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On the Intermittency of a Piecewise Linear Map (Takahashi Model)

Makoto MORI

National Defense Academy
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1. Introduction.

We will consider a piecewise linear mapping F from the unit interval $[0, 1]$ into itself for which there exists a sequence $\{c_n\}$ such that $1=c_0>c_1>\cdots\to 0$ and

$$F(x)=\begin{cases}\dfrac{x-c_1}{\psi_0} & \text{if } x\in(c_1, c_0]\,,\\[2ex] \dfrac{x-c_{n+1}}{\psi_n}+c_n & \text{if } x\in(c_{n+1}, c_n]\,,\end{cases}$$

where

$$\begin{aligned}\psi_n= & \quad F'(x)^{-1} \qquad \text{if } x\in(c_{n+1}, c_n)\\ = & \begin{cases}\dfrac{c_0-c_1}{c_0} & \text{if } n=0\,,\\[2ex] \dfrac{c_n-c_{n+1}}{c_{n-1}-c_n} & \text{if } n\geq 1\,.\end{cases}\end{aligned}$$

Y. Takahashi ([7] and [8]) studied the ergodic properties of this mapping by calculating its autocorrelations $\int xF(x)dx$. In this paper, we will consider this problem in more general situation.

This mapping has a fixed point at 0, and there are various cases depending on its property at 0:

1. When $F'(0)>1$, the dynamical system is mixing and the decay rate of correlation is of exponential order. In particular, the central limit theorem holds.
2. When $F(x)-x\sim x^\alpha$ such that $\alpha<5/3$, the dynamical system is still mixing, but the decay rate of correlation is of polynomial order. Nevertheless, the central limit

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theorem holds.

3. When $5/3 \leq \alpha < 2$, the dynamical system is still mixing and the decay rate of correlation is of polynomial order, but the central limit theorem does not hold.
4. Finally, when $\alpha \geq 2$ ($F''(0) < \infty$ if $\alpha = 2$ and $F''(0) = 0$ if $\alpha > 2$). Then it has a σ-finite invariant measure only (A similar result is obtained for some continuous maps by Thaler [10], [11]).

Note that, if $c_n \sim n^{-\gamma}$, then $\alpha = 1 + 1/\gamma$. In this paper we consider "decay rate of correlation" or "central limit theorem" for a certain class of functions, which we will state in §4.

The main tools which we use are, as in the previous papers ([2], [3], [4], [5]), the Perron-Frobenius operator P defined by

$$\int Pf(x)g(x)dx = \int f(x)g(F(x))dx$$

and a renewal equation.

In §2, we summarize the notations, which are almost the same as in the previous papers. In §3 and §4, we calculate the invariant measure and the decay rate of correlation. Finally in §5, we prove the central limit theorem.

2. Fredholm matrix.

Since the mapping F is Markov, the renewal equation can be constructed on the usual symbolic dynamics, although it can for more general mappings as is discussed in [4]. We use the following notations, which are almost the same as in [4].

Let $A = \{0, 1, 2, \cdots\}$ be an index set, that is, A is the set of alphabets. Let

$$\langle n \rangle = (c_{n+1}, c_n],$$

so that

$$cl \bigcup_{n=0}^{\infty} \langle n \rangle = [0, 1],$$

where clJ stands for the closure of a set J. The notations concerning words are slightly different from those in the previous papers. For a word $w = n_1 \cdots n_k$ ($n_i \in A$), we denote

$$|w| = \sum_{i=1}^{k} (n_i + 1),$$

$$\theta w = \begin{cases} (n_1 - 1)n_2 \cdots n_k & \text{if } n_1 \geq 1, \\ n_2 n_3 \cdots n_k & \text{if } n_1 = 0, \end{cases}$$

$$\langle w \rangle = \bigcap_{i=1}^{k} F^{-N_{i-1}} \langle n_i \rangle,$$

$$\psi_w^i = \begin{cases} 1 & \text{if} \quad i=0\,, \\ \prod_{m=1}^{l-1} \prod_{j_m=0}^{n_m} \psi_{j_m} \prod_{j=N_l-i}^{n_l} \psi_j & \text{if} \quad N_{l-1}<i\leq N_l\,, \end{cases}$$

where

$$N_i = \begin{cases} 0 & \text{if} \quad i=0\,, \\ \sum_{j=1}^{i} (n_j+1) & \text{if} \quad i\geq 1\,. \end{cases}$$

Note that

$$\psi_n^i = \begin{cases} 1 & \text{if} \quad i=0\,, \\ \dfrac{c_n-c_{n+1}}{c_{n-i}-c_{n-i+1}} & \text{if} \quad 1\leq i\leq n\,, \\ \dfrac{c_n-c_{n+1}}{c_0} & \text{if} \quad i=n+1\,. \end{cases}$$

We denote by W the set of words w which satisfy $\langle w\rangle \neq \varnothing$. If, as usual, we express the subinterval $(c_1, c_0]$ by R and $[0, c_1]$ by L, a word $w=n_1 \cdots n_k$ in our new notations is expressed as

$$\underbrace{L\cdots L}_{n_1}R\underbrace{L\cdots L}_{n_2}R\cdots\underbrace{L\cdots L}_{n_k}R\,.$$

2-1. Renewal equation. Now we will construct a renewal equation. Since the mapping F is Markov, the Fredholm matrix can be constructed on the usual symbolic dynamics. In [4], we also stated the relations between the Fredholm determinant and the zeta function using the Fredholm matrix. We will state this in the next subsection.

For a subinterval J, a function $g\in L^\infty$ and a complex number $z\in C$, we define

$$s_g^J(z)=\sum_{n=0}^{\infty} z^n \int_J g(F^n(x))dx\,,$$

$$\chi_g^J=\int_J g(x)dx\,,$$

that is,

$$\begin{aligned} s_g^J(z)&=(1_J, g)(z) \\ &=\sum_{n=0}^{\infty} z^n \int 1_J(x)g(F^n(x))dx \end{aligned}$$

$$=\sum_{n=0}^{\infty} z^n \int P^n 1_J(x)g(x)dx$$

$$=\int (I-zP)^{-1} 1_J(x)g(x)dx\,,$$

where 1_J is the indicator function of J. For a word w we denote simply $s_g^w(z)$ and χ_g^w instead of $s_g^{\langle w\rangle}(z)$ and $\chi_g^{\langle w\rangle}$.

To study the properties of the Perron-Frobenius operator, it is enough to construct a renewal equation for $s_g^0(z)$ (cf. [8], [2], [5]):

LEMMA 2-1. (1) *The renewal equation for* $s_g^0(z)$ *has the following form*:

$$s_g^0(z)=D(z)^{-1}\psi_0\left(\sum_{k=0}^{\infty}\frac{c_k}{c_k-c_{k+1}}\chi_g^k-(1-z)\sum_{n=0}^{\infty}z^n\sum_{k=1}^{\infty}\frac{c_{n+k}}{c_k-c_{k+1}}\chi_g^k\right),$$

where

$$D(z)=\frac{1}{c_0}(1-z)\sum_{n=0}^{\infty}c_n z^n\,.$$

We call $D(z)$ *a Fredholm determinant.*

(2) *For any* $w\in W$ $(w\neq 0)$, *we get*

$$s_g^w(z)=\chi_g^w(z)+\psi^w(z)s_g^0(z)\,,$$

where

$$\chi_g^w(z)=\sum_{k=0}^{|w|-2} z^k\psi_w^k\chi_g^{\theta^k w}\,,$$

$$\psi^w(z)=z^{|w|-1}\psi_w^{|w|-1}\,.$$

PROOF. Since

$$s_g^n(z)=\begin{cases}\chi_g^0+\sum_{m=0}^{\infty} z\psi_0 s_g^m(z) & \text{if } n=0\,,\\ \chi_g^n+z\psi_n s_g^{n-1}(z) & \text{if } n\geq 1\,,\end{cases} \tag{2.1}$$

the proof of (1) follows by induction. We can prove (2) in a similar way.

REMARK. (1) Note that

$$\psi^w(1)=\psi_w^{|w|-1}$$

$$=\frac{1}{c_0-c_1}Lebes\langle w\rangle\,.$$

(2) As we will prove in the sequel,

$$\frac{c_k}{c_k - c_{k+1}} \chi_g^k = \begin{cases} \left(\sum_{n=0}^{\infty} c_n\right) \int_{\langle k \rangle} g d\mu & \text{if } \sum_{n=0}^{\infty} c_n < \infty , \\ \int_{\langle k \rangle} g d\mu_\infty & \text{if } \sum_{n=0}^{\infty} c_n = \infty , \end{cases}$$

where μ (resp. μ_∞) is the invariant probability measure (resp. σ-finite invariant measure).

(3) Suppose that $g \in L^\infty$, then

$$\begin{aligned} &\int_{\langle k \rangle} |g| d\mu \leq \| g \|_\infty \frac{c_k}{\sum c_n} & \text{if } \sum_{n=0}^{\infty} c_n < \infty , \\ &\int_{\langle k \rangle} |g| d\mu_\infty \leq \| g \|_\infty c_k , & \text{if } \sum_{n=0}^{\infty} c_n = \infty , \end{aligned} \tag{2.2}$$

where by $\| g \|_\infty$ we denote the L^∞ norm.

2-2. Fredholm determinant and zeta function. Now we can compute the Fredholm determinant $D(z)$ and the zeta function

$$\zeta(z) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{\substack{p : F^n(p) = p \\ p \in [0, 1]}} \log | F^{n\prime}(p) | \right) ,$$

using the results in [4]. It turns out that there appears a difference between them, which comes from the following facts:

(1) To calculate $D(z)$, we treat F as a mapping from $(0, 1]$ into itself.

(2) The point $x = 0$ is a fixed point of F.

THEOREM 2-2. *Assume that* $\psi = \lim_{n \to \infty} \psi_n$ *exists. Then the zeta function is of the form*:

$$\zeta(z)^{-1} = (1 - \psi z) D(z) .$$

PROOF. Let F_N be a piecewise linear transformation of the form:

$$F_N(x) = \begin{cases} F(x) & \text{if } x \in (c_N, c_0] , \\ \dfrac{c_{N-1}}{c_N} x & \text{if } x \in [0, c_N] . \end{cases}$$

Then by Theorem B in [4], we get

$$\zeta_N(z)^{-1} = \det(I - \Phi_N(z)) ,$$

where $\zeta_N(z)$ and $\Phi_N(z)$ are the zeta function and the Fredholm matrix corresponding to F_N, respectively. The Fredholm matrix $\Phi_N(z)$ is a $2N$ square matrix which is expressed in terms of signed symbolic dynamics as:

$$\Phi_N(z)_{a^\sigma, b^\tau} = z\psi_{|a|}\sigma(\delta[b^\tau < \theta a^\sigma] - 1/2) \qquad a, b \in A\,, \quad \sigma, \tau \in \{+, -\}\,,$$

where

$$\delta[L] = \begin{cases} 1 & \text{if the statement } L \text{ is true}\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

(for details, see [4] §3). Since $\zeta_N(z)$ converges to $\zeta(z)$ in $|z| < 1$, we only need to calculate $\det(I - \Phi_N(z))$. We get inductively

$$\det(I - \Phi_N(z)) = 1 - z(\psi_1 + \psi_N) - \sum_{i=2}^{n-1} z^i(\psi_i - \psi_N) \prod_{j=1}^{i-1} \psi_j\,.$$

This shows

$$\lim_{N \to \infty} \det(I - \Phi_N(z)) = (1 - \psi z)D(z)\,,$$

and the proof is complete.

3. Invariant measure.

In this section, we will construct the invariant measure which is absolutely continuous with respect to the Lebesgue measure. When $\sum_{n=0}^{\infty} c_n = \infty$, the dynamical system has a σ-finite invariant measure only which corresponds to the case in Thaler [10], [11]. We also calculate the decay rate of correlation.

LEMMA 3-1. (1) *Suppose that* $\sum_{k=0}^{\infty} c_k < \infty$. *Then there exists a unique absolutely continuous invariant measure* μ *and its density* $\rho(x)$ *is given by*

$$\rho(x) = \frac{c_n}{(c_n - c_{n+1}) \sum c_k} \qquad \textit{if} \quad x \in \langle n \rangle\,.$$

(2) *On the contrary, suppose that* $\sum_{k=0}^{\infty} c_k = \infty$. *Then there exists no finite invariant measure which is absolutely continuous with respect to the Lebesgue measure, but there exists a* σ*-finite invariant measure* μ_∞ *whose density is given by*

$$\rho_\infty(x) = \frac{c_n}{c_n - c_{n+1}} \qquad \textit{if} \quad x \in \langle n \rangle\,.$$

PROOF. The density $\rho(x)$ of the invariant measure μ is an eigenfunction corresponding to the eigenvalue 1 of the Perron-Frobenius operator P, that is, it satisfies $P\rho = \rho$. Therefore

$$(\rho, g)(z) = \sum_{n=0}^{\infty} z^n \int \rho(x) g(F^n(x)) dx$$

$$= \sum_{n=0}^{\infty} z^n \int P^n \rho(x) g(x) dx$$

$$= \frac{\int \rho(x) g(x) dx}{1-z} . \tag{3.1}$$

On the other hand, if we express $\rho(x) = \sum_{w \in W} C_w 1_w(x)$, then we get

$$(\rho, g)(z) = \sum_{w \in W} C_w s_g^w(z)$$

$$= \sum_{w \in W} C_w (\chi_g^w(z) + \psi^w(z) s_g^0(z)) . \tag{3.2}$$

Hence for any $g_1, g_2 \in L^\infty$, applying (3.1) and (3.2) to $\lim_{z \uparrow 1} (\rho, g_1)(z)/(\rho, g_2)(z)$, we get

$$\frac{\int g_1(x)\rho(x)dx}{\int g_2(x)\rho(x)dx} = \lim_{z \uparrow 1} \frac{\sum_{w \in W} (C_w \chi_{g_1}^w(z) + \psi^w(z) s_{g_1}^0(z))}{\sum_{w \in W} C_w (\chi_{g_2}^w(z) + \psi^w(z) s_{g_2}^0(z))}$$

$$= \lim_{z \uparrow 1} \frac{\sum_{w \in W} C_w \psi^w(z) \sum_k c_k \chi_{g_1}^k/(c_k - c_{k+1})}{\sum_{w \in W} C_w \psi^w(z) \sum_k c_k \chi_{g_2}^k/(c_k - c_{k+1})}$$

$$= \frac{\sum_k \int_{\langle k \rangle} g_1(x) c_k/(c_k - c_{k+1}) dx}{\sum_k \int_{\langle k \rangle} g_2(x) c_k/(c_k - c_{k+1}) dx} .$$

This proves the lemma.

COROLLARY 3-2. (1) *Suppose that* $\sum_{n=0}^{\infty} c_n < \infty$. *Then we get for any* g *with* $\int |g| d\mu < \infty$

$$s_g^0(z) = \frac{\int 1_{\langle o \rangle}(x)dx \cdot \int g(x) d\mu}{(1-z)} + \frac{\sum \tilde{c}_n z^n}{\sum c_n z^n} \int 1_{\langle o \rangle}(x) dx \cdot \int g \, d\mu$$

$$- \frac{\sum c_n}{\sum c_n z^n} \sum_{n=0}^{\infty} z^n \sum_{k=1}^{\infty} \frac{c_{n+k}}{c_k} \int_{\langle k \rangle} g \, d\mu , \tag{3.3}$$

where

$$\tilde{c}_n = \sum_{k=n+1}^{\infty} c_k .$$

(2) *Suppose that* $\sum_{n=0}^{\infty} c_n = \infty$. *Then we get for* $\int |g| d\mu_\infty < \infty$

$$\left(\sum_{n=0}^{\infty} c_n z^n \right) s_g^0(z) = \frac{\int 1_{\langle o \rangle}(x) dx \cdot \int g(x) d\mu_\infty}{1-z}$$

$$- \sum_{n=0}^{\infty} z^n \sum_{k=1}^{\infty} \frac{c_{n+k}}{c_k} \int 1_{\langle o \rangle}(x) dx \cdot \int_{\langle k \rangle} g \, d\mu_\infty . \tag{3.4}$$

We can get the proof by combining Lemma 2-1 and Lemma 3-1.

This corollary suggests that for the case (1) the dynamical system is mixing and

$$\int f(x)g(F^n(x))dx - \int f\,dx \cdot \int g\,d\mu$$

decays with the order $\tilde{c}_n$. On the contrary in the case (2), $\int f(x)g(F^n(x))dx$ converges to 0 as $n \to \infty$ but

$$\lim_{n\to\infty} d_n \int f(x)g(F^n(x))dx = \int f\,dx \cdot \int g\,d\mu_\infty ,$$

where the sequence d_n is defined by

$$\sum_{n=0}^{\infty} \frac{z^n}{d_n} = \frac{1}{(1-z)\sum c_n z^n} .$$

We will state the general theory and also give the detail in the next section.

4. Decay rate of correlation.

As we have stated in §3, we will show that the dynamical system is mixing and that the decay rate of correlation is determined by c_n. First we will consider the case when $\sum_{n=0}^{\infty} c_n < \infty$. For the simplest cases, we get:

THEOREM 4-1. *Assume that the radius of convergence of the sequence* $\sum_{n=0}^{\infty} c_n z^n$ *is greater than* 1, *and that* $F'(0) > 1$. *Then we get*:

(1) $$Spec(F) \cap \{z : |z| > e^{-\xi}\} = \{z^{-1} : D(z) = 0, |z| < e^{\xi}\} ,$$

where Spec(F) is the spectrum of the Perron-Frobenius operator P restricted to BV, the set of functions with bounded variation.

(2) *The dynamical system is mixing and for any* $f \in BV$ *and* $g \in L^\infty$ *the decay rate of the correlation is the reciprocal of the smallest zero in modulus of* $\sum_{n=0}^{\infty} c_n z^n$, *that is, let* η *be the smallest zero in modulus. Then*

$$\int f(x)g(F^n(x))dx - \int f(x)dx \cdot \int g(x)d\mu \sim \eta^{-n} \qquad as \quad n \to \infty.$$

PROOF. These are the direct consequences of Corollary 3-2 and Theorem A in [5].

Now we will proceed to our main aim. Let the radius of convergence of $\sum_{n=0}^{\infty} c_n z^n$ equal 1. Then, as we stated in §3, we will show that the decay rate of correlation for a certain class of functions equals $\tilde{c}_n$.

DEFINITION. Let $\{a_n\}_{n=1}^{\infty}$ be a positive sequence. Then we say that a function f belongs to $Class(a_n)$ or $class(a_n)$, if there exists a decomposition of $f(x) = \sum_{w \in W} C_w 1_w(x)$

such that

$$\sum_{|w|=n} C_w \psi^w(1) \sim \begin{cases} O(a_n) & \text{if } f \in Class(a_n), \\ o(a_n) & \text{if } f \in class(a_n). \end{cases} \tag{4.1}$$

THEOREM 4-2. *Suppose that c_n is of order $n^{-\gamma}$ with $\gamma > 0$. Then $BV \subset Class(n^{-\gamma-1})$.*

PROOF. Assume that $f \in BV$ is monotone increasing. Now we express a word as $w = k_1 \cdots k_i$ such that $k_1 + \cdots + k_i = n - i$ and get

$$\sum_{|w|=n} C_w \psi^w(1) = \sum_{i=1}^{\infty} \sum_{k_1+\cdots+k_i=n-i} C_{k_1\cdots k_i} \psi^{k_1\cdots k_i}(1).$$

Since f is monotone increasing, we get

$$\sum_{i=1}^{\infty} \sum_{k_1+\cdots+k_i=n-i} C_{k_1\cdots k_i} \le var(f),$$

where by $var(f)$ we denote the variation of f. On the other hand,

$$\begin{aligned} \psi^{k_1\cdots k_i}(1) &= \prod_{j=1}^{i} \frac{c_{k_j} - c_{k_j+1}}{c_0} \frac{c_0}{c_0 - c_1} \\ &\le \left[\sup_{m \ge 1} \frac{c_m - c_{m+1}}{c_0} \right]^{i-1} \frac{c_{n-i} - c_{n-i+1}}{c_0 - c_1}. \end{aligned}$$

Since $\sup_m (c_m - c_{m+1})/c_0 < 1$, for sufficiently large n there exists a constant K such that

$$\psi^{k_1\cdots k_i}(1) \le \frac{K}{n^{\gamma+1}},$$

because $c_n - c_{n+1} \sim n^{-\gamma-1}$. Combining the results, we get

$$\sum_{|w|=n} C_w \psi^w(1) \le \frac{K\, var(f)}{n^{\gamma+1}}.$$

This proves the theorem.

THEOREM 4-3. *Suppose that $\sum_{n=0}^{\infty} c_n < \infty$. Then the dynamical system is mixing, and for a function $f \in Class(c_n)$ and $g \in L^{\infty}$, the decay rate of correlation equals $\tilde{c}_n = \sum_{k=1}^{\infty} c_{n+k}$.*

EXAMPLE 4-4. (1) Suppose that c_n is of order $n^{-\gamma}$ ($\gamma > 1$, that is $\alpha < 2$). Then for functions $f \in Class(n^{-\gamma})$ and $g \in L^{\infty}$, the decay rate of correlation equals $n^{-\gamma+1}$.

(2) Suppose that $c_n \sim \{n(\log n)^s\}^{-1}$ for $s > 1$. Then for functions $f \in Class(c_n)$ and $g \in L^{\infty}$ the decay rate of the correlation equals $(\log n)^{-s+1}$.

We use the following lemmas.

LEMMA 4-5. *Let $g \in L^{\infty}$. Then*

$$\sum_{k=1}^{\infty} \frac{c_{n+k}}{c_k} \int_{\langle k \rangle} g \, d\mu \sim \tilde{c}_n . \tag{4.2}$$

PROOF. By (2.2), we get

$$\left| \sum_{k=1}^{\infty} \frac{c_{n+k}}{c_k} \int_{\langle k \rangle} g \, d\mu \right| \leq \sum_{k=1}^{\infty} c_{n+k} \frac{\| g \|_{\infty}}{\sum c_m}$$
$$= \bar{c}_n \frac{\| g \|_{\infty}}{\sum c_m} .$$

This proves (4.2).

LEMMA 4-6. *Suppose that $f \in Class(c_n)$. Then for $g \in L^{\infty}$, the both n-th coefficient of*

$$\sum_{w \in W} C_w \chi_g^w(z) \tag{4.3}$$

and

$$\sum_{w \in W} C_w \frac{\psi^w(z) - \psi^w(1)}{1 - z} \tag{4.4}$$

are of order $\tilde{c}_n$.

PROOF. Since

$$\left| \sum_{w \in W} C_w \chi_g^w(z) \right| = \left| \sum_{w \in W} C_w (\chi_g^w + z \psi_w^1 \chi_g^{\theta w} + \cdots + z^{|w|-1} \psi_w^{|w|-1} \chi_g^{\theta^{|w|-1} w}) \right|$$
$$\leq \sum_{n=1}^{\infty} |z|^n \sum_{|w| \geq n} |C_w| \, |\psi^w(1)| \, \| g \|_{\infty} (c_0 - c_1) ,$$

by the assumption that f belongs to $Class(c_n)$, this proves that the n-th coefficient of (4.3) is of order $\tilde{c}_n$, and we can show in a same way that the n-th coefficient of (4.4) is also of order $\tilde{c}_n$.

Now we will prove Theorem 4-3. We denote $f = \sum_{w \in W} C_w 1_w$. Then, noticing the fact that

$$\sum_{w \in W} C_w \psi^w(1) \int 1_{\langle 0 \rangle} dx = \int f \, dx ,$$

we get

$$(f, g)(z) = \sum_{w \in W} C_w s_g^w(z)$$

$$= \sum_{w \in W} C_w\{\chi_g^w(z) + \psi^w(z)s_g^0(z)\}$$

$$= \sum_{w \in W} C_w\chi_g^w(z) + \left[\frac{1}{1-z} + \frac{\sum \tilde{c}_n}{\sum c_n z^n}\right]\int f\,dx \cdot \int g\,d\mu$$

$$-\int f\,dx \frac{\sum c_n}{\sum c_n z^n}\frac{1}{c_0 - c_1}\sum_{n=0}^{\infty} z^n \sum_{k=1}^{\infty}\frac{c_{n+k}}{c_k}\int_{\langle k\rangle} g\,d\mu$$

$$+ \sum_{|w| \in W} C_w(\psi^w(z) - \psi^w(1))\int 1_{\langle 0\rangle}\,dx$$

$$\cdot\left(\left[\frac{1}{1-z} + \frac{\sum \tilde{c}_n z^n}{\sum c_n z^n}\right]\int g\,d\mu - \frac{\sum c_n}{\sum c_n z^n}\sum_{n=0}^{\infty} z^n \sum_{k=1}^{\infty}\frac{c_{n+k}}{c_k}\int_{\langle k\rangle} g\,d\mu\right).$$

Therefore,

$$(f, g)(z) = \frac{\int f\,dx \cdot \int g\,d\mu}{1-z} + R(f, g)(z)\,, \tag{4.5}$$

where

$$R(f, g)(z) = R_1(f, g)(z) + R_2(f, g)(z) + R_3(f, g)(z) + R_4(f, g)(z)\,,$$

$$R_1(f, g)(z) = \frac{\sum \tilde{c}_n z^n}{\sum c_n z^n}\int f\,dx \cdot \int g\,d\mu\,,$$

$$R_2(f, g)(z) = \sum_{w \in W} C_w\chi_g^w(z) + \sum_{w \in W} C_w\frac{\psi^w(z) - \psi^w(1)}{1-z}\int 1_{\langle 0\rangle}\,dx \cdot \int g\,d\mu\,,$$

$$R_3(f, g)(z) = -\int f\,dx \frac{\sum c_n}{\sum c_n z^n}\sum_{n=0}^{\infty} z^n \sum_{k=1}^{\infty}\frac{c_{n+k}}{c_k}\int_{\langle k\rangle} g\,d\mu\,,$$

$$R_4(f, g)(z) = \sum_{|w| \in W} C_w(\psi^w(z) - \psi^w(1))\int 1_{\langle 0\rangle}\,dx$$

$$\cdot\left(\frac{\sum \tilde{c}_n z^n}{\sum c_n z^n}\int g\,d\mu - \frac{\sum c_n}{\sum c_n z^n}\sum_{n=0}^{\infty} z^n \sum_{k=1}^{\infty}\frac{c_{n+k}}{c_k}\int_{\langle k\rangle} g\,d\mu\right).$$

If $\sum c_n < \infty$, then for a function $f = 1_w$ with some word w and for $g \in L^\infty$, the above equation (4.5) shows $\int f(x)g(F^n(x))dx$ converges to $\int f\,dx \int g\,d\mu$. Hence the dynamical system is mixing. By Lemma 4-6 and Lemma 4-5, the n-th coefficients of $R_2(f, g)(z)$ and $R_3(f, g)(z)$ are of order $\tilde{c}_n$. Moreover, it is easy to see that $R_1(f, g)(z)$ is of order $\tilde{c}_n$ and $R_4(f, g)(z)$ is of small order. This proves the theorem.

Proof of Corollary 4-4. Suppose that c_n is of order $n^{-\gamma}$ with $\gamma > 1$. Then by Theorem 4-3, the dynamical system is mixing and since $\tilde{c}_n$ is of order $n^{-\gamma+1}$, the decay rate of correlation equals $n^{-\gamma+1}$. This proves (1). To prove (2), we only need to show

$$\lim_{n\to\infty} \tilde{c}_n(\log n)^t = \begin{cases} +\infty & \text{if } t>s-1\,, \\ -\infty & \text{if } t<s-1\,. \end{cases} \tag{4.6}$$

If $t>s-1$, then

$$\begin{aligned} \tilde{c}_n(\log n)^t &\sim \sum_{k=1}^{\infty} \frac{(\log n)^t}{(1+k/n)(\log(n+k))^s}\frac{1}{n} \\ &\leq K(\log n)^{t-s}\sum_{k=1}^{n} \frac{1}{1+k/n}\frac{1}{n} \\ &\sim (\log n)^{t-s+1} \end{aligned}$$

for some constant K. This proves the upper half of (4.6). For the rest half, we appeal to the fact:

LEMMA 4-7. *For α, β such that $0<\alpha$, $\beta<1$ and $\alpha+\beta=1$, we get for $x, y>0$*

$$x+y \geq x^\alpha y^\beta\,.$$

The proof follows from an elementary calculation.

This lemma shows that for $0<t<s$

$$\{\log(nm)\}^s \geq (\log n)^t(\log m)^{s-t}\,.$$

Then

$$\tilde{c}_n \sim \sum_{k=1}^{\infty} \frac{1}{(1+k/n)(\log n(1+k/n))^s}\frac{1}{n}\,. \tag{4.7}$$

Therefore by Lemma 4-7, for $t<s-1$ the right hand term of (4.7) is less than or equal to

$$\sum_{k=1}^{\infty} \frac{1}{(1+k/n)(\log n)^t(\log(1+k/n))^{s-t}}\frac{1}{n} \sim \frac{1}{(\log n)^t}\int_1^\infty \frac{1}{x(\log x)^{s-t}}dx\,.$$

Since for $\alpha>1$

$$\int_1^\infty \frac{1}{x(\log x)^\alpha}dx<\infty\,,$$

this proves the latter part of (4.6). Therefore the proof of Corollary 4-4 is completed.

We now consider the case $\sum_{n=0}^{\infty} c_n=\infty$. In this case, as we mentioned in Corollary 3-2, $\int f(x)g(F^n(x))\,dx$ converges to 0.

THEOREM 4-8. *Assume that $\sum_{n=0}^{\infty} c_n=\infty$. Define d_n by*

$$\sum_{n=0}^{\infty}\frac{z^n}{d_n}=\frac{1}{(1-z)\sum c_n z^n},$$

and let $\tilde{d}_n$ *be any sequence such that* $\lim_{n\to\infty} n\tilde{d}_n d_n=0$. *Then for* $f\in Class(\tilde{d}_n)$ *and* $g\in L^{\infty}$ *with* $\int|g|\,d\mu_{\infty}<\infty$, *we obtain*

$$\lim_{n\to\infty} d_n\int f(x)g(F^n(x))dx=\int f\,dx\cdot\int g\,d\mu_{\infty}\,.$$

PROOF.

$$\sum_{n=0}^{\infty} z^n\int f(x)g(F^n(x))dx=\sum_{w\in W} C_w s_g^w(z)$$

$$=\sum_{w\in W} C_w\{\chi_g^w(z)+\psi^w(z)s_g^0(z)\}$$

$$=\sum_{w\in W} C_w\chi_g^w(z)+\sum_{w\in W} C_w\psi^w(z)\frac{c_0-c_1}{\sum c_n z^n}\left(\frac{\int g\,d\mu_{\infty}}{1-z}-\sum_{n=0}^{\infty} z^n\sum_{k=1}^{\infty}\frac{c_{n+k}}{c_k}\int_{\langle k\rangle} g\,d\mu_{\infty}\right)$$

$$=\sum_{w\in W} C_w\chi_g^w(z)+\sum_{w\in W} C_w\frac{(\psi^w(z)-\psi^w(1))}{1-z}\frac{c_0-c_1}{\sum c_n z^n}\int g\,d\mu_{\infty}$$

$$+\int f\,dx\cdot\int g\,d\mu_{\infty}\sum_{n=0}^{\infty}\frac{z^n}{d_n}+\text{small order}\,. \tag{4.8}$$

Since by the assumption that $f\in Class(\tilde{d}_n)$, the n-th coefficient of $\sum_{w\in W} C_w\chi_g^w(z)$ is of order $n\tilde{d}_n$. Therefore the first term of the right hand term of (4.8) is of smaller order than the third term. The second term of (4.8) is also of smaller order than the third term, and this proves the theorem.

Note that when $c_n\sim 1/n$ (i.e. $\gamma=1$ or, in other words $\alpha=2$), then $d_n\sim\log n$. This corresponds to the case in Thaler [10]:

$$F(x)=\begin{cases}\dfrac{x}{1-x} & 0\le x\le 1/2\,,\\ 2x-1 & 1/2<x\le 1\,.\end{cases}$$

5. Central limit theorem.

In [1] and [6], the validity of the central limit theorem for a function h was characterized by the perturbed Perron-Frobenius operator P_{th}:

$$P_{th}f(x)=\sum_{y:F(y)=x} e^{ith(y)}|F'(y)|^{-1}f(y)$$

$$=P(e^{ith}f)(x)\,,$$

because the characteristic function $S_nh(x)=\sum_{k=0}^{n-1}h(F^k(x))$ is expressed by

$$\int e^{itS_nh(x)}\,d\mu=\int P_{th}^n\rho(x)dx\,.$$

We will study the central limit theorem by appealing to the renewal equation (4.5).

LEMMA 5-1. *Let us denote the characteristic functions of S_nh-nm by $\phi_n(t)$. Then it is expressed in the form*:

$$\sum_{n=0}^{\infty} z^n\phi_n(t)=\frac{1}{1-z}+\frac{z}{(1-z)^2}\int h_t d\mu+\frac{z}{(1-z)^2}\int \hat{\phi}_z^{h_t}(x)d\mu\,, \tag{5.1}$$

where

$$m=\int h\,d\mu\,,$$

$$h_t(x)=e^{it(h(x)-m)}-1\,,$$

$$\hat{\phi}_z^g(x)=\sum_{l=1}^{\infty}\sum_{j_1=1}^{\infty} z^{j_1}\cdots\sum_{j_l=1}^{\infty} z^{j_l}g(x)g(F^{j_1}(x))\cdots g(F^{j_1+\cdots+j_l}(x))\,.$$

PROOF.

$$\sum_{n=0}^{\infty} z^n\phi_n(t)=\sum_{n=0}^{\infty} z^n\int e^{it(S_nh(x)-nm)}\,d\mu$$

$$=\sum_{n=0}^{\infty} z^n\int\left(\prod_{k=0}^{n-1} e^{it(h-m)\circ F^k}\right)(x)\,d\mu$$

$$=\sum_{n=0}^{\infty} z^n\int\left[\prod_{k=0}^{n-1}\{(e^{it(h-m)\circ F^k}-1)(x)+1\}\right]d\mu$$

$$=\frac{1}{1-z}+\frac{z}{(1-z)^2}\int h_t\,d\mu$$

$$+\frac{z}{(1-z)^2}\sum_{l=1}^{\infty}\sum_{j_1=1}^{\infty} z^{j_1}\sum_{j_2=1}^{\infty} z^{j_2}\cdots\sum_{j_l=1}^{\infty} z^{j_l}\int h_t(x)\cdots h_t(F^{j_1+\cdots+j_l}(x))\,d\mu$$

$$=\frac{1}{1-z}+\frac{z}{(1-z)^2}\int h_t\,d\mu+\frac{z}{(1-z)^2}\int \hat{\phi}_z^{h_t}(x)\,d\mu\,.$$

This proves Lemma.

DEFINITION. In the present paper we say that a power series $f(z,t)=\sum_{n=0}^{\infty}a_n(t)z^n$ is of small order in (z,t) when $\lim_{n\to\infty}a_n(\theta/\sqrt{n})=0$ for any fixed θ.

We need the following lemmas.

LEMMA 5-2. *Suppose that* $c_n \sim n^{-\gamma}$ $(\gamma > 1)$. *Then for* $h \in L^\infty$

$$\int h_t\,d\mu \frac{\sum c_n}{\sum c_n z^n} \sum_{n=0}^{\infty} z^n \sum_{k=1}^{\infty} \frac{c_{n+k}}{c_k} \int_{\langle k \rangle} h_t\,d\mu$$

is of small order in (z, t).

PROOF. The proof follows from Lemma 4-5.

LEMMA 5-3. (1) *Let real numbers* p, q *satisfy* $p \geq q$. *Define* r_n *by*

$$\sum_{n=1}^{\infty} r_n z^n = \sum_{n=1}^{\infty} n^p z^n \sum_{n=1}^{\infty} n^q z^n .$$

Then

$$r_n \sim \begin{cases} n^p & q < -1 \\ n^p \log n & q = -1 \\ n^{p+q+1} & q > -1 . \end{cases} \tag{5.2}$$

(2) *Suppose that*

$$\sum_{n=1}^{\infty} r_n z^n = \sum_{n=1}^{\infty} n z^n \sum_{n=1}^{\infty} n^q L(n) z^n ,$$

where $L(n)$ *is monotone decreasing and* $\lim_{n\to\infty} L(n) = 0$, *and* $q > -1$. *Then* $r_n = o(n^{2+q})$.

PROOF. It follows from the elementary calculation.

Now we will calculate $\int \hat{\phi}_z^g\,d\mu$. Since

$$\begin{aligned} \hat{\phi}_z^g(x) &= \sum_{j=1}^{\infty} z^j g(x) g(F^j(x)) + \sum_{j_1}^{\infty} z^{j_1} g(x) \sum_{l=2}^{\infty} z^{j_2+\cdots+j_l} (g \cdot g \circ F^{j_2} \cdots g \circ F^{j_2+\cdots+j_l})(F^{j_1}(x)) \\ &= \sum_{j=0} z^j g(x) g(F^j(x)) - g^2(x) + \sum_{j=0}^{\infty} z^j g(x) \hat{\phi}_z^g(F^j(x)) - g(x)\hat{\phi}_z^g(x) , \end{aligned}$$

we get

$$(1 + g(x))\hat{\phi}_z^g(x) = \sum_{j=0} z^j g(x) g(F^j(x)) - g^2(x) + \sum_{j=0}^{\infty} z^j g(x) \hat{\phi}_z^g(F^j(x)) .$$

Therefore, by (4.5)

$$\begin{aligned} \int \hat{\phi}_z^g\,d\mu &= \left(\frac{g}{1+g}\rho, g\right)(z) - \int \frac{g^2}{1+g}\,d\mu + \left(\frac{g}{1+g}\rho, \hat{\phi}_z^g\right) \\ &= \frac{1}{1-z} \int \frac{g}{1+g}\,d\mu \cdot \int g\,d\mu + R\left(\frac{g}{1+g}\rho, g\right)(z) - \int \frac{g^2}{1+g}\,d\mu \end{aligned}$$

$$+\frac{1}{1-z}\int\frac{g}{1+g}d\mu\cdot\int\hat{\phi}_z^g d\mu+R\left(\frac{g}{1+g}\rho,\hat{\phi}_z^g\right),$$

that is,

$$\left(1-\frac{1}{1-z}\int\frac{g}{1+g}d\mu\right)\int\hat{\phi}_z^g d\mu=\frac{1}{1-z}\int\frac{g}{1+g}d\mu\cdot\int g d\mu+R\left(\frac{g}{1+g}\rho,g\right)(z)$$
$$-\int\frac{g^2}{1+g}d\mu+R\left(\frac{g}{1+g}\rho,\hat{\phi}_z^g\right). \tag{5.3}$$

Therefore, substituting (5.3) to (5.1), we get

$$\sum_{n=0}^{\infty}z^n\phi_n(t)=\frac{1}{1-z}+\frac{z}{(1-z)^2}\int h_t d\mu$$
$$+\frac{z}{(1-z)^2}\left[\frac{1}{1-z}\int\frac{h_t}{1+h_t}d\mu\cdot\int h_t d\mu+R\left(\frac{h_t}{1+h_t}\rho,h_t\right)(z)\right.$$
$$\left.-\int\frac{h_t^2}{1+h_t}d\mu+R\left(\frac{h_t}{1+h_t}\rho,\hat{\phi}_z^{h_t}\right)\right]\left(1-\frac{1}{1-z}\int\frac{h_t}{1+h_t}d\mu\right)^{-1}$$
$$=\frac{1}{1-z}+\frac{z}{(1-z)^2}\int h_t d\mu$$
$$+\frac{z}{(1-z)^2}\left[\frac{1}{1-z}\left(\int h_t d\mu\right)^2+R(h_t\rho,h_t)(z)\right]\left(1-\frac{\int h_t d\mu}{1-z}\right)^{-1}$$
$$+\text{small order in }(z,t)$$
$$=\frac{1}{(1-z)\{1-z\int h_t d\mu/(1-z)\}}$$
$$+\frac{z}{(1-z)^2}R(h_t\rho,h_t)(z)\left(1-\frac{1}{1-z}\int h_t d\mu\right)^{-1}+\text{small order in }(z,t). \tag{5.4}$$

Appealing to Lemma 5-2 and Lemma 5-3, we get if $\gamma>3/2$ and if $h\in L^\infty\cap$ *class*(n^{-1}),

$$\sum_{n=0}^{\infty}z^n\phi_n(t)=\frac{1}{(1-z)\{1-z\int h_t d\mu/(1-z)\}}+\text{small order in }(z,t)$$
$$=\frac{1}{1-z}\sum_{k=0}^{\infty}\left(\frac{z\int h_t d\mu}{1-z}\right)^k+\text{small order in }(z,t)$$
$$=\sum_{n=1}^{\infty}z^n\sum_{k=0}^{n}\left(\int h_t d\mu\right)^k\frac{n\cdots(n-k+1)}{k!}+\text{small order in }(z,t). \tag{5.5}$$

Therefore, when we put $t=\theta/\sqrt{n}$, the n-th coefficient of (5.5) converges to $\exp[-\theta^2 v/2]$, where $v=\int(h-m)^2\,d\mu$. Hence the central limit theorem holds for the case where $\gamma>3/2$ and $h\in L^\infty\cap class(n^{-1})$.

Now we will consider the case $\gamma\le 3/2$. Set

$$\hat{c}_n=\lim_{t\to 0}\frac{1}{t}\sum_{k=1}^{\infty}\frac{c_{n+k}}{c_k}\int_{\langle k\rangle}h_t\,d\mu$$
$$=\frac{1}{\sum c_m}\lim_{t\to 0}\frac{1}{t}\sum_{k=1}^{\infty}c_{n+k}\frac{\int_{\langle k\rangle}h_t\,dx}{c_k-c_{k+1}}.$$

If the order of $\hat{c}_n$ is greater than or equal to $n^{-1/2}$, then, by Lemma 5-3 (1), the term

$$\frac{z}{(1-z)^2}R_3(h_t\rho,h_t)(z)\left(1-\frac{\int h_t\,d\mu}{1-z}\right)^{-1}$$
$$=-\frac{z}{(1-z)^2}\int h_t\,d\mu\frac{\sum c_n}{\sum c_n z^n}\sum_{n=0}^{\infty}z^n\sum_{k=1}^{\infty}\frac{c_{n+k}}{c_k}\int_{\langle k\rangle}h_t\,d\mu\sum_{k=0}^{\infty}\left(\frac{\int h_t\,d\mu}{1-z}\right)^k$$
$$\sim-\frac{z}{(1-z)^2}\sum_{n=0}^{\infty}z^n\hat{c}_n t\int h_t\,d\mu\sum_{k=0}^{\infty}\left(\frac{\int h_t\,d\mu}{1-z}\right)^k \tag{5.6}$$

is no more of small order in (z,t). Because if the order of $\hat{c}_n$ is greater than or equal to $n^{-1/2}$, the n-th coefficient of (5.6) does not vanish when we put $t=\theta/\sqrt{n}$ and taking $n\to\infty$. In particular, if the order of $\hat{c}_n$ is greater than $n^{-1/2}$, then the n-th coefficient of (5.6) diverges, and if $\hat{c}_n$ is of order $n^{-1/2}$, it approximately equals

$$c\sum_{k=0}^{\infty}\frac{(\theta^2 v/2)^{k+3/2}}{k!(k+3/2)}\exp\left(-\frac{\theta^2 v}{2}\right),$$

if $c=\lim_{n\to\infty}\hat{c}_n n^{1/2}(2/v)^{3/2}$ exists. Therefore the central limit theorem does not hold. On the contrary, suppose that $\hat{c}_n=o(n^{-1/2}L(n))$, with $L(n)$ monotone decreasing to 0. Then, by Lemma 5-3 (2), the central limit theorem holds.

Summarizing the results stated above, we get:

THEOREM 5-4. *Let $h\in L^\infty\cap class(n^{-1})$.*

(1) *Suppose that $\gamma>3/2$, or $\hat{c}_n=n^{-1/2}L(n)$ for which $L(n)$ is monotone decreasing to* 0. *Then*

$$\frac{1}{\sqrt{n}}(S_n h(x)-nm)=\frac{1}{\sqrt{n}}\left(\sum_{k=0}^{n-1}h(F^k(x))-n\int h\,d\mu\right)$$

satisfies the central limit theorem with the variance $\{\int h^2 d\mu-(\int h\,d\mu)^2\}$.

(2) *Suppose that $\hat{c}_n=n^{-1/2}L(n)$ and $\liminf_{n\to\infty}L(n)>0$. Then the central limit theorem does not hold.*

COROLLARY 5-5. (1) *Suppose that* $c_n = o(n^{-3/2}L(n))$ *where* $L(n)$ *is monotone decreasing to* 0. *Then the central limit theorem holds.*

(2) *On the contrary, suppose that the order of* c_n *is greater than or equal to* $n^{-3/2}$. *Then for a function h which satisfies*

$$\liminf_{k\to\infty} \lim_{t\to 0} \frac{1}{t} \frac{1}{(c_k - c_{k+1})} \int_{\langle k\rangle} (e^{it(h(x)-m)} - 1)dx > 0 ,$$

the central limit theorem does not hold. In particular, for any word w the central limit theorem does not hold for the indicator 1_w.

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Present Address:
DEPARTMENT OF MATHEMATICS, NATIONAL DEFENSE ACADEMY
YOKOSUKA 239, JAPAN