TOKYO J. MATH.
VOL. 15, NO. 2, 1992

Symplectic Manifolds with Semi-Free Hamiltonian S^1-Action

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(Communicated by T. Nagano)

1. Introduction

In a previous paper [H1] we have shown that a connected closed symplectic S^1-manifold with the properties (i), (ii) and (iii) listed below must be simply connected and have the same homology groups as $S^2\times\cdots\times S^2$.

(i) The action admits a moment map.
(ii) The fixed points are all isolated.
(iii) The action is semi-free.

In this paper we shall prove the following

THEOREM 1.1. *Let M be a connected closed symplectic S^1-manifold satisfying the conditions* (i), (ii) *and* (iii) *listed above. Then M has the same cohomology ring and the same Chern classes as $S^2\times\cdots\times S^2$.*

Semi-free actions are the simplest and, in a sense, the basic type among S^1-actions. For example, if M is any S^1-manifold and if $\boldsymbol{Z}/m$ is a maximal finite isotropy subgroup then each component of the fixed point set of the restricted $\boldsymbol{Z}/m$-action is an invariant symplectic S^1-submanifold on which the S^1-action, made effective, is free or semi-free. Thus our result above may be considered as the first step to investigate general symplectic S^1-manifolds admitting moment map.

The main idea of proof can be stated as follows. The critical points of the moment map are precisely the fixed points of the action. Let $\Sigma_1,\cdots,\Sigma_n$ be a suitably chosen homology basis of $H_2(M)$ corresponding to the fixed points of index 2 and let $x_1,\cdots,x_n$ be the dual basis of $H^2(M)$. Let $\xi=\xi(h_1,\cdots,h_n)$ be the complex line bundle over M with $c_1(\xi)=x=h_1x_1+\cdots+h_nx_n$ where $h_i\in\boldsymbol{Z}$. The S^1-action on M can be lifted to an action on ξ and defines a weight $a_i=a_i(h_1,\cdots,h_n)$ at each fixed point P_i. These weights a_i satisfy certain relations coming from a fixed point formula. Also they are related to $x^n[M]$. Using these two facts and the linearity of x and a_i with respect to $h_1,\cdots,h_n$ we determine all the values $x_{i_1}x_{i_2}\cdots x_{i_n}[M]$ which, in turn,

Received June 25, 1991

are enough to determine the ring $H^*(M)$ in view of the Poincaré duality.

The choice of homology basis $\Sigma_1, \cdots, \Sigma_n$ is made so that each Σ_i, as a cycle, contains the unique fixed point P^0 of index 0. If P_i^1 is the fixed point of index 2 contained in Σ_i then the weight a_i^1 corresponding to P_i^1 becomes equal to h_i with this choice.

The cycles $\Sigma_1, \cdots, \Sigma_n$ are realized as the closures of orbits of a C^*-action extending the given S^1-action. Such a C^*-action can be obtained by using a pair of S^1-invariant almost complex structure and S^1-invariant Riemannian metric on M compatible with the given symplectic structure ω. The homotopy class of such pairs is uniquely determined by that of ω. Hence the Chern classes of M are well defined.

Finally the equivariant tangent bundle is determined in the localized equivariant $K_{S^1}(M)$ using the localization theorem and the Chern classes of M are identified.

In Section 2 we discuss the relations among moment maps, C^*-orbits and complex line bundles. The main results will be proved in Section 3 and Section 4.

2. Moment map, C^*-orbits and complex line bundles.

Throughout this section M will denote a connected closed symplectic manifold of dimension $2n$ admitting a Hamiltonian S^1-action. This means an action of S^1 preserving the given symplectic form ω endowed with a C^∞ map $\mu\colon M \to \boldsymbol{R}$ such that

$$d\mu = i(X)\omega$$

where X is the vector field generating the action of $e^{it} \in S^1$.

We take a pair of invariant almost complex structure J and invariant Riemannian metric $\langle\ ,\ \rangle$ such that

$$\omega(JY, JZ) = \omega(Y, Z) \quad \text{and} \quad \omega(Y, JZ) = \langle Y, Z\rangle$$

for any tangent vectors Y and Z. The existence of such pairs is well known. Such a pair will be called a compatible pair. It can be shown [H1] that the gradient vector field $\operatorname{grad}\mu$ with respect to $\langle\ ,\ \rangle$ coincides with JX and the $\boldsymbol{R}_+$-action $\psi_r = \operatorname{Exp}(-(\log r)JX)$ commutes with the given S^1-action. Hence an action of C^* on M is defined by

$$re^{i\theta}\cdot x = e^{i\theta}\psi_r(x)\,.$$

It is easy to see that a fixed point P is a critical point of μ and vice versa. The function μ is a non-degenerate function in the sense of Bott and the index of P is equal to twice the number of positive weights around P, cf. e.g. [At] or [AH]. Let P be a fixed point. Then its unstable manifold P^u is defined to be the set

$$\{x \in M\,;\ \lim_{z\to 0} z\cdot x = P,\ z \in C^*\}\,.$$

If the index of μ at P is equal to 2 then the closure $\overline{P^u}$ of P^u coincides with the closure of the C^*-orbit of a point near P. In general, the closure $M(p)$ of a C^*-orbit C^*p is a topologically embedded 2-sphere in M and the set $M(p)$ minus C^*p consists of two fixed points; one is $n(p)=\lim_{z\to 0} z(p)$ and the other is $s(p)=\lim_{z\to\infty} z(p)$, the former being called the north pole and the latter the south pole. If $M(p)$ coincides with the closure of the unstable manifold P^u then $n(p)$ coincides with P.

Note that the compatible pairs are not unique but their equivariant homotopy class is uniquely determined by the symplectic form ω. The class depends even only on the equivariant homotopy class of ω. The statement of Theorem 1.1 is eventually the one which depends only on the equivariant homotopy class of the symplectic structure. Therefore, in order to prove Theorem 1.1, we may take a suitable symplectic form ω in the given equivariant homotopy class and an associated compatible pair J and $\langle\ ,\ \rangle$. If we perturb symplectic structures and compatible pairs then the corresponding stable and unstable manifolds vary. We shall see a particular case of such variations in a moment. Recall that if M satisfies the additional conditions (ii) and (iii) then there are exactly $\binom{n}{q}$ fixed points of index $2q$ by [H1]. In particular we have n fixed points of index 2. Also there is a unique fixed point of index 0.

Let P be a fixed point such that $0<\text{index at } P<2n$ and suppose that there are fixed points $P_1, \cdots, P_k$ of index 2 such that the closures of their unstable manifolds all have P as their south poles with respect to the given symplectic form and the compatible pair. Under this situation we shall show the following

LEMMA 2.1. *There is a small change of symplectic forms and compatible pairs such that the closures of the unstable manifolds of $P_1, \cdots, P_k$ get apart from P after that change.*

PROOF. Take a point q_i near P lying on the unstable manifold of each P_i. Thus the closure of the unstable manifold of P_i coincides with $M(q_i)$. It is sufficient to show that, after a small perturbation of symplectic forms and compatible pairs near P, we can arrange so that $M'(q_i)$ (the closure of the C^*-orbit of q_i after the perturbation) does not pass through P but some $M'(q_i')$ does for each $i=1, \cdots, k$ where q_i' is sufficiently close to q_i.

We shall proceed by induction on k. Thus we may assume that we have already chosen points $q_2', \cdots, q_k'$ where q_i' is sufficiently close to q_i such that $M'(q_i')$ passes through P but $M'(q_i)$ does not for $2\le i\le k$ after a small perturbation.

With a further perturbation and by virtue of the equivariant Darboux theorem [We] we may assume that there exists a complex coordinate system $(z_1, \cdots, z_\kappa, w_1, \cdots, w_\lambda)$ ($2\lambda=\text{index at } P$, $\kappa+\lambda=n$, $0<\kappa, \lambda$) such that the S^1-action is of the form

$$z(z_1, \cdots, z_\kappa, w_1, \cdots, w_\lambda)=(z^{-1}z_1, \cdots, z^{-1}z_\kappa, zw_1, \cdots, zw_\lambda),$$

the unstable manifold P_i^u of P_i passes through $e_i=(0, \cdots, 0, 1, 0, \cdots, 0)$ (1 at the i-th place) for $i=1, \cdots, \kappa$ and the symplectic form ω is the standard one:

$$\omega=\sum_{j=1}^{\kappa} dx_j \wedge dy_j+\sum_{j=1}^{\lambda} du_j \wedge dv_j$$

where $z_j=x_j+\sqrt{-1}\,y_j$ and $w_j=u_j+\sqrt{-1}\,v_j$. Here we may assume that $q_1=e_1$ and $q_i'=e_i$ for $i\geq 2$.

The vector field X generating the above action is

$$X=\sum_{j=1}^{\kappa}\left(y_j\frac{\partial}{\partial x_j}-x_j\frac{\partial}{\partial y_j}\right)+\sum_{j=1}^{\lambda}\left(-v_j\frac{\partial}{\partial u_j}+u_j\frac{\partial}{\partial v_j}\right). \tag{2.2}$$

Let ρ be a C^∞ function defined on $[0, \infty)$ such that $0\leq\rho\leq 1$ and $\rho(r^2)=1$ for $r^2\leq 1/4$ and $\rho(r^2)=0$ for $r^2\geq 1/2$.

At a point $(x_1, y_1, \cdots, x_\kappa, y_\kappa, u_1, v_1, \cdots, u_\lambda, v_\lambda)$ we define a linear transformation A by the matrix

$$A=\frac{1}{\sqrt{1-\alpha^2}}D(\varphi)^{-1}BD(\varphi)$$

where B is given by

$$B=\left(\begin{array}{c|c} E_{2\kappa} & \begin{matrix}\alpha & 0 & \\ 0 & -\alpha & \\ \vdots & \vdots & 0 \\ \alpha & 0 & \\ 0 & -\alpha & \end{matrix} \\ \hline \begin{matrix}\alpha & 0 & \cdots & \alpha & 0 \\ 0 & -\alpha & \cdots & 0 & -\alpha \\ & & 0 & & \end{matrix} & E_{2\lambda} \end{array}\right)$$

where $z_1=x_1+iy_1=re^{-i\varphi}$, $\alpha=tr^2\rho(r^2)$ with $0\leq t\leq 1$, E_m is the identity matrix of degree m and $D(\varphi)$ is the orthogonal matrix corresponding to the given action of $e^{i\varphi}$ on $(z_1, \cdots, z_\kappa, w_1, \cdots, w_\lambda)$. Here we are regarding linear transformations as acting on the row vectors from the right by matrix multiplication.

Then we define a new almost complex structure J_t' at $(x_1, y_1, \cdots, x_\kappa, y_\kappa, u_1, v_1, \cdots, u_\lambda, v_\lambda)$ by

$$J_t'=AJA^{-1}$$

where J is the complex multiplication by $\sqrt{-1}$ on $(z_1, \cdots, z_\kappa, w_1, \cdots, w_\lambda)$. It can be easily seen that J_t' is S^1-invariant and compatible with ω, i.e. $\omega(J_t'Y, J_t'Z)=\omega(Y, Z)$

for any vectors Y and Z. Explicitly J_t' is given by the matrix

$$\frac{1}{1-\alpha^2}D(\varphi)^{-1}CD(\varphi)$$

where C is given by

$$\left(\begin{array}{cccccc|cccc}
0 & 1+\alpha^2 & & & & & 0 & 2\alpha & & \\
-(1+\alpha^2) & 0 & & & 0 & & 2\alpha & 0 & & \\
 & & \ddots & & & & \vdots & \vdots & & 0 \\
 & 0 & & & 0 & 1+\alpha^2 & 0 & 2\alpha & & \\
 & & & & -(1+\alpha^2) & 0 & 2\alpha & 0 & & \\
\hline
0 & 2\alpha & \cdots & & 0 & 2\alpha & 0 & 1+\alpha^2 & & 0 \\
2\alpha & 0 & \cdots & & 2\alpha & 0 & -(1+\alpha^2) & 0 & & \\
 & & & & & & & & \ddots & \\
 & & & 0 & & & & 0 & 0 & 1+\alpha^2 \\
 & & & & & & & & -(1+\alpha^2) & 0
\end{array}\right).$$

The gradient vector field $\mathrm{grad}_t\,\mu$ with respect to the Riemannian metric $\langle\ ,\ \rangle_t$ defined by $\langle Y, Z\rangle_t=\omega(Y, J_t'Z)$ coincides with $J_t'X$ (cf. [H1]). From this together with (2.2) it follows that

$$\begin{aligned}
(2.3)\qquad -\mathrm{grad}_t\,\mu_{(x_1,0,x_2,0,\cdots,x_\kappa,0,u_1,0,\cdots,0)}=\frac{1}{1-\alpha^2}\Big\{&-((1+\alpha^2)x_1+2\alpha u_1)\frac{\partial}{\partial x_1}\\
&-((1+\alpha^2)x_2+2\alpha u_1)\frac{\partial}{\partial x_2}-\cdots\\
&-((1+\alpha^2)x_\kappa+2\alpha u_1)\frac{\partial}{\partial x_\kappa}\\
&+\left(2\alpha\sum_{j=1}^{\kappa}x_j+(1+\alpha^2)u_1\right)\frac{\partial}{\partial u_1}\Big\}.
\end{aligned}$$

Here we recall $\alpha=t|z_1|^2\rho(|z_1|^2)$. Therefore the trajectory of $-\mathrm{grad}_t\,\mu=-J_t'X$ passing through $e_1=q_1$ tends to $+\infty$ along the u_1-axis for $t>0$. This shows that the unstable manifold of P_1 is not adherent to $P=(0,\cdots,0)$. On the other hand the trajectory passing through $e_j=q_j'$ with $2\leq j\leq\kappa$ tends to P via the x_j-axis. This means that the closure $M_t(q_j')$ of the orbit of q_j' with respect to the C^* action corresponding to J_t' passes through P. Moreover $M_t(q_j)$ which is different from but close to $M_t(q_j')$ does not pass through P for $2\leq j\leq k$.

On the other hand it follows from (2.3) that there exists a point q_1' near q_1 such that $M_t(q'_1)$ passes through P. If we take t small enough then q'_1 can be arbitrarily close to q_1. Therefore $M_t(q'_1)$ is different from $M_t(q'_j)$ and also from $M_t(q_j)$ for $2 \leq j \leq k$ for sufficiently small t. This means that $M_t(q_1), M_t(q_2), \cdots, M_t(q_k)$ cannot pass through P. $\square$

We apply Lemma 2.1 to the fixed points P in the downward inductive order with respect to the values $\mu(P)$ and arrive to the following

PROPOSITION 2.4. *Let M be a connected closed symplectic S^1-manifold satisfying the conditions* (i), (ii) *and* (iii). *Then, with a suitable choice of symplectic forms and compatible pairs within the given equivariant homotopy class, we can arrange so that the closures of the unstable manifolds of all the fixed points of index 2 have the fixed point P^0 of index* 0 *as their south poles.*

As mentioned in Section 1 the symplectic manifold M we are considering has $\binom{n}{q}$ fixed points of index $2q$. We shall label them as P_I^q where $I=(i_1, \cdots, i_q)$ ranges over all sequences of $1 \leq i_1 < \cdots < i_q \leq n$. We shall abbreviate $P_\emptyset^0$ simply as P^0.

We choose a symplectic form and associated compatible pair as stated in Proposition 2.4 so that the closure of the unstable manifold of P_i^1, denoted by Σ_i, has P^0 as its south pole for each $i=1, \cdots, n$. The oriented 2-spheres $\Sigma_1, \cdots, \Sigma_n$ give a homology basis of $H_2(M; \boldsymbol{Z})$. Let $x_1, \cdots, x_n$ be the dual basis of $H^2(M; \boldsymbol{Z})$.

For a sequence $h=(h_1, \cdots, h_n) \in \boldsymbol{Z}^n$ let $\xi(h)$ be the complex line bundle over M such that

$$c_1(\xi(h)) = x(h) = h_1 x_1 + \cdots + h_n x_n .$$

Since M is simply connected as mentioned in Section 1 the S^1-action on M can be lifted to the action on $\xi = \xi(h)$ and the lifting is uniquely determined by the S^1-module structure of ξ restricted to a specified fixed point [HY]. Set $e_j = (0, \cdots, 0, 1, 0, \cdots, 0)$ (1 at the j-th place) and we shall determine the lifting of action on $\xi(e_i)$ by requiring $\xi(e_i)|P^0$ to be the trivial S^1-module. Then the lifting to the general $\xi(h)$ is also determined since we have

$$\xi(h) = \xi(e_1)^{h_1} \cdots \xi(e_n)^{h_n} .$$

We set

$$t^{a_I^q(h)} = \xi(h)|P_I^q$$

where t denotes the standard S^1-module. It is easy to see that

$$a_I^q(h) = \sum_{j=1}^{n} h_j a_I^q(e_j) \tag{2.5}$$

and

$$a^0(h) = 0 \tag{2.6}$$

for all h ($a^0 = a^0_{\varnothing}$). Moreover we have

$$a_i^1(h) = h_i \,. \tag{2.7}$$

In fact, since $c_1(\xi(e_j)) = x_j$, $\xi(e_j)|\Sigma_i$ is trivial for $i \neq j$ and is equal to the standard line bundle over the oriented 2-sphere Σ_j for $i = j$. Since we have normalized so that $\xi(e_j)|P^0 = 1$ we have

$$\xi(e_j)|P_i^1 = t^{\delta_{ij}} \,.$$

Hence $a_i^1(e_j) = \delta_{ij}$ and $a_i^1(h) = \sum_j h_j \delta_{ij} = h_i$.

We now recall an integrality theorem of Atiyah-Segal [AS], see also [H2]. Let, in general, M be an almost complex S^1-manifold with only isolated fixed points $P_0, P_1, \cdots, P_{\chi-1}$. Here χ is the Euler number of M. Let $\{m_{ik}\}_{k=1}^n$ be the weights around P_i. Moreover Let ξ be a complex line bundle with a lifted S^1-action. Then, setting

$$t^{a_i} = \xi | P_i$$

we have

LEMMA 2.8. *The sum*

$$\varphi(t) = \sum_i \frac{1 - t^{a_i - a_0}}{\prod_k (1 - t^{m_{ik}})}$$

must be a Laurent polynomial in t.

We apply Lemma 2.8 in our present situation with $P_0 = P^0$ and $\xi = \xi(h)$. In particular $a_0 = 0$ by (2.6). At $P_i = P_I^q$ the weights are $+1$ with multiplicity q and -1 with multiplicity $n-q$. Therefore

$$\prod (1 - t^{m_{ik}}) = (1-t)^q (1 - t^{-1})^{n-q} = (1-t)^n (1 - t^{-1})^{n-q} \,.$$

Hence we finally get

PROPOSITION 2.9. *In the present situation*

$$\varphi(t) = \frac{1}{(1-t)^n} \sum_{\nu=0}^{n-1} (-1)^\nu t^\nu \sum_{i_1 < \cdots < i_{n-\nu}} (1 - t^{a^{n-\nu}_{i_1 \cdots i_{n-\nu}}(h)})$$

must be a Laurent polynomial in t.

3. Proof of Theorem 1.1. First part.

We continue with the situation of Section 2. First we shall deduce the following theorem from Proposition 2.9.

THEOREM 3.1. *With a suitable indexing of the fixed points $\{P_I^q\}$ we have*

$$a_I^q(h)=h_I=h_{i_1}+\cdots+h_{i_q}$$

where $I=(i_1, \cdots, i_q)$, $i_1<\cdots<i_q$.

PROOF. We set

$$X^{q,k}=\sum_I (a_I^q(h))^k .$$

$X^{q,k}$ is a homogeneous polynomial of degree k in $h_1, \cdots, h_n$ by (2.5). From (2.7) it follows

$$X^{1,k}=h_1^k+\cdots+h_n^k . \tag{3.2}$$

We also set

$$Q^{q,k}(h_1, \cdots, h_n)=\sum_I h_I^k .$$

We shall show the following assertions.

ASSERTION 1.

$$X^{q,1}=Q^{q,1} \qquad \textit{for all} \quad q .$$

ASSERTION 2. *Given* q, *the assumptions*

$$X^{q,k}=Q^{q,k} \qquad \textit{for} \quad 1\leq k\leq n-q+1$$

imply

$$a_I^q(h)=h_I \qquad \textit{for all} \quad k$$

and hence also

$$X^{q,k}=Q^{q,k} \qquad \textit{for all} \quad k .$$

ASSERTION 3. *Given* q, *the assumptions*

$$a_I^r(h)=h_I \qquad \textit{for all} \quad r>q \textit{ and } I$$

imply

$$X^{s,n-q+1}=Q^{s,n-q+1} \qquad \textit{for} \quad s\leq q .$$

If we admit these assertions for a moment then the descending induction with respect to q starting from $q=n$ and also using (2.7) or (3.2) yields Theorem 3.1 as can be seen easily.

We set

$$f(t)=\sum_{\nu=0}^{n-1}(-1)^\nu t^\nu \sum_J (1-t^{a_J^{n-\nu}(h)}) .$$

Then, by Proposition 2.9, we must have

$$f'(1)=0\,,\quad f''(1)=0\,,\quad \cdots,\quad f^{(n-1)}(1)=0\,. \tag{3.3}$$

But

$$f^{(i)}(1)=-i!\sum_{\nu=0}^{n-1}(-1)^{\nu}\sum_{j=0}^{i}\sum_{J}\binom{a_J^{n-\nu}(h)}{j}\binom{\nu}{i-j}\,. \tag{3.4}$$

On the other hand $\binom{a}{j}$ is a polynomial of degree j in a. If we write it as

$$\binom{a}{j}=m_j^j a^j+\cdots+m_1^j a \tag{3.5}$$

then we have

$$\sum_J\binom{a_J^{n-\nu}(h)}{j}=m_j^j X^{n-\nu,j}+\cdots+m_1^j X^{n-\nu,1}\,.$$

Hence the homogeneous part of degree k in (3.4) is

$$-i!\sum_{\nu=0}^{n-1}(-1)^{\nu}\left\{m_k^i\binom{\nu}{0}+m_k^{i-1}\binom{\nu}{1}+\cdots+m_k^k\binom{\nu}{i-k}\right\}X^{n-\nu,k}\,. \tag{3.6}$$

Now (3.3) holds for any sequence $h=(h_1,\cdots,h_n)$ of integers. Therefore, if we put (3.4) and (3.6) in (3.3), then each homogeneous part must vanish independently. Thus

$$\sum_{\nu=0}^{n-1}(-1)^{\nu}\left\{m_k^i\binom{\nu}{0}+m_k^{i-1}\binom{\nu}{1}+\cdots+m_k^k\binom{\nu}{i-k}\right\}X^{n-\nu,k}=0 \tag{3.7}$$

for $k=1,\cdots,n-1$ and $i=k,k+1,\cdots,n-1$.

At this point we remark that the polynomials $Q^{q,k}$ also satisfy the same equations (3.7). This could be checked by calculation. But a more conceivable way is to consider the standard S^1-action on the product $S^2\times\cdots\times S^2$. In this case we have $a_I^q(h)=h_I$ and, consequently, $X^{q,k}=Q^{q,k}$.

Fixing k, we shall view (3.7) as linear equations for the unknowns $X^{n,k},\cdots,X^{1,k}$. Since $m_k^j\neq 0$ the equations (3.7) are equivalent to the equations

$$\sum_{\nu=0}^{n-1}(-1)^{\nu}\binom{\nu}{l}X^{n-\nu,k}=0\,,\qquad l=0,1,\cdots,n-1-k\,, \tag{3.8}$$

where $1\leq k\leq n-1$.

We first consider the case $k=1$. From (3.8) it follows that $X^{q,1}$ must be a multiple of $X^{1,1}$ for all q. But $X^{1,1}=h_1+\cdots+h_n=Q^{1,1}$ by (3.2). Hence we must have

$$X^{q,1}=Q^{q,1}=\binom{n-1}{q-1}(h_1+\cdots+h_n)\,.$$

This proves Assertion 1.

The proof of Assertion 3 is similar. Namely suppose that $a_I^r(h)=h_I$ and hence $X^{r,k}=Q^{r,k}$ for all $r>q$ and k. We consider $X^{q,n-q+1}, \cdots, X^{2,n-q+1}$ as the unknowns and $X^{n,n-q+1}, \cdots, X^{q+1,n-q+1}$ and $X^{1,n-q+1}$ as the knowns in (3.8) with $k=n-q+1$. The coefficients matrix for the unknowns is

$$A=\begin{pmatrix} \binom{n-q}{0} & \binom{n-q+1}{0} & \cdots & \binom{n-2}{0} \\ \binom{n-q}{1} & \binom{n-q+1}{1} & \cdots & \binom{n-2}{1} \\ & & \ddots & \\ \binom{n-q}{q-2} & \binom{n-q+1}{q-2} & \cdots & \binom{n-2}{q-2} \end{pmatrix}.$$

However $\det A=1$ as is well known and can be easily proved. Hence $X^{q,n-q+1}, \cdots, X^{2,n-q+1}$ are uniquely determined. As $X^{r,n-q+1}=Q^{r,n-q+1}$ for $r>q$ and $r=1$ we have

$$X^{s,n-q+1}=Q^{s,n-q+1} \qquad \text{for} \quad 2\leq s\leq q\,.$$

This proves Assertion 3.

Finally we shall prove Assertion 2. Expanding $Q^{q,k}$ we have

$$\begin{aligned} Q^{q,k}&=\sum_{j_1<\cdots<j_q}(h_{j_1}+\cdots+h_{j_q})^k \\ &=\binom{n-1}{q-1}\sum_j h_j^k+k!\binom{n-k}{q-k}\sum_{j_1<\cdots<j_k} h_{j_1}\cdots h_{j_k}+\text{remaining terms}\,. \end{aligned}$$

On the other hand

$$\begin{aligned} X^{q,k}&=\sum_I(h_1a_I^q(e_1)+\cdots+h_na_I^q(e_n))^k \\ &=\sum_j\left(\sum_I a_I^q(e_j)^k\right)h_j^k+k!\sum_{j_1<\cdots<j_k}\left(\sum_I a_I^q(e_{j_1})\cdots a_I^q(e_{j_k})\right)h_{j_1}\cdots h_{j_k}+\text{remaining terms}\,. \end{aligned}$$

Comparing the coefficients of h_i^k and $h_{j_1}\cdots h_{j_k}$ in the equalities

$$X^{q,1}=Q^{q,1}, \quad \cdots, \quad X^{q,n-q+1}=Q^{q,n-q+1}$$

we see that

$$\text{(3.9)} \qquad \sum_I a_I^q(e_j)^k=\binom{n-k}{q-1} \qquad \text{for} \quad 1\leq k\leq n-q+1$$

and

$$\sum_I a_I^q(e_{j_1})\cdots a_I^q(e_{j_k})=\binom{n-k}{q-k} \quad \text{for} \quad 2\leq k\leq n-q+1\,. \tag{3.10}$$

Then, from (3.9) with $k=1$ and $k=2$, we deduce easily that $a_I^q(e_i)$ is equal to 1 or 0 and that

$$\#\{I\,;\, a_I^q(e_i)=1\}=\binom{n-1}{q-1} \tag{3.11}$$

for each i, where $\#$ denotes the number of the elements.

Assertion 2 readily follows from the following claim. Hence the proof of Theorem 3.1 is reduced to that of the claim.

CLAIM. *The $a_I^q(e_i)$ which are equal to* 1 *or* 0 *and satisfy* (3.10) *and* (3.11) *must be of the form*

$$a_I^q(e_i)=\begin{cases}1 & \text{if} \quad i\in I\\ 0 & \text{if} \quad i\notin I\end{cases}$$

with a suitable indexing of $\{I\}$.

PROOF. We proceed by induction on n. We see that

$$\#\{I\,;\, a_I^q(e_n)=0\}=\binom{n}{q}-\binom{n-1}{q-1}=\binom{n-1}{q}$$

by (3.11). We decompose the set $\{I\}$ of indices into two parts $\{I^0\}$ and $\{I^1\}$ where

$$a_{I^0}^q(e_n)=0 \quad \text{and} \quad a_{I^1}^q(e_n)=1\,.$$

Then we see, for each $i<n$, that

$$\#\{I^0\,;\, a_{I^0}^q(e_i)=1\}=\binom{n-2}{q-1}=\binom{n-1-1}{q-1}. \tag{3.11$'$}$$

In fact, using (3.11) and (3.10) for $k=2$, we have

$$\begin{aligned}\#\{I^0\,;\, a_{I^0}^q(e_i)=1\}&=\#\{I\,;\, a_I^q(e_i)=1\}-\#\{I\,;\, a_I^q(e_i)=a_I^q(e_n)=1\}\\ &=\binom{n-1}{q-1}-\sum_I a_I^q(e_i)a_I^q(e_n)\\ &=\binom{n-1}{q-1}-\binom{n-2}{q-2}\\ &=\binom{n-2}{q-1}.\end{aligned}$$

Similarly we have

$$\sum_{I^0} a^q_{I^0}(e_{j_1})\cdots a^q_{I^0}(e_{j_k})=\binom{n-1-k}{q-k} \tag{3.10$'$}$$

for $2\leq k\leq n-1-q+1$ where $j_1<\cdots<j_k\leq n-1$. In fact

$$\begin{aligned}\sum_{I^0} a^q_{I^0}(e_{j_1})\cdots a^q_{I^0}(e_{j_k}) &= \sum_{I} a^q_I(e_{j_1})\cdots a^q_I(e_{j_k})-\sum_{I} a^q_I(e_{j_1})\cdots a^q_I(e_{j_k})a^q_I(e_n)\\ &=\binom{n-k}{q-k}-\binom{n-(k+1)}{q-(k+1)}\\ &=\binom{n-1-k}{q-k},\end{aligned}$$

for $2\leq k\leq n-q$.

Therefore, by inductive assumption, we see that

$$a^q_{I^0}(e_i)=\begin{cases}1 & \text{if } i\in I^0\\ 0 & \text{if } i\notin I^0\end{cases}$$

where $I^0=(i_1,\cdots,i_q)$ with $1\leq i_1<\cdots<i_q\leq n-1$. From the very definition of $\{I^0\}$ we have

$$a^q_{I^0}(e_n)=0\ .$$

Thus Claim is proved for I^0.

Similar calculations show that

$$\sum_{I^1} a^q_{I^1}(e_{j_1})\cdots a^q_{I^1}(e_{j_k})=\binom{n-1-k}{q-1-k} \tag{3.10$''$}$$

for $2\leq k\leq n-1-(q-1)+1$, $1\leq i_1<\cdots<i_{q-1}\leq n-1$ and

$$\#\{I^1\ ;\ a^q_{I^1}(e_i)=1\}=\binom{n-1-1}{q-1-1} \tag{3.11$''$}$$

for $1\leq i\leq n-1$. Hence, if we regard I^1 as a sequence $(i_1,\cdots,i_{q-1})$ with $1\leq i_1<\cdots<i_{q-1}\leq n-1$, we also see that

$$a^q_{I^1}(e_i)=\begin{cases}1 & \text{if } i\in I^1\\ 0 & \text{if } i\notin I^1\end{cases}$$

with a suitable indexing of $\{I^1\}$ where $1\leq i\leq n-1$. From the definition of $\{I^1\}$ we have

$$a^q_{I^1}(e_n)=1$$

where we regard I^1 as a sequence $(i_1,\cdots,i_{q-1},n)$. Thus Claim also holds for I^1.

This completes the proof of Claim and hence of Theorem 3.1. $\square$

PROPOSITION 3.12. *Let $x=x(h)=\sum h_i x_i$ as in Section 2. Then we have*

$$x^n[M]=n!h_1\cdots h_n .$$

PROOF. Let ξ be the complex line bundle over m with $c_1(\xi)=x(h)$ endowed with a lifted S^1-action such that $\xi|P^0=1$ (trivial S^1-module). Consider the element u in $K_{S^1}(M)$ defined by

$$u=\prod_{i=1}^{n}(1-\xi^{-1}t^{a_i^1(h)})$$

and set

$$\varphi(t)=p_!(u)\in K_{S^1}(\mathrm{pt})=R(S^1)=\boldsymbol{Z}[t, t^{-1}] ,$$

where $p_!$ is the Gysin homomorphism of the projection $p: M\to\mathrm{pt}$. Then it can be shown that $\varphi(1)=x^n[M]$, cf. e.g. [H2]. On the other hand we have

$$\varphi(1)=\sum_{q\neq 1}\sum_{I}\frac{\prod_{i=1}^{n}(1-t^{a_i^1(h)-a_I^q(h)})}{(1-t^{-1})^q(1-t)^{n-q}} .$$

This shows that $\varphi(1)$ depends only on $\{a_I^q(h)\}$.

At our present situation the $\{a_I^q(h)\}$ are the same as the standard S^1-action on $S^2\times\cdots\times S^2$. Therefore $x^n[M]=\varphi(1)$ must coincide with the standard one. Since $x_i^2=0$ and $x_1\cdots x_n$ is the positive generator of $H^{2n}(S^2\times\cdots\times S^2)$ we have

$$\begin{aligned}(h_1x_1+\cdots+h_nx_n)^n[S^2\times\cdots\times S^2]&=n!h_1\cdots h_nx_1\cdots x_n[S^2\times\cdots\times S^2]\\&=n!h_1\cdots h_n .\end{aligned}$$

Hence we have $x^n[M]=n!h_1\cdots h_n$ in the present situation. $\square$

COROLLARY 3.13. *We have*

$$H^*(M ; \boldsymbol{Z})\cong\boldsymbol{Z}[x_1, \cdots, x_n]/(x_1^2, \cdots, x_n^2) .$$

In particular M has the same cohomology ring as $S^2\times\cdots\times S^2$.

PROOF. By Proposition 3.12 we have

$$(h_1x_1+\cdots+h_nx_n)^n[M]=n!h_1\cdots h_n$$

for any sequence $(h_1, \cdots, h_n)\in\boldsymbol{Z}^n$. It follows that

$$x_{i_1}\cdots x_{i_n}[M]=\begin{cases}1 & \text{if } i_1, \cdots, i_n \text{ are mutually distinct} ,\\ 0 & \text{otherwise} .\end{cases}\tag{3.14}$$

From (3.14) and the Poincaré duality it follows easily that the $x_I = x_{i_1} \cdots x_{i_q}$ where $I = (i_1, \cdots, i_q)$ with $i_1 < \cdots < i_q$ form a basis of $H^{2q}(M; Z)$ which is known to be a free module of rank $\binom{n}{q}$ by [H1]. Then we also have

$$x_i^2 = 0$$

since $x_i^2 x_I = 0$ for all the basis elements x_I of $H^{2(n-2)}(M; Z)$. □

Corollary 3.13 is nothing but the first part of Theorem 1.1.

4. Proof of Theorem 1.1. Second part.

We use the same notations as in Sections 2 and 3. We shall prove the following theorem which is clearly a statement equivalent to the second part of Theorem 1.1.

THEOREM 4.1. *The Chern class of M is given by*

$$c(TM) = \prod_{i=1}^{n} (1 + 2x_i)$$

where TM is the complex tangent bundle of M.

PROOF. Since $H^*(M; Z)$ has no torsion it is sufficient to determine $ch(TM)$ which should be equal to

$$ch(TM) = \sum_{i=1}^{n} e^{2x_i}. \tag{4.2}$$

We consider the following natural commutative diagram:

$$\begin{array}{ccccc}
S^{-1}K_{S^1}(M) & \xleftarrow{\varphi_K} & K_{S^1}(M) & \xrightarrow{F} & K(M) \\
\downarrow ch & & \downarrow ch & & \downarrow ch \\
S^{-1}H_{S^1}^{**}(M; Q) & \xleftarrow{\varphi_H} & H_{S^1}^{**}(M; Q) & \longrightarrow & H^*(M; Q)
\end{array}$$

where S^{-1} denotes the localization.

Let $T_{S^1}M$ denote the equivariant complex tangent bundle. We need two lemmas.

LEMMA 4.3.

$$\varphi_K(T_{S^1}M) = \varphi_K(\sum \xi_i^2 t^{-1})$$

where $\xi_i = \xi(e_i)$ is the equivariant complex line bundle as defined in Section 2.

LEMMA 4.4. *The natural homomorphism φ_H is injective.*

Before proving these lemmas we shall show that they in fact imply (4.2). From Lemma 4.3 and Lemma 4.4 we see that

$$ch(T_{S^1}M)=ch(\sum \xi_i^2 t^{-1}) .$$

Hence we get

$$ch(TM)=ch(\sum \xi_i^2)=\sum e^{2x_i} .$$

It remains to prove Lemmas 4.3 and 4.4.

By the localization theorem the $S^{-1}K_{S^1}(\mathrm{pt})$-module $S^{-1}K_{S^1}(M)$ can be identified with

$$\sum_{P_I^q} S^{-1}K_{S^1}(P_I^q) .$$

Then $\varphi_K(T_{S^1}M)$ is given by

$$\begin{aligned}\varphi_K(T_{S^1}M)&=\sum_{P_I^q} T_{S^1}M|P_I^q\\ &=\sum_{P_I^q}(qt+(n-q)t^{-1})\end{aligned}$$

which is equal to $\varphi_K(\sum_{i=1}^n \xi_i^2 t^{-1})$ since

$$\xi_i^2 t^{-1}|P_I^q=\begin{cases} t^2t^{-1}=t & \text{if } i\in I\\ t^{-1} & \text{if } i\notin I\end{cases}$$

by Theorem 3.1. This proves Lemma 4.3.

As to Lemma 4.4 we consider the Gysin sequence of the S^1 fibering $ES^1\times M\to M_{S^1}$ where $ES^1\to BS^1$ is the universal S^1 bundle. Using the fact that $H^{\mathrm{odd}}(M)=0$ we see easily that the Gysin sequence is split, i.e.

$$0\longrightarrow H^q(M_{S^1})\xrightarrow{\ \cup e\ } H^{q+2}(M_{S^1})\longrightarrow H^{q+2}(M)\longrightarrow 0$$

is exact where

$$e\in H_{S^1}^2(\mathrm{pt})=H^2(BS^1)\,, \qquad H_{S^1}^{**}(\mathrm{pt})=Q[e] .$$

This means that $H_{S^1}^{**}(M)=H^{**}(M_{S^1})$ is $H_{S^1}^{**}(\mathrm{pt})$-torsion free and consequently that φ_H is injective.

This completes the proof of Theorem 4.1. $\square$

REMARK. *Actually it can be shown that*

$$T_{S^1}M=\sum \xi_i^2 t^{-1}$$

in $K_{S^1}(M)$ strengthening Lemma 4.3.

5. Concluding remarks.

Let m be as above. Using the fact that M is simply connected and has the same cohomology ring as $S^2 \times \cdots \times S^2$ it can be shown that M has the same homotopy type as $S^2 \times \cdots \times S^2$.

In dimension 4 it is known that M is diffeomorphic to $S^2 \times S^2$, cf. [Au], [AH].

In general M has the same Pontrjagin classes as $S^2 \times \cdots \times S^2$ since they have the same Chern classes. In particular, in dimension 6, theorem of Wall [Wa] implies that M is diffeomorphic to $S^2 \times S^2 \times S^2$.

References

[AS] M. F. Atiyah and G. B. Segal, The index of elliptic operators: II, Ann. of Math., **87** (1968), 531–545.

[At] M. F. Atiyah, Convexity and commuting Hamiltonians, Bull. London Math. Soc., **14** (1982), 1–15.

[Au] M. Audin, Hamiltoniens périodiques sur les variétés symplectiques compactes de dimension 4, Lecture Notes in Math., **1416** (1990), 1–25, Springer.

[AH] K. Ahara and A. Hattori, 4-dimensional symplectic S^1 manifolds admitting moment map, J. Fac. Sci. Univ. Tokyo, **38** (1991), 251–298.

[H1] A. Hattori, Circle actions on symplectic manifolds, Lecture Notes in Math., **1375** (1985), 89–97, Springer.

[H2] A. Hattori, S^1-actions on unitary manifolds and quasi-ample line bundles, J. Fac. Sci. Univ. Tokyo, **31** (1985), 433–486.

[HY] A. Hattori and T. Yoshida, Lifting compact group actions into fiber bundles, Japan. J. Math., **2** (1976), 13–25.

[Wa] C. T. C. Wall, Classification problems in differential topology. V, On certain 6-manifolds, Invent. Math., **1** (1966), 355–374.

[We] A. Weinstein, *Lectures on Symplectic Manifolds*, Regional Conf. Ser. in Math., **29** (1977).

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