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Logarithmic Projective Connections

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In this note, we consider logarithmic version of the results in [K3]. Let X be a complex manifold of dimension $n\geq 2$ and D a reduced effective divisor on X with only normal crossing singularities. We define logarithmic projective Weyl forms $\bar{P}_k$, $0\leq k\leq n$, a kind of characteristic forms of the pair (X, D), by means of a C^∞-logarithmic projective connection and prove the following formula

$$\bar{c}_k(\bar{\theta})=\sum_{j=0}^{k}\binom{n+1-j}{k-j}((n+1)^{-1}\bar{c}_1(\bar{\theta}))^{k-j}\bar{P}_j(\bar{\pi}), \qquad 0\leq k\leq n,$$

where the $\bar{c}_k(\bar{\theta})$ are the logarithmic Chern forms defined by a suitable C^∞-logarithmic affine connection $\bar{\theta}$ (Theorem 3.1). If X is compact, Kähler and admits a holomorphic logarithmic projective connection, then the logarithmic projective Weyl forms are d-exact. Hence, in this case, our formula gives the formula on the logarithmic Chern classes

$$\bar{c}_k=\binom{n+1}{k}((n+1)^{-1}\bar{c}_1)^k, \qquad 1\leq k\leq n.$$

The latter is the logarithmic version of Gunning's formula [G], see also [K3].

In the last section, we shall reprove a formula on Chern classes of certain compact non-Kähler 3-folds which were constructed in [K1] as an application of our main result.

Notation.

Ω^p : the sheaf of germs of holomorphic p-forms on a complex manifold,

$\Omega^p(\log D)$: the sheaf of germs of logarithmic p-forms along a divisor D on a complex manifold,

$\mathcal{O}\simeq\Omega^0$: the sheaf of germs of holomorphic functions on a complex manifold,

Θ : the sheaf of germs of holomorphic vector fields on a complex manifold,

$\Theta(-\log D)$: the sheaf of germs of logarithmic vector fields along a divisor D on a complex manifold,

$\mathscr{A}^r(E)$: the sheaf of germs of differentiable r-form-valued sections of a vector

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bundle E on a manifold,

$\mathscr{A}^{p,q}(E)$: the sheaf of germs of differentiable (p, q)-form-valued sections of a vector bundle E on a manifold,

T : the sheaf of germs of differentiable vector fields on a manifold,

$\mathscr{F}^p = \Omega^p(\mathrm{End}(\Theta))$,

$\mathscr{F}^p\langle D\rangle = \Omega^p(\log D)(\mathrm{End}(\Theta(-\log D)))$,

$\mathscr{G}^{p,q} = \mathscr{A}^{p,q}(\mathrm{End}(\Theta))$,

$\mathscr{G}^r = \bigoplus_{p+q=r}\mathscr{G}^{p,q}$,

$\mathscr{G}^{p,q}\langle D\rangle = \mathscr{A}^{p,q}(\log D)(\mathrm{End}(\Theta(-\log D)))$,

$\mathscr{G}^r\langle D\rangle = \bigoplus_{p+q=r}\mathscr{G}^{p,q}\langle D\rangle$.

1. Preliminaries.

Let X be a complex manifold of dimension $n \geq 1$. Let D be a reduced effective divisor on X with only normal crossing singularities. We call (X, D) a *logarithmic pair*. Take a locally finite open covering $\mathscr{U} = \{U_\alpha\}$ of X such that on each U_α there defined a system of local coordinates $z_\alpha = (z_\alpha^1, z_\alpha^2, \cdots, z_\alpha^n)$ such that

$$U_\alpha \cap D = \{z_\alpha^{i_1} z_\alpha^{i_2} \cdots z_\alpha^{i_k} = 0\}\,, \qquad 1 \leq k \leq n\,,$$

where $i_1, i_2, \cdots, i_k$ are distinct integers satisfying $1 \leq i_\nu \leq n$. Such a local coordinate system is said to be a *logarithmic coordinate system along* D on U_α, [I, page 321]. Put

$$\varphi_{\alpha\beta} = z_\alpha \circ z_\beta^{-1}\,.$$

The transition function of the tangent bundle Θ is given by

$$\tau_{\alpha\beta} = \text{the Jacobian matrix of } \varphi_{\alpha\beta} \qquad \text{on } U_\alpha \cap U_\beta\,.$$

Put $w_\alpha^k = (z_\alpha^k)^{e_k}$, where

$$e_k = \begin{cases} 1 & \text{if } D \supset (\{z_\alpha^k = 0\} \cap U_\alpha) \\ 0 & \text{if } D \not\supset (\{z_\alpha^k = 0\} \cap U_\alpha)\,. \end{cases}$$

On each U_α, we consider logarithmic 1-forms

$$\omega_\alpha^j = \frac{dz_\alpha^j}{w_\alpha^j} \qquad j = 1, 2, \cdots, n\,, \tag{1}$$

and logarithmic vector fields

$$\theta_\alpha^j = w_\alpha^j \frac{\partial}{\partial z_\alpha^j} \qquad j = 1, 2, \cdots, n\,. \tag{2}$$

Let τ_α be an $n \times n$ matrix defined by

$$\tau_\alpha = \begin{pmatrix} w_\alpha^1 & & & 0 \\ & w_\alpha^2 & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & w_\alpha^n \end{pmatrix}. \tag{3}$$

Then, on $U_\alpha \cap U_\beta$, the transition function of the logarithmic tangent bundle $\Theta(-\log D)$ with respect to logarithmic coordinates on U_α and U_β is given by

$$\bar{\tau}_{\alpha\beta} = \tau_\alpha^{-1} \cdot \tau_{\alpha\beta} \cdot \tau_\beta , \tag{4}$$

and that of the logarithmic cotangent bundle $\Omega^1(\log D)$ is given by

$$\bar{\tau}^*_{\alpha\beta} = \tau_\alpha \cdot \tau^*_{\alpha\beta} \cdot \tau_\beta^{-1} , \qquad \tau^*_{\alpha\beta} = {}^t\tau_{\alpha\beta}^{-1} . \tag{5}$$

Here we note that each τ_α may be singular on U_α, but the product $\bar{\tau}_{\alpha\beta}$ is non-singular on $U_\alpha \cap U_\beta$.

Define an $n \times n$ matrix-valued holomorphic 1-form by

$$a_{\alpha\beta} = \tau_{\alpha\beta}^{-1} d\tau_{\alpha\beta} \tag{6}$$

and a scalar-valued holomorphic 1-form

$$\sigma_{\alpha\beta} = (n+1)^{-1} d \log(\det \tau_{\alpha\beta}) = (n+1)^{-1} \operatorname{Trace}(\tau_{\alpha\beta}^{-1} d\tau_{\alpha\beta}) . \tag{7}$$

Similarly, for logarithmic version we define $n \times n$ matrix-valued holomorphic 1-form by

$$\bar{a}_{\alpha\beta} = \bar{\tau}_{\alpha\beta}^{-1} d\bar{\tau}_{\alpha\beta} \tag{8}$$

and a scalar-valued holomorphic 1-form

$$\bar{\sigma}_{\alpha\beta} = (n+1)^{-1} d \log(\det \bar{\tau}_{\alpha\beta}) = (n+1)^{-1} \operatorname{Trace}(\bar{\tau}_{\alpha\beta}^{-1} d\bar{\tau}_{\alpha\beta}) . \tag{9}$$

We write $\sigma_{\alpha\beta}$ (respectively for logarithmic version, $\bar{\sigma}_{\alpha\beta}$) as

$$\sigma_{\alpha\beta} = \sigma_{\alpha\beta j} dz_\beta^j$$

$$\text{resp.} \quad \bar{\sigma}_{\alpha\beta} = \bar{\sigma}_{\alpha\beta j} \omega_\beta^j$$

and define $\rho_{\alpha\beta}$ by

$$(\rho_{\alpha\beta})_k^j = \sigma_{\alpha\beta k} dz_\beta^j \tag{10}$$

$$\text{resp.} \quad (\bar{\rho}_{\alpha\beta})_k^j = \bar{\sigma}_{\alpha\beta k} \omega_\beta^j . \tag{11}$$

We consider another $n \times n$ matrix-valued holomorphic 1-form by

$$p_{\alpha\beta} = a_{\alpha\beta} - \rho_{\alpha\beta} - I \cdot \sigma_{\alpha\beta} \tag{12}$$

$$\text{resp.} \quad \bar{p}_{\alpha\beta} = \bar{a}_{\alpha\beta} - \bar{\rho}_{\alpha\beta} - I \cdot \bar{\sigma}_{\alpha\beta} . \tag{13}$$

LEMMA 1.1. *On $U_\alpha \cap U_\beta \cap U_\gamma$, we have the relations*

$$\sigma_{\alpha\beta}+\sigma_{\beta\gamma}=\sigma_{\alpha\gamma}\,, \tag{14}$$

$$\tau_{\beta\gamma}^{-1}a_{\alpha\beta}\tau_{\beta\gamma}+a_{\beta\gamma}=a_{\alpha\gamma}\,, \tag{15}$$

$$\tau_{\beta\gamma}^{-1}\rho_{\alpha\beta}\tau_{\beta\gamma}+\rho_{\beta\gamma}=\rho_{\alpha\gamma}\,, \tag{16}$$

$$\bar{\sigma}_{\alpha\beta}+\bar{\sigma}_{\beta\gamma}=\bar{\sigma}_{\alpha\gamma}\,, \tag{17}$$

$$\bar{\tau}_{\beta\gamma}^{-1}\bar{a}_{\alpha\beta}\bar{\tau}_{\beta\gamma}+\bar{a}_{\beta\gamma}=\bar{a}_{\alpha\gamma}\,, \tag{18}$$

$$\bar{\tau}_{\beta\gamma}^{-1}\bar{\rho}_{\alpha\beta}\bar{\tau}_{\beta\gamma}+\bar{\rho}_{\beta\gamma}=\bar{\rho}_{\alpha\gamma}\,. \tag{19}$$

By the above lemma, we have the 1-cocycles

$$\{\sigma_{\alpha\beta}\}\in H^1(X,\Omega^1)\,,$$

$$\{a_{\alpha\beta}\},\ \{\rho_{\alpha\beta}\},\ \{p_{\alpha\beta}\}\in H^1(X,\Omega^1(\mathrm{End}(\Theta)))\,,$$

$$\{\bar{\sigma}_{\alpha\beta}\}\in H^1(X,\Omega^1(\log D))\,,$$

and

$$\{\bar{a}_{\alpha\beta}\},\ \{\bar{\rho}_{\alpha\beta}\},\ \{\bar{p}_{\alpha\beta}\}\in H^1(X,\Omega^1(\log D)(\mathrm{End}(\Theta(-\log D))))\,.$$

It is easy to check that these cohomology classes are determined independently of the choice of logarithmic local coordinate systems along D. We put

$$\mathscr{F}^p=\Omega^p(\mathrm{End}(\Theta))\,,$$

and

$$\mathscr{F}^p\langle D\rangle=\Omega^p(\log D)(\mathrm{End}(\Theta(-\log D)))\,.$$

Let $\mathscr{A}^{p,q}(E)$ denote the sheaf of germs of C^∞ sections of a locally free sheaf E. We put

$$\mathscr{G}^{p,q}=\mathscr{A}^{p,q}(\mathrm{End}(\Theta))\,,$$

$$\mathscr{G}^r=\bigoplus\nolimits_{p+q=r}\mathscr{G}^{p,q}\,,$$

$$\mathscr{G}^{p,q}\langle D\rangle=\mathscr{A}^{p,q}(\log D)(\mathrm{End}(\Theta(-\log D)))\,,$$

and

$$\mathscr{G}^r\langle D\rangle=\bigoplus\nolimits_{p+q=r}\mathscr{G}^{p,q}\langle D\rangle\,.$$

DEFINITION 1.1. The cohomology class

$$a_X=\{a_{\alpha\beta}\}\in H^1(X,\mathscr{F}^1)$$

$$\text{resp.}\quad \bar{a}_X\langle D\rangle=\{\bar{a}_{\alpha\beta}\}\in H^1(X,\mathscr{F}^1\langle D\rangle)$$

is called the obstruction to the holomorphic affine connection (resp. the obstruction to the logarithmic affine connection with respect to D) of X. The cohomology class

$$p_X=\{p_{\alpha\beta}\}\in H^1(X,\mathscr{F}^1)$$

$$\text{resp.}\quad \bar{p}_X=\{\bar{p}_{\alpha\beta}\}\in H^1(X,\mathscr{F}^1\langle D\rangle)$$

is called the obstruction to the holomorphic projective connection (resp. the obstruction to the logarithmic projective connection with respect to D) of X.

DEFINITION 1.2. For a complex manifold X with $a_X=0$ (resp. $\bar{a}_X=0$), there exists a (holomorphic) 0-cochain $\{a_\alpha\}$ (resp. $\{\bar{a}_\alpha\}$) such that $\delta\{a_\alpha\}=\{a_{\alpha\beta}\}$ (resp. $\delta\{\bar{a}_\alpha\}=\{\bar{a}_{\alpha\beta}\}$), which is called a holomorphic affine connection (resp. holomorphic logarithmic affine connection with respect to D) of X. There always exists a C^∞ 0-cochain $\{a_\alpha\}$ (resp. $\{\bar{a}_\alpha\}$) in the natural sense, where the a_α (resp. $\bar{a}_\alpha$) is an element of

$$\Gamma(U_\alpha,\mathscr{G}^{1,0})$$

$$\text{resp.}\quad \Gamma(U_\alpha,\mathscr{G}^{1,0}\langle D\rangle)\,.$$

The 0-cochain is called a smooth affine connection (resp. smooth logarithmic affine connection with respect to D).

DEFINITION 1.3. For a complex manifold X with $p_X=0$ (resp. $\bar{p}_X=0$), there exists a (holomorphic) 0-cochain $\{p_\alpha\}$ (resp. $\{\bar{p}_\alpha\}$) such that $\delta\{p_\alpha\}=\{p_{\alpha\beta}\}$ (resp. $\delta\{\bar{p}_\alpha\}=\{\bar{p}_{\alpha\beta}\}$), which is called a holomorphic projective connection (resp. holomorphic logarithmic projective connection with respect to D) of X. There always exists a C^∞ 0-cochain $\{p_\alpha\}$ (resp. $\{\bar{p}_\alpha\}$) in a natural sense, where p_α (resp. $\bar{p}_\alpha$) is an element of

$$\Gamma(U_\alpha,\mathscr{G}^{1,0})$$

$$\text{resp.}\quad \Gamma(U_\alpha,\mathscr{G}^{1,0}\langle D\rangle)\,.$$

The 0-cochain is called a smooth projective connection (resp. smooth logarithmic projective connection with respect to D).

Let $i\colon X\setminus D\to X$ be the natural inclusion. Since $i^*\mathscr{F}^p\langle D\rangle\simeq\mathscr{F}^p$, i induces a natural homomorphism

$$i_q^*\colon H^q(X,\mathscr{F}^p\langle D\rangle)\to H^q(X\setminus D,\mathscr{F}^p)\,.$$

Put

$$\bar{a}_\alpha=\tau_\alpha^{-1}d\tau_\alpha\,, \tag{20}$$

$$\bar{\sigma}_\alpha=(n+1)^{-1}d\log\det\tau_\alpha\,, \tag{21}$$

$$\bar{\rho}_\alpha=(\bar{\rho}^j_{\alpha k})\,, \tag{22}$$

$$\bar{\rho}^j_{\alpha k}=\bar{\sigma}_{\alpha k}\omega^j_\alpha\,, \tag{23}$$

$$\bar{q}_\alpha=\bar{a}_\alpha-\bar{\rho}_\alpha-I\cdot\bar{\sigma}_\alpha\,. \tag{24}$$

LEMMA 1.2. *On $U_\alpha\cap U_\beta\cap U_\gamma$, we have the relations*

$$\bar{a}_{\alpha\beta}=\tau_\beta^{-1}a_{\alpha\beta}\tau_\beta+\bar{a}_\beta-\bar{\tau}_{\alpha\beta}^{-1}\bar{a}_\alpha\bar{\tau}_{\alpha\beta}\,,$$
$$\bar{\sigma}_{\alpha\beta}=\sigma_{\alpha\beta}+\sigma_\beta-\bar{\sigma}_\alpha\,,$$
$$\bar{\rho}_{\alpha\beta}=\tau_\beta^{-1}\rho_{\alpha\beta}\tau_\beta+\bar{\rho}_\beta-\bar{\tau}_{\alpha\beta}^{-1}\bar{\rho}_\alpha\bar{\tau}_{\alpha\beta}\,,$$
$$\bar{p}_{\alpha\beta}=\tau_\beta^{-1}p_{\alpha\beta}\tau_\beta+\bar{q}_\beta-\bar{\tau}_{\alpha\beta}^{-1}\bar{q}_\alpha\bar{\tau}_{\alpha\beta}\,.$$

As a corollary to the above lemma, we have

PROPOSITION 1.1.

$$i_1^*\bar{a}_X\langle D\rangle=a_{X\setminus D}\,, \tag{25}$$
$$i_1^*\bar{p}_X\langle D\rangle=p_{X\setminus D}\,. \tag{26}$$

PROPOSITION 1.2. *The vanishing of a_X (resp. $\bar{a}_X\langle D\rangle$) implies the vanishing of p_X (resp. $\bar{p}_X\langle D\rangle$).*

PROOF. In the logarithmic case, given a holomorphic logarithmic affine connection $\{\bar{a}_\alpha\}$, then we can define a holomorphic logarithmic projective connection 0-cochain $\{\bar{p}_\alpha\}$ given by

$$\bar{p}_\alpha=\bar{a}_\alpha-\bar{\rho}_\alpha-I\cdot\bar{\sigma}_\alpha\,,$$

where

$$\bar{\sigma}_\alpha=(n+1)^{-1}\operatorname{Trace}\bar{a}_\alpha=\bar{\sigma}_{\alpha j}\omega_\alpha^j\,,$$
$$(\bar{\rho}_\alpha)_k^j=\bar{\sigma}_{\alpha k}\omega_\alpha^j\,.$$

The (ordinary) affine connection case is settled by the same manner. ∎

2. Logarithmic projective Weyl forms.

Let X be a complex manifold of dimension $n\geq 2$ and D a reduced effective divisor on X with only normal crossing singularities. In the following, logarithmic projective connections (resp. logarithmic affine connections) imply *smooth* logarithmic projective connections with respect to D (resp. *smooth* logarithmic affine connections with respect to D) unless otherwise stated explicitly. Suppose that $\bar{p}_X\langle D\rangle$ is represented by a cocycle $\{\bar{p}_{\alpha\beta}\}$. Let $\bar{\pi}=\{\bar{p}_\alpha\}$ be a (smooth) logarithmic projective connection on X with respect to D, that is, on each U_α there is an $n\times n$ matrix-valued (1, 0)-form $\bar{p}_\alpha$ such that

$$\bar{p}_\beta=\bar{p}_{\alpha\beta}+\bar{\tau}_{\alpha\beta}^{-1}\bar{p}_\alpha\bar{\tau}_{\alpha\beta}\,, \tag{27}$$

where

$$(\bar{p}_\alpha)_j^k=\bar{p}_{\alpha ij}^k\omega_\alpha^i\,, \tag{28}$$

with C^∞-function $\bar{p}_{\alpha ij}^k$ defined on U_α. Similarly, the (k, j)-component of $\bar{p}_{\alpha\beta}$ is written as

$$(\bar{p}_{\alpha\beta})^k_j = \bar{p}^k_{\alpha\beta ij}\omega^i_\beta , \tag{29}$$

with C^∞-function $\bar{p}^k_{\alpha\beta ij}$ defined on $U_{\alpha\beta}$. Using the basis of (1) and (2), we have

$$\bar{a}_{\alpha\beta} = \bar{a}^k_{\alpha\beta ij}\omega^i_\beta \otimes \omega^j_\beta \otimes \theta_{\beta k} ,$$

$$\bar{\rho}_{\alpha\beta} = \delta^k_i \bar{\sigma}_{\alpha\beta j}\omega^i_\beta \otimes \omega^j_\beta \otimes \theta_{\beta k} ,$$

and

$$\bar{\sigma}_{\alpha\beta} = \delta^k_j \bar{\sigma}_{\alpha\beta i}\omega^i_\beta \otimes \omega^j_\beta \otimes \theta_{\beta k} .$$

Therefore the equality

$$\bar{p}^k_{\alpha\beta ij} = \bar{p}^k_{\alpha\beta ji}$$

holds. In view of this equality, it is easy to see that

$$\bar{q}_\alpha = (\bar{q}^k_{\alpha j}) ,$$

where

$$\bar{q}^k_{\alpha j} = \bar{p}^k_{\alpha ji}\omega^i_\alpha$$

is also a logarithmic projective connection. Hence $\{2^{-1}(\bar{p}_\alpha + \bar{q}_\alpha)\}$ is a logarithmic projective connection. Therefore we may assume that

$$\bar{p}^k_{\alpha ij} = \bar{p}^k_{\alpha ji} . \tag{30}$$

Since $\mathrm{Trace}(\bar{\rho}_{\alpha\beta}) = 0$, it follows from (27) that $\mathrm{Trace}(\bar{p}_\alpha) = \mathrm{Trace}(\bar{p}_\beta)$. Since $\{\bar{p}_\alpha - n^{-1}\,\mathrm{Trace}(\bar{p}_\alpha)\cdot I\}$ is also a projective connection, we may assume that

$$\bar{p}^j_{\alpha ij} = 0 . \tag{31}$$

A logarithmic projective connection satisfying (30) is said to be *normal*. A logarithmic projective connection satisfying (31) is said to be *reduced*. Thus *any complex manifold admits normal reduced* (*smooth*) *logarithmic projective connections*. As we see from the above argument, if *a complex manifold admits a holomorphic logarithmic projective connection, then the manifold admits a normal reduced holomorphic logarithmic projective connection*. In the following, *we consider only normal reduced* (*holomorphic or smooth*) *logarithmic projective connections*.

Now we shall calculate the logarithmic projective Weyl curvature tensor. As in [K3], we follow the argument of Eisenhart [E]. By (8) and (13), we have

$$\bar{p}_\beta = \bar{\tau}^{-1}_{\alpha\beta} d\bar{\tau}_{\alpha\beta} - \bar{\rho}_{\alpha\beta} - I\cdot\bar{\sigma}_{\alpha\beta} + \bar{\tau}^{-1}_{\alpha\beta}\bar{p}_\alpha\bar{\tau}_{\alpha\beta} .$$

Taking the exterior derivative, we get

$$\begin{aligned} d\bar{p}_\beta = & -\bar{\tau}^{-1}_{\alpha\beta} d\bar{\tau}_{\alpha\beta} \wedge \bar{\tau}^{-1}_{\alpha\beta} d\bar{\tau}_{\alpha\beta} - d\bar{\rho}_{\alpha\beta} - \bar{\tau}^{-1}_{\alpha\beta} d\bar{\tau}_{\alpha\beta} \wedge \bar{\tau}^{-1}_{\alpha\beta}\bar{p}_\alpha\bar{\tau}_{\alpha\beta} \\ & + \bar{\tau}^{-1}_{\alpha\beta} d\bar{p}_\alpha\bar{\tau}_{\alpha\beta} - \bar{\tau}^{-1}_{\alpha\beta}\bar{p}_\alpha d\bar{\tau}_{\alpha\beta} , \end{aligned} \tag{32}$$

where we have used the equation $d\bar{\sigma}_{\alpha\beta}=0$. Therefore we obtain

$$\begin{aligned} d\bar{p}_\beta+\bar{p}_\beta\wedge\bar{p}_\beta=\bar{\tau}_{\alpha\beta}^{-1}(d\bar{p}_\alpha+\bar{p}_\alpha\wedge\bar{p}_\alpha)\bar{\tau}_{\alpha\beta}-d\bar{\rho}_{\alpha\beta}+\bar{\rho}_{\alpha\beta}\wedge\bar{\rho}_{\alpha\beta}-\bar{\tau}_{\alpha\beta}^{-1}d\bar{\tau}_{\alpha\beta}\wedge\bar{\rho}_{\alpha\beta} \\ -\bar{\rho}_{\alpha\beta}\wedge\bar{\tau}_{\alpha\beta}^{-1}d\bar{\tau}_{\alpha\beta}-\bar{\rho}_{\alpha\beta}\wedge\bar{\tau}_{\alpha\beta}^{-1}\bar{p}_\alpha\bar{\tau}_{\alpha\beta}-\bar{\tau}_{\alpha\beta}^{-1}\bar{p}_\alpha\bar{\tau}_{\alpha\beta}\wedge\bar{\rho}_{\alpha\beta} \\ -\bar{\tau}_{\alpha\beta}^{-1}d\bar{\tau}_{\alpha\beta}\wedge I\cdot\bar{\sigma}_{\alpha\beta}-I\cdot\bar{\sigma}_{\alpha\beta}\wedge\bar{\tau}_{\alpha\beta}^{-1}d\bar{\tau}_{\alpha\beta}-\bar{\tau}_{\alpha\beta}^{-1}\bar{p}_\alpha\bar{\tau}_{\alpha\beta}\wedge I\cdot\bar{\sigma}_{\alpha\beta} \\ -I\cdot\bar{\sigma}_{\alpha\beta}\wedge\bar{\tau}_{\alpha\beta}^{-1}\bar{p}_\alpha\bar{\tau}_{\alpha\beta}+I\cdot\bar{\sigma}_{\alpha\beta}\wedge I\cdot\bar{\sigma}_{\alpha\beta}+\bar{\rho}_{\alpha\beta}\wedge I\cdot\bar{\sigma}_{\alpha\beta}+I\cdot\bar{\sigma}_{\alpha\beta}\wedge\bar{\rho}_{\alpha\beta}\,. \end{aligned} \tag{33}$$

LEMMA 2.1.

$$\bar{p}_\alpha\wedge\bar{\tau}_{\alpha\beta}\bar{\rho}_{\alpha\beta}=0\,, \tag{34}$$

$$d\bar{\tau}_{\alpha\beta}\wedge\bar{\rho}_{\alpha\beta}=0\,, \tag{35}$$

$$\begin{aligned} I\cdot\bar{\sigma}_{\alpha\beta}\wedge I\cdot\bar{\sigma}_{\alpha\beta}&=\bar{\rho}_{\alpha\beta}\wedge I\cdot\bar{\sigma}_{\alpha\beta}+I\cdot\bar{\sigma}_{\alpha\beta}\wedge\bar{\rho}_{\alpha\beta} \\ &=\bar{\tau}_{\alpha\beta}^{-1}d\bar{\tau}_{\alpha\beta}\wedge I\cdot\bar{\sigma}_{\alpha\beta}+I\cdot\bar{\sigma}_{\alpha\beta}\wedge\bar{\tau}_{\alpha\beta}^{-1}d\bar{\tau}_{\alpha\beta} \\ &=I\cdot\bar{\sigma}_{\alpha\beta}\wedge\bar{\tau}_{\alpha\beta}^{-1}\bar{p}_\alpha\bar{\tau}_{\alpha\beta}+\bar{\tau}_{\alpha\beta}^{-1}\bar{p}_\alpha\bar{\tau}_{\alpha\beta}\wedge I\cdot\bar{\sigma}_{\alpha\beta} \\ &=0\,, \end{aligned} \tag{36}$$

$$\bar{\rho}_{\alpha\beta}\wedge\bar{\rho}_{\alpha\beta}=\bar{\rho}_{\alpha\beta}\wedge I\cdot\bar{\sigma}_{\alpha\beta}\,. \tag{37}$$

PROOF. Since $\bar{\sigma}_{\alpha\beta}$ is a scalar-valued form, the equality (36) follows from the fact that $\bar{\sigma}_{\alpha\beta}$, $d\bar{\tau}_{\alpha\beta}$ and $\bar{p}_\alpha$ are 1-forms. By the definition of $\bar{\rho}_{\alpha\beta}$, $\bar{\rho}_{\alpha\beta}\wedge\bar{\rho}_{\alpha\beta}$ is given by

$$\begin{aligned} (\bar{\sigma}_{\alpha\beta m}\omega_\beta^k\wedge\bar{\sigma}_{\alpha\beta j}\omega_\beta^m)&=(\bar{\sigma}_{\alpha\beta j}\omega_\beta^k\wedge\bar{\sigma}_{\alpha\beta m}\omega_\beta^m) \\ &=\bar{\rho}_{\alpha\beta}\wedge I\cdot\bar{\sigma}_{\alpha\beta}\,. \end{aligned}$$

Hence (37) is proved. Note that by (1.1) we have

$$\begin{aligned} d\bar{\tau}_{\alpha\beta}\wedge\bar{\rho}_{\alpha\beta}=\tau_\alpha^{-1}\{&-d\tau_\alpha\tau_\alpha^{-1}\tau_{\alpha\beta}\wedge\rho_{\alpha\beta}+\tau_{\alpha\beta}d\tau_\beta\tau_\beta^{-1}\wedge\rho_{\alpha\beta}+d\tau_{\alpha\beta}\wedge\rho_{\alpha\beta} \\ &-d\tau_\alpha\tau_\alpha^{-1}\tau_{\alpha\beta}\rho_\beta+\tau_{\alpha\beta}d\tau_\beta\tau_\beta^{-1}\rho_\beta+d\tau_{\alpha\beta}\rho_\beta+d\tau_\alpha\tau_\alpha^{-1}\rho_\alpha\tau_{\alpha\beta} \\ &-\tau_{\alpha\beta}d\tau_\beta\tau_\beta\tau_{\alpha\beta}^{-1}\rho_\alpha\tau_{\alpha\beta}-d\tau_{\alpha\beta}\tau_{\alpha\beta}^{-1}\rho_\alpha\tau_{\alpha\beta}\}\tau_\beta\,. \end{aligned} \tag{38}$$

The (j, k)-component of $d\tau_{\alpha\beta}\wedge\rho_{\alpha\beta}$ is

$$\frac{\partial\tau_{\alpha\beta m}^j}{\partial z_\beta^s}dz_\beta^s\wedge\sigma_{\alpha\beta k}dz_\beta^m=\sigma_{\alpha\beta k}\left(\frac{\partial^2 z_\alpha^j}{\partial z_\beta^s\partial z_\beta^m}\right)dz_\beta^s\wedge dz_\beta^m=0\,. \tag{39}$$

Hence

$$d\tau_{\alpha\beta}\wedge\rho_{\alpha\beta}=0\,. \tag{40}$$

Similarly, we have

$$d\tau_{\alpha\beta}\wedge\rho_\beta=0\,, \tag{41}$$

$$d\tau_{\alpha\beta}\tau_{\alpha\beta}^{-1}\wedge\rho_\alpha=0\,. \tag{42}$$

The (j, k)-component of $\tau_{\alpha\beta}\rho_{\alpha\beta}$ is given by

$$\tau^j_{\alpha\beta m}\rho_{\alpha\beta k}dz^m_\beta=\rho_{\alpha\beta k}dz^j_\alpha\,.$$

Therefore we have

$$d\tau_\alpha\tau_\alpha^{-1}\tau_{\alpha\beta}\wedge\rho_{\alpha\beta}=0\,. \tag{43}$$

Similarly, we have

$$d\tau_\alpha\tau_\alpha^{-1}\tau_{\alpha\beta}\wedge\rho_\beta=0\,. \tag{44}$$

It is easy to see that

$$d\tau_\beta\tau_\beta^{-1}\wedge\rho_{\alpha\beta}=0\,, \tag{45}$$

$$d\tau_\beta\tau_\beta^{-1}\wedge\rho_\beta=0\,, \tag{46}$$

$$d\tau_\beta\tau_\beta^{-1}\tau_{\alpha\beta}^{-1}\wedge\rho_\alpha=0\,. \tag{47}$$

Then (35) follows from (40)–(47) and (38). By (30), the (j,k)-component of $\bar{p}_\alpha\wedge\bar{\tau}_{\alpha\beta}\bar{\rho}_{\alpha\beta}$ is

$$\bar{p}^j_{\alpha ts}\omega^t_\alpha\wedge\bar{\tau}^s_{\alpha\beta m}\bar{\sigma}_{\alpha\beta k}\omega^m_\beta=\bar{\sigma}_{\alpha\beta k}(\bar{p}^j_{\alpha ts}\omega^t_\alpha\wedge\omega^s_\alpha)=0\,.$$

Thus (34) is proved. ■

By the above lemma and (33), we have

$$d\bar{p}_\beta+\bar{p}_\beta\wedge\bar{p}_\beta=\bar{\tau}_{\alpha\beta}^{-1}(d\bar{p}_\alpha+\bar{p}_\alpha\wedge\bar{p}_\alpha)\bar{\tau}_{\alpha\beta}-\bar{r}_{\alpha\beta}\,, \tag{48}$$

where, using (13) and (27)

$$\begin{aligned}\bar{r}_{\alpha\beta}&=\bar{\rho}_{\alpha\beta}\wedge\bar{\tau}_{\alpha\beta}^{-1}\bar{p}_\alpha\bar{\tau}_{\alpha\beta}+d\bar{\rho}_{\alpha\beta}-\bar{\rho}_{\alpha\beta}\wedge I\cdot\bar{\sigma}_{\alpha\beta}+\bar{\rho}_{\alpha\beta}\wedge\bar{\tau}_{\alpha\beta}^{-1}d\bar{\tau}_{\alpha\beta}\\&=\bar{\rho}_{\alpha\beta}\wedge\bar{p}_\beta+d\bar{\rho}_{\alpha\beta}+\bar{\rho}_{\alpha\beta}\wedge I\cdot\bar{\sigma}_{\alpha\beta}\,.\end{aligned} \tag{49}$$

Note that $\bar{r}_{\alpha\beta}$ is a matrix-valued smooth (2, 0)-form. Put

$$\bar{r}_{\alpha\beta}=(\bar{r}^h_{\alpha\beta i})=(\bar{r}^h_{\alpha\beta ijk}\omega^j_\beta\wedge\omega^k_\beta)\,. \tag{50}$$

Then by (49) we have

$$\bar{r}^h_{\alpha\beta ijk}=\delta^h_j\bar{r}_{\alpha\beta ik}-\delta^h_k\bar{r}_{\alpha\beta ij}\,, \tag{51}$$

where

$$2\bar{r}_{\alpha\beta ik}=\bar{\sigma}_{\alpha\beta m}\bar{p}^m_{\beta ik}-\frac{1}{n+1}\,\theta_{\beta i}\theta_{\beta k}(\log\det\bar{\tau}_{\alpha\beta})+\bar{\sigma}_{\alpha\beta i}\bar{\sigma}_{\alpha\beta k}\,. \tag{52}$$

Letting

$$\partial\bar{p}_\beta+\bar{p}_\beta\wedge\bar{p}_\beta=(\bar{X}^h_{\beta ijk}\omega^j_\beta\wedge\omega^k_\beta)\,, \tag{53}$$

we have

$$\bar{X}^h_{\beta ijk}=\theta_{\beta j}(\bar{p}^h_{\beta ik})-\theta_{\beta k}(\bar{p}^h_{\beta ij})+\bar{p}^h_{\beta mj}\bar{p}^m_{\beta ik}-\bar{p}^h_{\beta mk}\bar{p}^m_{\beta ij}\,. \tag{54}$$

Put $\bar{X}_{\beta ij}=\bar{X}^k_{\beta ijk}$. Then

$$\bar{X}_{\beta ij}=-\theta_{\beta k}(\bar{p}^k_{\beta ij})+\bar{p}^k_{\beta mj}\bar{p}^m_{\beta ik}\,. \tag{55}$$

It follows from (48), (50) and (51) that

$$\bar{r}_{\alpha\beta ik}=\frac{1}{n-1}\,(\bar{X}_{\beta ik}-\bar{X}_{\alpha jm}\bar{\tau}^j_{\alpha\beta i}\bar{\tau}^m_{\alpha\beta k})\,. \tag{56}$$

Thus we obtain

$$\bar{W}_\beta=\bar{\tau}_{\alpha\beta}^{-1}\bar{W}_\alpha\bar{\tau}_{\alpha\beta}\,, \tag{57}$$

where

$$\bar{W}_\alpha=d\bar{p}_\alpha+\bar{p}_\alpha\wedge\bar{p}_\alpha+\frac{1}{n-1}\,\bar{X}_\alpha\,, \tag{58}$$

$$\bar{X}_\alpha=(\delta^h_j\bar{X}_{\alpha ik}\omega^j_\alpha\wedge\omega^k_\alpha)\,. \tag{59}$$

Thus we have

PROPOSITION 2.1. *The tensor field* $\{\bar{W}_\alpha\}$ *defined by* (58) *and* (59), *called the (smooth) logarithmic projective Weyl curvature tensor, is an element of* $\Gamma(X, \mathscr{G}^2\langle D\rangle)$.

If, further, the projective connection $\bar{p}_\alpha$ *is holomorphic, then the tensor field* $\{\bar{W}_\alpha\}$, *called the holomorphic logarithmic projective Weyl curvature tensor, is an element of* $\Gamma(X, \mathscr{F}^2\langle D\rangle)$.

Let t be an indeterminate and A be an $n\times n$ matrix. Define polynomials $\varphi_0, \varphi_1, \cdots, \varphi_n$ by

$$\det\left(I-\frac{1}{2\pi i}\,tA\right)=\sum_{k=0}^{n}\varphi_k(A)t^k\,.$$

We put

$$\bar{P}_k(\bar{\pi})=\varphi_k(\bar{W}_\alpha)\,,\qquad k=0, 1, \cdots, n\,,$$

where $\bar{\pi}$ stands for the normal reduced logarithmic projective connection $\{\bar{p}_\alpha\}$. In view of (57), $\bar{P}_k(\bar{\pi})$ are smooth $2k$-forms defined globally on X.

THEOREM 2.1. i) *$\bar{P}_k(\bar{\pi})$ is a d-closed smooth $2k$-form.*

ii) *The de Rham cohomology class $[\bar{P}_k(\bar{\pi})]$ is a real cohomology class and is independent of the choice of the normal reduced logarithmic projective connection.*

DEFINITION 2.1. The d-closed smooth $2k$-form $\bar{P}_k=\bar{P}_k(\bar{\pi})$ is called the k-th logarithmic projective Weyl form.

Since $\bar{\pi}$ is reduced, we have

$$\bar{P}_1 = 0 . \tag{60}$$

The above theorem is a corollary (Corollary 3.1) to Theorem 3.1 in the next section.

REMARK 1. In holomorphic conformal geometry, we can define conformal Weyl forms by means of conformal Weyl curvature tensor (cf. [K4]). Therefore, to avoid confusion, we call by k-th *projective* Weyl form the d-closed smooth $2k$-form $P_k = P_k(\pi)$ defined in [K3, Definition 2.31].

3. A formula on logarithmic projective Weyl forms and logarithmic Chern forms.

In this section, we shall prove the following theorem.

THEOREM 3.1. *Let (X, D) be a logarithmic pair of a complex manifold X of dimension $n \geq 2$. Let $\bar{\pi}$ be any normal reduced smooth logarithmic projective connection with respect to D on X. Then there is a smooth logarithmic affine connection $\bar{\theta}$ on X with respect to D which satisfies the following equality;*

$$\sum_{q=0}^{n} \bar{c}_q(\bar{\theta}) t^q = (1 + \bar{a}t)^{n+1} \sum_{q=0}^{n} \bar{P}_q(\bar{\pi}) \left(\frac{t}{1 + \bar{a}t} \right)^q ,$$

or, equivalently

$$\sum_{q=0}^{n} \bar{P}_q(\bar{\pi}) t^q = (1 - \bar{a}t)^{n+1} \sum_{q=0}^{n} \bar{c}_q(\bar{\theta}) \left(\frac{t}{1 - \bar{a}t} \right)^q ,$$

where $\bar{c}_k(\bar{\theta})$ = the k-th logarithmic Chern form associated with $\bar{\theta}$, $\bar{a} = (n+1)^{-1} \bar{c}_1(\bar{\theta})$, $\bar{P}_j(\bar{\pi})$ = the j-th logarithmic projective Weyl form associated with $\bar{\pi}$.

We shall give a proof of the above theorem by almost the same method as that of [K3, Theorem 3.1]. In the logarithmic case, however, we cannot expect an analogue of [K3, Lemma 3.14].

To prove the theorem, first we choose the smooth logarithmic affine connection $\bar{\theta}$. Put

$$\bar{K}_X = K_X + D ,$$

where K_X is the canonical line bundle of X. We may assume that $\bar{K}_X$ is represented by a 1-cocycle $\{\bar{K}_{\alpha\beta}\}$, $\bar{K}_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X^*)$. Let $\{\bar{h}_\alpha\}$ be a smooth metric of $\{\bar{K}_{\alpha\beta}\}$. That is, each $\bar{h}_\alpha$ is a real-valued positive smooth function on U_α satisfying

$$\bar{h}_\beta = |\bar{K}_{\alpha\beta}|^2 \bar{h}_\alpha \qquad \text{on} \quad U_\alpha \cap U_\beta . \tag{61}$$

By (9), we have

$$\bar{\sigma}_{\alpha\beta} = -(n+1)^{-1} \partial \log \bar{K}_{\alpha\beta} . \tag{62}$$

Put

$$\bar{\sigma}_\alpha = -(n+1)^{-1}\bar{\partial}\log \bar{h}_\alpha . \tag{63}$$

Then

$$\bar{\sigma}_{\alpha\beta} = \bar{\sigma}_\beta - \bar{\sigma}_\alpha . \tag{64}$$

We write $\bar{\sigma}_\alpha$ as

$$\bar{\sigma}_\alpha = \bar{\sigma}_{\alpha j}\omega_\alpha^j \tag{65}$$

and define

$$\bar{\rho}_\alpha = (\bar{\sigma}_{\alpha k}\omega_\alpha^j) . \tag{66}$$

Then $\bar{\rho}_\alpha$ is an $n\times n$ matrix-valued smooth (1, 0)-form, where the (j, k)-component is $\bar{\sigma}_{\alpha k}\omega_\alpha^j$, and satisfies

$$\bar{\rho}_{\alpha\beta} = \bar{\rho}_\beta - \bar{\tau}_{\alpha\beta}^{-1}\bar{\rho}_\alpha\bar{\tau}_{\alpha\beta} . \tag{67}$$

Now let $\bar{\pi} = \{\bar{p}_\alpha\}$ be any normal reduced (smooth) logarithmic projective connection. Recall that the $\bar{p}_\alpha$ satisfy the relation (27). Define an $n\times n$ matrix valued smooth (1, 0)-form $\bar{\theta}_\alpha$ on U_α by

$$\bar{\theta}_\alpha = \bar{p}_\alpha + \bar{\rho}_\alpha + \bar{\sigma}_\alpha \cdot I . \tag{68}$$

Using (13), (64) and (67), we have

$$\bar{a}_{\alpha\beta} = \bar{\theta}_\beta - \bar{\tau}_{\alpha\beta}^{-1}\bar{\theta}_\alpha\bar{\tau}_{\alpha\beta} . \tag{69}$$

This shows that $\bar{\theta} = \{\bar{\theta}_\alpha\}$ is a logarithmic affine connection of X. We calculate the Chern forms associated with $\bar{\theta}$. The curvature form of the logarithmic affine connection

$$\bar{\Theta}_\alpha = d\bar{\theta}_\alpha + \bar{\theta}_\alpha \wedge \bar{\theta}_\alpha \tag{70}$$

satisfies the equation

$$\bar{\Theta}_\beta = \bar{\tau}_{\alpha\beta}^{-1}\bar{\Theta}_\alpha\bar{\tau}_{\alpha\beta} \qquad \text{on} \quad U_\alpha \cap U_\beta . \tag{71}$$

The logarithmic Chern forms are given by

$$\bar{c}_q(\bar{\theta}) = \varphi_q(\bar{\Theta}_\alpha) . \tag{72}$$

LEMMA 3.1.

$$\bar{p}_\alpha \wedge \bar{\rho}_\alpha = d\bar{p}_\alpha \wedge \bar{\rho}_\alpha = \bar{p}_\alpha \wedge d\bar{\rho}_\alpha = 0 . \tag{73}$$

PROOF. These equalities follow immediately from the condition (30). ∎

By (68), (70) and Lemma 3.1, we have

$$\bar{\Theta}_\alpha = d\bar{\sigma}_\alpha \cdot I + d\bar{p}_\alpha + \bar{p}_\alpha \wedge \bar{p}_\alpha + \bar{\rho}_\alpha \wedge \bar{p}_\alpha + d\bar{\rho}_\alpha + \bar{\rho}_\alpha \wedge \bar{\rho}_\alpha . \tag{74}$$

In what follows, we omit the subscript α. Put

$$\lambda=1-\frac{t}{2\pi i}\,d\bar{\sigma}=1-\frac{t}{2\pi i}\,\bar{\partial}\bar{\sigma}\,. \tag{75}$$

It follows from Lemma 3.1 that

$$I-\frac{t}{2\pi i}\,\bar{\Theta}=\lambda\left(I-\frac{t}{2\pi i\lambda}\,(d\bar{p}+\bar{p}\wedge\bar{p})\right)\left(I-\frac{t}{2\pi i\lambda}\,\bar{\rho}\wedge\bar{p}\right)\left(I-\frac{t}{2\pi i\lambda}\,(d\bar{\rho}+\bar{\rho}\wedge\bar{\rho})\right). \tag{76}$$

LEMMA 3.2.

$$\det\left(I-\frac{t}{2\pi i}\,\bar{X}\right)=1\,.$$

PROOF. Note that

$$\det\left(I-\frac{t}{2\pi i}\,\bar{X}\right)=\sum_{q=0}^{n}\left(-\frac{t}{2\pi i}\right)^{q}{\sum_{J}}'\det\bar{X}_J^J\,, \tag{77}$$

where J runs all q-tuples $\{j_1, j_2, \cdots, j_q\}$ with $j_1<j_2<\cdots<j_q$, $1\leq j_\lambda\leq n$, and $\bar{X}_J^J$ is the $q\times q$-principal minor corresponding to J. Let S_q denote the symmetric group of degree q and put $k_\lambda=j_\sigma(\lambda)$, $\sigma\in S_q$. Then by

$$\bar{X}_k^j=\omega^j\wedge\bar{X}_{km}\omega^m\,,$$

we have

$$\begin{aligned}{\sum_{J}}'\det\bar{X}_J^J&={\sum_{J}}'\sum_{\sigma\in S_q}(\operatorname{sgn}\sigma)(\omega^{j_1}\wedge\bar{X}_{k_1m_1}\omega^{m_1}\wedge\cdots\wedge\omega^{j_q}\wedge\bar{X}_{k_qm_q}\omega^{m_q})\\&=q!\,{\sum_{J}}'\omega^{j_1}\wedge\bar{X}_{j_1m_1}\omega^{m_1}\wedge\cdots\wedge\omega^{j_q}\wedge\bar{X}_{j_qm_q}\omega^{m_q}\\&=\sum_{I}\omega^{i_1}\wedge\bar{X}_{i_1m_1}\omega^{m_1}\wedge\cdots\wedge\omega^{i_q}\wedge\bar{X}_{i_qm_q}\omega^{m_q}\\&=(\bar{X}_{im}\omega^i\wedge\omega^m)^q\,.\end{aligned}$$

By (55), $\bar{X}_{im}\omega^i\wedge\omega^m=0$ holds. Hence we have the lemma by (77) and (78). ■

LEMMA 3.3.

$$\det\left(I-\frac{t}{2\pi i}\,(d\bar{p}+\bar{p}\wedge\bar{p})\right)=\det\left(I-\frac{t}{2\pi i}\,\bar{W}\right). \tag{78}$$

PROOF. Note that

$$\bar{p}\wedge\bar{X}=0 \tag{79}$$

holds. By (59), the (j, k)-component of $\bar{\rho} \wedge \bar{X}$ is

$$\bar{\rho}^j_l \wedge \bar{X}^l_k = \bar{\rho}^j_{il}\omega^i \wedge \omega^l \bar{X}_{km} \wedge \omega^m . \tag{80}$$

The equality $\bar{\rho}^j_{il}\omega^i \wedge \omega^l = 0$ follows from (30). Hence we get (79). Similarly we have the equality

$$d\bar{\rho} \wedge \bar{X} = 0 . \tag{81}$$

By (79), (81) and (58), we have the equality

$$I - \frac{t}{2\pi i} \bar{W} = \left(I - \frac{t}{2\pi i} (d\bar{\rho} + \bar{\rho} \wedge \bar{\rho})\right)\left(I - \frac{t}{2\pi i} \frac{1}{n-1} \bar{X}\right).$$

Then the lemma follows from Lemma 3.2. ∎

LEMMA 3.4.

$$\det\left(I - \frac{t}{2\pi i} \bar{\rho} \wedge \bar{\rho}\right) = 1 . \tag{82}$$

PROOF. By (66), the (j, k)-component of $\bar{\rho} \wedge \bar{\rho}$ is

$$\bar{\rho}^j_l \wedge \bar{\rho}^l_k = \omega^j \wedge \bar{\sigma}_l \bar{\rho}^l_{mk} \omega^m .$$

Hence by the same calculation as in the proof of Lemma 3.2, we have

$$\det\left(I - \frac{t}{2\pi i} \bar{\rho} \wedge \bar{\rho}\right) = \sum_{q=0}^{n} \left(-\frac{t}{2\pi i}\right)^q (\bar{\sigma}_l \bar{\rho}^l_{mi} \omega^i \wedge \omega^m)^q .$$

Since $\bar{\rho}^l_{mi}\omega^i \wedge \omega^m = 0$ by (30), we have the lemma. ∎

LEMMA 3.5.

$$\det\left(I - \frac{t}{2\pi i} (d\bar{\rho} + \bar{\rho} \wedge \bar{\rho})\right) = \left(I - \frac{1}{n+1} \bar{c}_1(\bar{\theta}) t\right)^{-1} . \tag{83}$$

PROOF. By (66), the (j, k)-component of $d\bar{\rho} + \bar{\rho} \wedge \bar{\rho}$ is

$$d\bar{\rho}^j_k + \bar{\rho}^j_l \wedge \bar{\rho}^l_k = (d\bar{\sigma}_k - \bar{\sigma}_k \bar{\sigma}_l \omega^l) \wedge \omega^j .$$

Hence, by the same calculation as in the proof of Lemma 3.2, we have

$$\begin{aligned}
\det\left(I - \frac{t}{2\pi i} (d\bar{\rho} + \bar{\rho} \wedge \bar{\rho})\right) &= \sum_{q=0}^{n} \left(\frac{t}{2\pi i}\right)^q ((d\bar{\sigma}_j - \bar{\sigma}_j \bar{\sigma}_l \omega^l) \wedge \omega^j)^q \\
&= \sum_{q=0}^{n} \left(-\frac{t}{2\pi i}\right)(\bar{\partial}\bar{\sigma})^q \\
&= \left(1 + \frac{t}{2\pi i} \bar{\partial}\bar{\sigma}\right)^{-1} .
\end{aligned}$$

Since $\frac{1}{2\pi i}\bar{\partial}\bar{\sigma}=-\frac{1}{n+1}\bar{c}_1(\bar{\theta})$, we have the lemma. ■

PROOF OF THEOREM 3.1. Put

$$\bar{a}=\frac{1}{n+1}\,\bar{c}_1(\bar{\theta})=-\frac{1}{2\pi i}\,\bar{\partial}\bar{\sigma}\,.$$

Then

$$\lambda=1+\bar{a}t\,.$$

Hence from Lemmas 3.2, 3.3, 3.4, 3.5 and the equalities (75) and (76), it follows easily that

$$\sum_{q=0}^{n}\bar{c}_q(\bar{\theta})t^q=(1+\bar{a}t)^{n+1}\sum_{q=0}^{n}\bar{P}_q(\bar{\pi})\left(\frac{t}{1+\bar{a}t}\right)^q\,.$$

If $t/(1+\bar{a}t)$ is replaced by t, we obtain the latter formula of the theorem. ■

REMARK 2. The first formula of Theorem 3.1 can be rewritten as

$$\bar{c}_k(\bar{\theta})=\sum_{j=0}^{k}\binom{n+1-j}{k-j}((n+1)^{-1}\bar{c}_1(\bar{\theta}))^j\bar{P}_{k-j}(\bar{\pi})\,. \tag{84}$$

By Theorem 3.1, we have immediately the following

COROLLARY 3.1. *The logarithmic projective Weyl forms are d-closed and correspond to real de Rham cohomology classes. For a fixed logarithmic pair (X, D), these de Rham classes are determined independently of the choice of normal reduced logarithmic projective connections.*

In view of this corollary, we can define *the logarithmic projective Weyl classes* as the de Rham cohomology classes of the logarithmic projective Weyl forms.

REMARK 3. In what follows, the Weyl classes defined in [K3, page 437] are called the *projective* Weyl classes.

Since the logarithmic projective Weyl forms are holomorphic for holomorphic logarithmic projective connections, we have

COROLLARY 3.2. *If a complex manifold admits a holomorphic logarithmic projective connection, then the logarithmic projective Weyl forms are holomorphic and d-closed.*

COROLLARY 3.3. *If a compact (not necessarily Kähler) complex manifold with dimension $n\geq 2$ admits a holomorphic logarithmic projective connection, then all kth logarithmic projective Weyl forms with $2k\geq n$ vanish. If, further, it is of Kähler, then all kth logarithmic projective Weyl forms with $k\geq 1$ vanish.*

PROOF. All kth logarithmic projective Weyl forms are holomorphic $2k$-forms. Therefore if $2k>n$ then the kth projective Weyl form vanishes. Since d-closed holomorphic n-form represents a real de Rham cohomology class only if it represents a zero class, we see that the nth logarithmic projective Weyl class also vanishes. If, further, the manifold is of Kähler then we can apply Hodge theory. Since the logarithmic projective Weyl forms are holomorphic, they are harmonic. On the other hand, the logarithmic projective Weyl classes are real by Corollary 3.1. Therefore they vanish by Hodge theory. ■

4. An example.

In this section, we shall give an example of compact complex manifolds which does admit a holomorphic *logarithmic* projective connection with respect to certain divisor but does not admit any holomorphic projective connection. We take a series of compact (simply connected, non-Kähler) complex 3-folds $\{M_n\}_{n=1}^{\infty}$ given in [K1]. The series of manifolds is constructed by a succession of complex analytic connected sum of copies of M_1. See [K1], [KY] for the details. We recall here briefly the construction of M_1.

Let $\boldsymbol{P}^3$ be a projective space of dimension 3 and $[z_0:z_1:z_2:z_3]$ its standard system of homogeneous coordinates. Let l_0 and l_1 be the two projective lines in $\boldsymbol{P}^3$ defined by $z_0=z_1=0$ and $z_2=z_3=0$. Put $W=\boldsymbol{P}^3-l_0-l_1$. Let α be a constant satisfying $0<|\alpha|<1$, and $g\in PGL(4, \boldsymbol{C})$ the holomorphic automorphism of W defined by

$$[z_0:z_1:z_2:z_3]\mapsto[\alpha z_0:\alpha z_1:z_2:z_3]\,.$$

The quotient space $M=W/\langle g\rangle$ is a compact complex 3-fold with a holomorphic flat projective structure. Define an elliptic fibre bundle structure on M by the projection

$$f\colon M\to\boldsymbol{P}^1\times\boldsymbol{P}^1\,,$$

where f is the induced map of

$$[z_0:z_1:z_2:z_3]\mapsto([z_0:z_1],[z_2:z_3])\,.$$

Take a point p, say $p=([1:0],[1:0])$, on $\boldsymbol{P}^1\times\boldsymbol{P}^1$ and consider the fibre $E:=f^{-1}(p)\simeq\boldsymbol{C}^*/\langle\alpha\rangle$.

Let $x=z_1/z_0$ and $y=z_3/z_2$. Then (x, y) is a system of local coordinates on $\boldsymbol{P}^1\times\boldsymbol{P}^1$ with $p=(0, 0)$. Take a small disk

$$\Delta=\{(x, y)\in\boldsymbol{P}^1\times\boldsymbol{P}^1 : |x|^2+|y|^2<\varepsilon^2\}\,,$$

where $\varepsilon>0$. The map

$$[z_0:z_1:z_2:z_3]\mapsto(x, y, z_0/z_2)$$

induces an isomorphism

$$\psi_1 : f^{-1}(\Delta) \to \Delta \times E .$$

Put $\tilde{V} = (C^2 - \{O\}) \times C$, where $O = (0, 0)$. Let β be the automorphism of $\tilde{V}$ defined by

$$(u_1, u_2, u_3) \mapsto (\beta_0 u_1, \beta_0 u_2, \beta_0^{-1} u_3) ,$$

where $\beta_0 = \exp(2\pi i/a)$, $\alpha = \exp(2\pi i a)$. Denote by $\tilde{S}$ the hypersurface $u_3 = 0$ in $\tilde{V}$. Denote by V the factor space $\tilde{V}/\langle \beta \rangle$ and by S the Hopf surface $\tilde{S}/\langle \beta \rangle$. We indicate by $[u_1, u_2, u_3]$ the point corresponding to $(u_1, u_2, u_3) \in \tilde{V}$. Put $\Delta^* = \Delta - \{p\}$. We consider a holomorphic map

$$\psi_2 : \Delta^* \times E \to V$$

by

$$(x, y, \xi) \mapsto [x\xi^{1/a}, y\xi^{1/a}, (\xi^{1/a})^{-1}] ,$$

which is biholomorphic onto its image. It is easy to see that

$$\psi_2(\Delta^* \times E) = \{[u_1, u_2, u_3] \in V : 0 < (|u_1|^2 + |u_2|^2)|u_3|^2 < \varepsilon^2\}$$

and that

$$\psi_2(\partial \Delta \times E) = \{[u_1, u_2, u_3] \in V : (|u_1|^2 + |u_2|^2)|u_3|^2 = \varepsilon^2\} .$$

Put

$$\tilde{N} = \{(u_1, u_2, u_3) \in \tilde{V} : (|u_1|^2 + |u_2|^2)|u_3|^2 < \varepsilon^2\} ,$$

$$\tilde{N}^* = \{(u_1, u_2, u_3) \in \tilde{V} : (|u_1|^2 + |u_2|^2)|u_3|^2 < \varepsilon^2, u_3 \neq 0\} ,$$

and

$$N = \tilde{N}/\langle \beta \rangle ,$$

$$N^* = \tilde{N}^*/\langle \beta \rangle .$$

Obviously, N is a tubular neighborhood of S and $N^* = N \setminus S$. The manifold M_1 is defined to be the union

$$M_1 = (M \setminus E) \cup N ,$$

where N^* is identified with $f^{-1}(\Delta^*)$ by $\psi := \psi_2 \circ \psi_1$. Thus the elliptic curve E is replaced by the Hopf surface S.

We remark that the logarithmic pair (M_1, S) admits at most one holomorphic logarithmic projective connection. Indeed, the difference of two such connections defines an element of $H^0(M_1, \mathscr{F}^1\langle S \rangle)$. Since

$$H^0(M_1, \mathscr{F}^1\langle S \rangle) \subset H^0(M_1 \setminus S, \mathscr{F}^1\langle S \rangle) \simeq H^0(M \setminus E, \mathscr{F}^1) \simeq H^0(M, \mathscr{F}^1) = 0 ,$$

we see that any two holomorphic logarithmic projective connections coincide. On

$M\setminus E$, we introduce systems of local coordinates that can be defined naturally by the inhomogeneous coordinates on $\boldsymbol{P}^3$ associated with the homogeneous coordinates $[z_0:z_1:z_2:z_3]$. Then we see that the obstruction 1-cocycle which represents $p_{M\setminus E}$ with respect to these systems of local coordinates vanishes. On N, we introduce systems of local coordinates that can be defined naturally by the global coordinates (u_1, u_2, u_3) on $\tilde{V}$. Then the obstruction 1-cocycle which represents $p_N\langle S\rangle$ with respect to these systems of local coordinates vanishes. On the total space $M_1=(M\setminus E)\cup N$, we consider the union of the systems of local coordinates on $M\setminus E$ and N introduced above. Now we shall calculate $p_{M_1}\langle S\rangle$ with respect to these systems of local coordinates. Consider the Mayer-Vietoris exact sequence of cohomology with coefficient $\mathscr{F}^1\langle S\rangle$;

$$\begin{aligned}\cdots \longrightarrow H^0(M_1\setminus S)\oplus H^0(N) \longrightarrow H^0(N^*) \longrightarrow H^1(M_1)\\ \longrightarrow H^1(M_1\setminus S)\oplus H^1(N) \longrightarrow H^1(N^*) \longrightarrow H^2(M_1) \longrightarrow \cdots .\end{aligned}$$

Note that

$$p_{M_1\setminus S}\langle S\rangle = p_{M_1\setminus S}=0\,,$$
$$p_N\langle S\rangle=0\,,$$

and

$$H^0(M_1\setminus S)\simeq H^0(M\setminus E)=0\,.$$

Then we see by the exact sequence above, the obstruction class $p_{M_1}\langle S\rangle\in H^1(M_1)$ can be represented by an element $\bar{p}\in H^0(N^*)$.

LEMMA 4.1. $p_{M_1}\langle S\rangle=0$.

PROOF. To prove the lemma, it is enough to show that $\bar{p}$ extends to an element of $H^0(N)$. The element $\bar{p}$ can be calculated by the identification map ψ. Consider the inhomogeneous coordinates $(z_1/z_0, z_2/z_0, z_3/z_0)$ on $\boldsymbol{P}^3$. Put $x_1=z_1/z_0$, $x_3=z_2/z_0$, and $x_2=z_3/z_0$. We regard (x_1, x_2, x_3) as a system of local coordinates on $M\setminus E$. Then the inverse ϕ of ψ can be written as

$$\begin{cases} x_1=u_1u_3\\ x_2=u_2u_3^{1-a}\\ x_3=u_3^{-a}\,. \end{cases}$$

Then the transition function $\bar{\Phi}$ of $\Theta(-\log S)$ on $(M\setminus E)\cup N$ is given by

$$\bar{\Phi}=\begin{pmatrix} u_3 & 0 & u_1u_3\\ 0 & u_3^{1-a} & (1-a)u_2u_3^{1-a}\\ 0 & 0 & -au_3^{-a}\end{pmatrix},$$

where $\{\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3\}$ and $\{\partial/\partial u_1, \partial/\partial u_2, u_3\partial/\partial u_3\}$ are local frames. Put $\omega_1=du_1$,

$\omega_2 = du_2$, $\omega_3 = du_3/u_3$, $\theta_1 = \partial/\partial u_1$, $\theta_2 = \partial/\partial u_2$, and $\theta_3 = u_3 \partial/\partial u_3$. Then we have

$$\bar{\Phi}^{-1} d\bar{\Phi} = \begin{pmatrix} \omega_3 & 0 & \omega_1 + (1+a)u_1\omega_3 \\ 0 & (1-a)\omega_3 & (1-a)u_2\omega_3 + (1-a)\omega_2 \\ 0 & 0 & -a\omega_3 \end{pmatrix},$$

$$\bar{\sigma} := \frac{1}{4} \operatorname{Trace}(\bar{\Phi}^{-1} d\bar{\Phi}) = \frac{1}{2}(1-a)\omega_3 ,$$

$$\bar{\rho} := \begin{pmatrix} 0 & 0 & \frac{1}{2}(1-a)\omega_1 \\ 0 & 0 & \frac{1}{2}(1-a)\omega_2 \\ 0 & 0 & \frac{1}{2}(1-a)\omega_3 \end{pmatrix},$$

where $\bar{\rho} = (\bar{\sigma}_j \omega^k)$, $\bar{\sigma} = \bar{\sigma}_j \omega^j$. The element $\bar{p}$ which represents $p_{M_1}\langle S \rangle$ is given by $\bar{\Phi}^{-1} d\bar{\Phi} - \bar{\rho} - \bar{\sigma} \cdot I$. Therefore $\bar{p}$ can be written as a tensor field defined on the whole N^* of the form:

$$\begin{aligned} \bar{p} = & \tfrac{1}{2}(1+a)\omega_1 \otimes \omega_3 \otimes \theta_1 + \tfrac{1}{2}(1+a)\omega_3 \otimes \omega_1 \otimes \theta_1 \\ & + \tfrac{1}{2}(1-a)\omega_2 \otimes \omega_3 \otimes \theta_2 + \tfrac{1}{2}(1-a)\omega_3 \otimes \omega_2 \otimes \theta_2 \\ & + (1+a)u_1\omega_3 \otimes \omega_3 \otimes \theta_1 + (1-a)u_2\omega_3 \otimes \omega_3 \otimes \theta_2 - \omega_3 \otimes \omega_3 \otimes \theta_3 . \end{aligned}$$

This shows that $\bar{p}$ extends to an element of $H^0(N)$. ∎

As we see by the construction, each M_n, $n \geq 1$, contains disjoint n Hopf surfaces $S_1, \cdots, S_n$, which are copies of S in M_1, and every S_j, $1 \leq j \leq n$, has a neighborhood which is biholomorphic to N.

Since the connected sum operation preserves the flat projective structure on $M_1 \setminus S$, we have the following.

THEOREM 4.1. *$p_{M_n}\langle D \rangle = 0$ for every $n \geq 1$, where $D = \bigcup_{j=1}^n S_j$.*

LEMMA 4.2. *Each cycle represented by S_j, $1 \leq j \leq n$, is homologous to zero in M_n.*

PROOF. By the proof of [K1, Theorem], we see that a non-singular rational curve (called *line* in [K1]) which has a neighborhood biholomorphic to that of a projective line in $\boldsymbol{P}^3$ in $M_n \setminus \bigcup_{j=1}^n S_j$ generates $H_2(M_n, \boldsymbol{Z})$. From this fact the lemma follows immediately. ∎

COROLLARY 4.1 [KY, Proposition 7]. *All positive dimensional projective Weyl classes of M_n vanish. Namely,*

$$c_2[M_n] = \frac{3}{8} c_1^2[M_n]$$

and

$$c_3[M_n] = \frac{1}{16} c_1^3[M_n] = 4(1-n)$$

for every $n \geq 1$.

PROOF. By Theorems 3.1 and 4.1, we have

$$\bar{P}_j(M_n, D)=0\,, \qquad 1\leq j\leq 3\,. \tag{85}$$

On the other hand, by the exact sequences of sheaves

$$0\longrightarrow \Theta_{M_n}(-\log D)\longrightarrow \Theta_{M_n}\longrightarrow N_D\longrightarrow 0\,,$$

and

$$0\longrightarrow \mathcal{O}_{M_n}\longrightarrow \mathcal{O}_{M_n}([D])\longrightarrow N_D\longrightarrow 0\,,$$

we have

$$ch(\Theta_{M_n}(-\log D))=ch(\Theta_{M_n})-ch(\mathcal{O}_{M_n}[D])+ch(\mathcal{O}_{M_n})\,,$$

where $D=\bigcup_{j=1}^{n} S_j$, and $ch(\mathscr{F})$ indicates the Chern character of a sheaf $\mathscr{F}$. Then the formula

$$ch(\Theta_{M_n}(-\log D))=ch(\Theta_{M_n})$$

follows from Lemma 4.2. Thus we obtain the corollary by (85). ■

REMARK 4. For every $n\geq 1$, M_n does not admit holomorphic projective connections. In fact, if M_n admits a holomorphic projective connection, then it is integrable. Indeed, $M_n\backslash D$ admits one and only one holomorphic projective connection, which is integrable. Therefore the projective Weyl curvature tensor of the holomorphic projective connection vanishes on $M_n\backslash D$ and hence on the whole space by the identity theorem. Thus any holomorphic projective connection on M_n is integrable. Since M_n is simply connected, this implies that M_n is biholomorphic to $\boldsymbol{P}^3$, which is absurd.

REMARK 5. The construction of M and M_1 above can be generalized easily to a higher dimensional case. In this case, M is an infinite cyclic compact quotient space of $\boldsymbol{P}^n\backslash(L_1\cup L_2)$, where L_1 and L_2 are linear subspaces with $\dim L_1+\dim L_2=n-1$. Then M has an elliptic fibre bundle structure on $\boldsymbol{P}^{\dim L_2}\times \boldsymbol{P}^{\dim L_1}$. We construct M_1 by replacing a fibre of M by an $(n-1)$-dimensional Hopf manifold S. Then M_1 is a simply connected compact complex manifold and the pair (M_1, S) admits a holomorphic logarithmic projective connection. Since $M\backslash E$ admits no non trivial holomorphic forms, all logarithmic Weyl forms on M_1 vanish by Corollary 3.2. Moreover S is homologous to zero in M. Therefore we have the equality among (ordinary) Chern classes;

$$c_k[M_1]=\binom{n+1}{k}((n+1)^{-1}c_1[M_1])^k\,, \qquad 1\leq k\leq n\,.$$

Remark 4 applies also in this case. Note that, if $n=3$ and $\dim L_1=0$, M_1 is the Calabi-Eckmann structure on $S^3\times S^3$ (cf. [T]).

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