+O(t^{-1})=c+O(t^{-m(\varepsilon)}),$$

which implies

$$c_1c_2=c\geq 0. \tag{41}$$

From (39), (40) and (30), it follows that

$$\begin{aligned}U(x)=U(it)&=\frac{1}{2}((1+O(t^{-1}))y(t)+(1+O(t^{-1}))z(t))\\&=C_1(x)\cos(x+\gamma\log x)-C_2(x)\sin(x+\gamma\log x)\end{aligned}$$

for $x\geq x_0$, where

$$C_1(x)=\tilde{c}_1+\tilde{c}_2+O(x^{-m(\varepsilon)}),\qquad C_2(x)=i(\tilde{c}_1-\tilde{c}_2)+O(x^{-m(\varepsilon)}),$$
$$\tilde{c}_1=(c_1/2)\exp(-\gamma i\log(-i)),\qquad \tilde{c}_2=(c_2/2)\exp(\gamma i\log(-i)).$$

Since $U(x)$ is a real-valued function for $x\geq x_0$,

$$U(x)=\operatorname{Re}U(x)=(\operatorname{Re}C_1(x))\cos(x+\gamma\log x)-(\operatorname{Re}C_2(x))\sin(x+\gamma\log x), \tag{42}$$

$$\operatorname{Im}U(x)\equiv 0 \tag{43}$$

for $x\geq x_0$. It follows from (43) that

$$\operatorname{Re}\tilde{c}_1=\operatorname{Re}\tilde{c}_2=a,\qquad \operatorname{Im}\tilde{c}_1=-\operatorname{Im}\tilde{c}_2=b.$$

Note that $\tilde{c}_1\tilde{c}_2=a^2+b^2=c_1c_2/4=c/4$, and put $\rho=c^{1/2}$ (≥ 0). Then, from (42), we can derive

$$U(x)=R_0(\rho,x)\cos(x+\Lambda(\rho)\log x+\Theta_0(\theta,x))$$

with

$$\begin{aligned}R_0(\rho,x)&=((\mathrm{Re}\,C_1(x))^2+(\mathrm{Re}\,C_2(x))^2)^{1/2}\\&=2(a^2+b^2+O(x^{-m(\varepsilon)}))^{1/2}=\rho+O(x^{-m(\varepsilon)})\,,\\\Theta_0(\theta,x)&=\theta+O(x^{-m(\varepsilon)})\end{aligned}$$

for $x\geq x_0$, where θ is a real constant satisfying $0\leq\theta<2\pi$, and $\Lambda(\rho)=\gamma(\rho^2)$ (cf. (33)). The asymptotic expression of $U'(x)$ can be obtained in a similar way. Thus the proof of Theorem 1 is completed.

§5. Example I.

We give an example to which Theorem 1 is applicable. Consider an equation of the form

$$u''+u(1+x^{-1}u^2-x^{-2}u^3\sin(u^3))=0\,. \tag{44}$$

Using our theorem we can show the existence of an oscillatory solution. If we put $\varepsilon=1$, $p(u)=u^2$, then conditions (A), (B) and (B1) are satisfied for every $r>0$. Since

$$\begin{aligned}F(x,u)&=2\int_0^u(-v^4\sin(v^3))dv\\&=\frac{2}{3}u^2\cos(u^3)-\frac{4}{3}\int_0^u v\cos(v^3)dv\\&\geq\frac{2}{3}(\cos(u^3)-1)u^2\geq-\frac{4}{3}u^2\end{aligned}$$

for $u\in\boldsymbol{R}$, equation (44) satisfies condition (B2) with $\alpha_1=4/3$, $\alpha'_1=0$, $\beta_1=0$. Inequality (10) is valid for $\alpha_2=\beta_2=0$. Therefore, by Theorem 1, if $x_0>(4/3)^{1/2}$, then the solution $u=U(x)$ of equation (44) satisfying (11) is expressible in the form

$$U(x)=(\rho+O(x^{-1}))\cos\left(x+\frac{3}{8}\rho^2\log x+\theta+O(x^{-1})\right)$$

as $x\to+\infty$, where $\rho=\rho(x_0,u_0,u'_0)$ and $\theta=\theta(x_0,u_0,u'_0)$ are integral constants satisfying $\rho\geq 0$, $0\leq\theta<2\pi$.

§6. Example II.

Consider an equation of the form

$$u''+x^{-1}u'+u(g_0(u)+x^{-1}g_1(u)+x^{-1-\varepsilon}g_2(x, u))=0\,. \tag{45}$$

In order to apply Theorem 1 we impose the following conditions corresponding to (A), (B), (B1) and (B2) respectively.

(C) *$g_0(u)$ and $g_1(u)$ are expressed as*

$$g_0(u)=1+\lambda u^2+u^{2(1+\varepsilon)}h_0(u)\,, \tag{46}$$

$$g_1(u)=\mu+u^{2\varepsilon}h_1(u)\,, \tag{47}$$

where λ is a nonnegative constant, μ is a real constant and $h_0(u)$ and $h_1(u)$ are real-valued continuous functions for $-\infty<u<+\infty$.

(D) *The function*

$$h(x, v)=x^{m(\varepsilon)-\varepsilon}(v^{2(1+\varepsilon)}h_0(x^{-1/2}v)+v^{2\varepsilon}h_1(x^{-1/2}v)+g_2(x, x^{-1/2}v))+(1/4)x^{m(\varepsilon)-1}$$

($m(\varepsilon)=\min\{1, \varepsilon\}$) and its derivative $(\partial/\partial x)h(x, v)$ are real-valued continuous functions in $D(r)$ ($\in(x, v)$) with the properties:

(D1) *For every positive constant r', there exists a positive constant $L'(r')$ such that*

$$|h(x, v)|\le L'(r')\,, \qquad \left|\frac{\partial}{\partial x}h(x, v)\right|\le L'(r')$$

for $x>r$, $|v|<r'$;

(D2) *If we put*

$$H(x, v)=2\int_0^v wh(x, w)dw\,, \tag{48}$$

$$K(x, v)=2\int_0^v w\frac{\partial}{\partial x}h(x, w)dw\,, \tag{49}$$

then there exist real constants α_3, α'_3 and β_3 such that

$$\inf_{|s|\le|v|} H(x, s)\ge-\alpha_3 v^2-\beta_3\,, \tag{50}$$

$$\inf_{|s|\le|v|} (-K(x, s))\ge-\alpha'_3 v^2-\beta_3 \tag{51}$$

for $x>r$, $-\infty<v<+\infty$.

By $u=V(x)$ we denote a solution of equation (45) satisfying

$$V(x_0)=v_0\,,\qquad V'(x_0)=v'_0\,, \tag{52}$$

where x_0 ($>r$), v_0 and v'_0 are arbitrary real constants. Then we have

PROPOSITION 7. *Assume that $x_0>r$ and that*

$$\alpha_3 x_0^{-1-m(\varepsilon)}+(1/2)(|\alpha'_3|+\alpha'_3)m(\varepsilon)^{-1}x_0^{-m(\varepsilon)}<1\,.$$

Then the function $V(x)$ can be prolonged over $x_0\le x<+\infty$ and satisfies

$$V(x)=S_0(\rho, x)x^{-1/2}\cos(x+\kappa(\rho)\log x+\phi_0(\theta, x))\,, \tag{53}$$

$$V'(x)=-S_1(\rho, x)x^{-1/2}\sin(x+\kappa(\rho)\log x+\phi_1(\theta, x))\,, \tag{54}$$

$$S_i(\rho, x)=\rho+O(x^{-m(\varepsilon)})\,,\quad \phi_i(\theta, x)=\theta+O(x^{-m(\varepsilon)})\qquad (i=0, 1)$$

as $x\to+\infty$. Here $\rho=\rho(x_0, v_0, v'_0)$ and $\theta=\theta(x_0, v_0, v'_0)$ are integral constants satisfying $\rho\ge 0$, $0\le\theta<2\pi$, and $\kappa(\rho)$ is a constant defined by

$$\kappa(\rho)=\frac{\mu}{2}+\frac{3}{8}\lambda\rho^2\,.$$

PROOF. If we put $u=x^{-1/2}v$, equation (45) becomes

$$v''+vH(x, v)=0\,,$$

where

$$H(x, v)=g_0(x^{-1/2}v)+x^{-1}g_1(x^{-1/2}v)+x^{-1-\varepsilon}g_2(x, x^{-1/2}v)+(1/4)x^{-2}\,.$$

By condition (C), $H(x, v)$ is written in the form

$$H(x, v)=1+x^{-1}(\lambda v^2+\mu)+x^{-1-m(\varepsilon)}h(x, v)$$

with

$$h(x, v)=x^{m(\varepsilon)-\varepsilon}(v^{2(1+\varepsilon)}h_0(x^{-1/2}v)+v^{2\varepsilon}h_1(x^{-1/2}v)+g_2(x, x^{-1/2}v))+(1/4)x^{m(\varepsilon)-1}\,.$$

Since λ is a nonnegative constant, using condition (D), we can easily verify that the polynomial $\lambda v^2+\mu$ and the function $h(x, v)$ satisfy conditions (A), (B), (B1) and (B2). Furthermore the constant α_2 in (10) can be taken to be 0. Therefore the conclusion follows from Theorem 1.

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